Resurgence[1]; convergence from divergence

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[1] An overview, and guide to literature.

Divergence and Dyson's argument

Asymptotic series and perturbation expansions are almost invariably divergent in practice, understood as zero radius of convergence. E.g. in Stirling's formula, as $n \to \infty$, $(2\pi n)^{-1} e^n n^{-n} \Gamma(n+1) \sim \sum_{k=0}^{\infty} c_k n^{-k}$, $\{c_j\}_j = \{1, \frac{1}{12}, \frac{1}{288}, -, -, +, +, ...\}$, where $|c_j| \sim j!/(2\pi)^j$. Factorial divergence occurs in virtually all special functions asymptotics.

Two more examples: $\mathrm{Ai}(x) \sim e^{-\frac{2}{3}x^{\frac{3}{2}}} \sum c_k x^{-\frac{3k}{2}-\frac{1}{4}}$ (a transseries) as $x \to +\infty$ and

$$\mathrm{Ai}(x) \sim e^{-\frac{2}{3}x^{\frac{3}{2}}} \sum c_k x^{-\frac{3k}{2}-\frac{1}{4}} - ie^{\frac{2}{3}x^{\frac{3}{2}}} \sum c_k x^{-\frac{3k}{2}-\frac{1}{4}},\ x \to -\infty$$

Finally, take $e^{-x}\mathrm{Ei}(x) = PV \int_0^{\infty} \frac{e^{-xp}}{1-p} dp$. As $x \to +\infty, e^{-x}\mathrm{Ei}(x) \sim \sum k! x^{-k-1}$. A calculation shows that $e^{-x}\mathrm{Ei}(x) \sim \pi i e^{x} + \sum_{k=0}^{\infty} k! x^{-k-1},\ x \to +\infty e^{2\pi i}$.

What do these examples have in common? Qualitative changes in behavior as ∞ is approached from different directions, **Stokes phenomena** (Dyson's argument). Convergent series at infinity clearly *cannot exhibit Stokes phenomena, hence the asymptotic series most special functions must have zero radius of convergence*: infinity is an **essential singularity**.

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When do we get essential singularities?

Such singularities result from perturbative expansions when, to leading order, the highest derivative is small, and would be eliminated in leading approximation:

$$-\hbar^2\Delta\psi - V\psi = \lambda\psi \ \ (\hbar \to 0)$$

meaning, in a first approximation we discard the highest derivative, as above, or,

$$y' + y = 1/x \ \ (t \to \infty)$$

(leading approximation $y \sim 1/x$) or when removing the perturbed term changes the nature of the problem,

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x,t)\psi(x,t)$$

Essential singularities

or even renders the physical quantity meaningless, s.a. in a path integral

$$\int_{-1}^{\infty} \cos\left(\varepsilon^{-\frac{3}{2}}\left(\frac{1}{3}t^3 + t^2 - \frac{2}{3}\right)\right) \mathrm{d}t$$

and of course in all realistic path integrals.

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So, what happens if we need to go ahead, nonetheless?

$$y' + y = \frac{1}{x} \to y = \frac{1}{x} - y' \to y \overset{[0]}{\approx} \frac{1}{x} \overset{[1]}{\approx} \frac{1}{x} - \overset{[0]}{y'} = \frac{1}{x} + \frac{1}{x^2}$$

$$\to \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} \to \cdots \to \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$$

Divergence! (Mathematically, we are **iterating on an unbounded operator**, and such iteration leads to factorial divergence. $\left(-\frac{d}{dx}\right)^n \frac{1}{x} = \frac{n!}{x^{n+1}}$) In all examples I mentioned before, we would get roughly the same phenomenon. We can check that the divergent series is a formal solution of the equation, nevertheless.

But is this the whole perturbation expansion?

It can't be. The ODE $y' + y = \frac{1}{x}$ must have a one-parameter family of solutions. The general solution is a particular one plus the general solution of the homogeneous equation:

$$\sum_{k\geq 0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x}$$

instanton expansions. The resummation methods in physics were pioneered by Bogomolny & Zinn-Justin. These two branches of what we now know as **resurgence theory** made contact around 2005, and since then there has been intense activity, and many workshops and programmes to exploit these points of contact.

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a **transseries**. All essential singularities resulting from perturbation expansions of otherwise mostly analytic problems are described by transseries, combinations of power series (divergent, in general) exponentials, and logs (sometimes). In theory, but almost never in practice, iterated exponentials occur $e^{-e^{x}}$ ...

Transseries were discovered in the late 70's by J. Écalle (Orsay!), who also found the way to resum them (accelero-summability) and independenly (for decades so) and simultaneously by Polyakov, Polyakov & 't Hooft, who called them multi-instanton expansions. The resummation methods in physics were pioneered by Bogomolny & Zinn-Justin. These two branches of what we now know as **resurgence theory** made contact around 2005, and since then there has been intense activity, and many workshops and programmes to exploit these points of contact.

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Two examples of transseries

The transseries expansion for the N-th energy level in the anharmonic oscillator as a function of the coupling:

$$E^{(N)}(g) = \sum_{n=0}^{\infty} \sum_{l=1}^{n-1} \sum_{m=0}^{\infty} c_{n,l,m} \left(e^{-S/g} g^{-(N+1/2)}\right)^{n} \left(\ln\left[\frac{a}{g}\right]\right)^{l} g^{m}\,, \tag{1}$$

with coefficients $c_{n,l,m}$ and constants a, S.

The general transseries of a generic system of nonlinear ODEs, with meromorphic coefficients, brought to normal form:

$$\mathbf{y}' = \Lambda \mathbf{y} + \frac{1}{x} B + \mathbf{g}(x, \mathbf{y});\ \mathbf{y} \in \mathbb{C}^d$$

is

$$\sum_{\mathbf{k}\in\mathbb{N}^d}^{\infty} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\lambda x} x^{\mathbf{k}\cdot\beta} (\ln x)^{|\mathbf{k}|} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{2}$$

where $\tilde{\mathbf{y}}_{\mathbf{k}}(x)$ are divergent power series in $1/x$.

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Least term truncation

The Stirling series is alternating; $\Gamma(x)$ is between successive truncates of the series. For x not too small, the terms $j!/(2\pi x)^j$ decrease to a minimum (*the least term*) before growing again. Truncating to get at the least term we get

$$1! \approx 0.9997, 2! = 2 \pm 10^{-6}, 3! = 6(1 \pm 10^{-9}), \ldots$$

Truncation to the least term goes back to Cauchy, and is quite accurate for this and many other functions.

Less known, least term truncation gives an accuracy of the order of the least term even for non-alternating series (say, all coefficients are positive). The requirements are *resurgence*[2] and that the terms beyond all orders (instanton corrections) vanish, as is often the case; and even if not the error is still exponentially small[3].

Using correction terms, there are ways to obtain even higher accuracy [3]. But not arbitrarily high.

Least term truncation is to be used as a "last resort"–when the number of terms is really small, or the accuracy is low.

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[2] Explained in the sequel.

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Classical Borel summation

Borel summation ($\mathcal{LB}$) is essentially based on *refinements* of Fourier analysis. Why Fourier? Ultimately, factorial divergence originates in applying $(\partial)^n$ which in Fourier space (the spectral measure unitary for ∂) becomes $(ik)^n$ which behaves geometrically, not factorially. Instead of Fourier, $\mathcal{L}^{-1}$ in Écalle critical time: if the exponential correction is say e^{-cx^a} the critical time is x^a.

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Back to the toy-model: $\mathcal{L}p^k = \frac{k!}{x^{k+1}}$ or $\mathcal{L}^{-1}\frac{k!}{x^{k+1}} = p^k$. Thus

$$\sum_{k=0}^{\infty}\frac{k!}{(-x)^{k+1}} = \hat{1}\sum_{k=0}^{\infty}\frac{k!}{(-x)^{k+1}} = \mathcal{L}\mathcal{L}^{-1}\sum_{k=0}^{\infty}\frac{k!}{(-x)^{k+1}} =: \mathcal{LB}\sum_{k=0}^{\infty}\frac{k!}{(-x)^{k+1}}$$
$$= \mathcal{L}\sum_{k=0}^{\infty}\mathcal{L}^{-1}\frac{k!}{(-x)^{k+1}} = \mathcal{L}\sum_{k=0}^{\infty}(-p)^k = \mathcal{L}\frac{1}{1+p} = \int_0^{\infty}\frac{e^{-xp}}{1+p}dp$$

Definition

A series $\tilde{f} = \sum_{k=0}^{\infty}\frac{a_k}{x^{k+1}}$ is Borel summable if $\mathcal{B}\tilde{f} = \sum_{k=0}^{\infty}\frac{a_k}{k!}p^k$ converges, to a function $F(p)$ which is real-analytic and exponentially bounded. Then, $\mathcal{LB}\tilde{f} =: \mathcal{L}F(p)$.

Classical Borel summation, features and limitations

$\mathcal{LB}$ is formally the identity **1**. Thus, whatever properties a series $\tilde{f}$ has, $f = \mathcal{LB}\tilde{f}$ has them too. Borel summation behaves like usual, convergent summation.

More interestingly, with $\tilde{f} = \sum k!(-x)^{-k-1}$ we have $\tilde{f}' - \tilde{f} = x^{-1}$, thus $(\mathcal{LB}\tilde{f})' - \mathcal{LB}\tilde{f} = \mathcal{LB}x^{-1}$, meaning that $f' - f = 1/x$, f is the actual solution of the ODE decaying at infinity, and thus $\mathcal{LB}\tilde{f} = e^x \mathrm{Ei}(-x)$.

Even more interestingly, the problem could be nonlinear too, since $\mathcal{LB}(\tilde{f}\tilde{g}) = (\mathcal{LB}\tilde{f})(\mathcal{LB}\tilde{g})$, and a power series solution of the Painlevé equation $y'' = y^2 + z$ becomes an actual solution; this allowed us to prove some important conjectures. This applies to PDEs too, such as the time-dependent Schrödinger equation.

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More interestingly, with $\tilde{f} = \sum k!(-x)^{-k-1}$ we have $\tilde{f}' - \tilde{f} = x^{-1}$, thus $(\mathcal{LB}\tilde{f})' - \mathcal{LB}\tilde{f} = \mathcal{LB}x^{-1}$, meaning that $f' - f = 1/x$, f is the actual solution of the ODE decaying at infinity, and thus $\mathcal{LB}\tilde{f} = e^x \mathrm{Ei}(-x)$.

Even more interestingly, the problem could be nonlinear too, since $\mathcal{LB}(\tilde{f}\tilde{g}) = (\mathcal{LB}\tilde{f})(\mathcal{LB}\tilde{g})$, and a power series solution of the Painlevé equation $y'' = y^2 + z$ becomes an actual solution; this allowed us to prove some important conjectures. This applies to PDEs too, such as the time-dependent Schrödinger equation.

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Which functions are resurgent?

All functions which occur naturally in mathematics (and mathematical physics) have been shown to be resurgent [3] It means: these functions are represented by Écalle-Borel summable transseries. This is of course remarkable.

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Therefore, if a transseries expansion solves a problem of "natural origin", then it is Écalle-Borel to a unique solution of the problem it originated in.

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Some outstanding questions where resurgence was instrumental: Nonlinear stability of self-similar singularity formation in supercritical[4] Wave Maps and Yang Mills [8], the solution of the time-periodic Schrödinger equation in external fields which are $O(1)$ [4] proof of Dubrovin's conjecture (pole positions of special solutions of Painlevé P1) [9].

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$$i\frac{\partial\psi}{\partial t} = -\Delta\psi - \frac{b}{r}\psi + V(x)\cos(\omega t)\psi = 0$$

This setting is relevant for atoms interacting with radiation (such as laser fields). At small V, the theory goes back to the 1930's (atoms ionize, and the exponential decay obeys the Fermi Golden Rule). For moderate-to-large amplitudes there are of course numerical methods, as well as semi-classical approximations (Keldysh theory) which a not always in qualitative agreement with the experiment. At this time, the only mathematical theory to date is based on resurgence (in t [16]). The phenomena in larger fields are much more subtle: islands of "stabilization", of power-law instead of exponential decay etc. This is in very good qualitative (sometimes quantitative) agreement with experiments [4] and references therein.

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Example of resurgent analysis: P1 [5] In normalized form, a modified Boutroux form, P1 reads

$$h'' - \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \quad (*)$$

When possible, instead of Borel transforming the asymptotic series we Borel transform the source of the series. The **Borel transform of (*)** is

$$H = (p^2-1)^{-1}\left(\frac{196}{1875}p^3 - \int_0^p sH(s)ds + \frac{1}{2}\int_0^p H(s)H(p-s)ds\right) \quad (**)$$

We "see" that $p = \pm 1$ are singular points. Looking more carefully, both are $1/\sqrt{}$ branch points. If we iterate (**), convolution spreads these two singularities at all nonzero integers. Generated by convolution, the singularities are related to each-other. The above is mechanism is typical of any order ODEs.

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General (small) transseries in P1

The general solution decaying along $\mathbb{R}^+$ (the **tronquées**) depends on a constant C and has the transseries

$$h(C,x) = \sum_{k\geq 0} C^k h_k x^{-k/2} e^{-kx}$$

where h_k are (generalized) Borel sums of divergent series; the h_k satisfy linear nonhomogeneous second order ODEs. Across a Stokes line $C \to C + S$, where $S = i\sqrt{6/(5\pi)}$ is the Stokes constant [6].

Resurgence. Let $H_k = \mathcal{L}^{-1} h_k$. The Borel plane jump at the j singularity of H_k is related to H_{k+j} through a formula independent of the ODE

$$(H_k^+ - H_k^-)_j = \binom{k+j}{j} S^j H_{k+j}^-$$

In particular, the whole structure of H_0 on the universal covering of $\mathbb{C} \setminus \mathbb{N}$ [7] is contained in H_k. Since it's all reduced to the first sheet, **endless continuation** also follows.

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Transasymptotics [8], [9] a sketch

We can view the transseries

$$h(x) = \sum_{j,k} c_{kj} C^k x^{-k/2} e^{-kx} x^{-j}$$

as a formal function of two variables $\xi = Cx^{-1/2}e^{-x}, \eta = 1/x$,

$$h(x) = F(\xi, \eta) = \sum_{k,j} c_{k,j} \xi^k \eta^j \quad (*)$$

When $\xi \ll \eta$ $(e^{-x} \ll 1/x)$, (*) was conveniently written in the standard "multiinstanton" form

$$\sum_{k \geq 0} h_k(\eta) \xi^k$$

[8] Instanton condensation!

However, when an antistokes line is approached (here $\pm i\mathbb{R}^+$, where the exponential becomes oscillatory), it is natural to write it in the form

$$(*)\ \sum_j F_j(\xi)\eta^k$$

Plugging (*) in P_1 and solving perturbatively in η we get

$$F_0(\xi) = \frac{\xi}{(\xi/12-1)^2}$$

and all F_k **are rational functions**. We see formation of singularities near antistokes lines, at the points $Ce^{-x}x^{-1/2} \approx 12$, infinitely many of them due to the periodicity of e^{-x}.

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A simple PDE example [15]

Simplest example: the heat equation, $f_t = f_{xx}$. Because the equation is parabolic, if we solve the initial value problem by a series expansion $f = \sum t^k f_k(x)$, $f_0 = f(0,x)$, the PDE implies $f_{k+1}(x) = f_k''(x)/k$, that is

$$f(x,t) = \sum_{k\geq 0} \frac{f_0^{(2k)}(x)}{k!} t^k$$

which diverges factorially even if f_0 is analytic (but not entire). Instead of Borel transforming the solution it is much better to Borel transform the equation, in $1/t$. This gives better analytic control, and more importantly we can allow non-analytic initial conditions. With $f(t,x) = t^{-1/2} g(1/t, x)$ and $\mathcal{L}^{-1}_{\frac{1}{t}} g(q) = q^{-1/2} G(x, 2q^{1/2})$, $2q^{1/2} = p$, the equation becomes

$$G_{pp} - G_{xx} = 0$$

the wave equation, for which power series solutions converge.

Cont, and more general PDEs

Using the elementary solution of the wave equation $G_1(x+p)+G_2(x-p)$ and the initial and boundary conditions, one gets, after returning to f by Laplace transform and changes of variables,

$$f(t,x) = t^{-1/2}\int_{-\infty}^{\infty} f(0,s)\exp(-(x-s)^2/(4t))ds$$

The point here, of course, is not to solve the heat equation in closed form. It is, rather, like in most applications of resurgence, to transform divergent series into convergent ones, more generally singular perturbations into regular perturbations. This approach allows $f(0,s)$ to be general, say in L^1 and also shows when resurgence is obtained: essentially iff $f(0,s)$ is analytic.

A conceptually similar approach applies to **very general systems of nonlinear PDEs** (Navier-Stokes included) [11,8], resulting in Laplace representations of actual solutions, proving (at least local) existence of solutions and the possibility to control solutions more globally.

Because of dependence on initial conditions, one studies resurgence of the **Green's function or of the unitary propagator**. Fairly well understood for time-periodic d-dim Schrödinger equations. In these models, the Borel sum of the series is insufficient; one needs the full transseries. [16]

The Borel transform as a regularizing operator

Another interesting property of Borel summation is that it is a **regularizing transformation**. The derivative in $y' + y = \frac{1}{x}$ is singularly perturbed at ∞. Its Borel transform ($\approx$ inverse Laplace) is $-pY + Y = 1$, an algebraic equation where the previously singularly perturbed term is not singularly perturbed anymore.

$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2}$ is parabolic (singular for small t); its Borel transform, $\frac{\partial^2 H}{\partial q^2} = \frac{\partial^2 H}{\partial x^2}$, its Borel transform, is a regular, hyperbolic, PDE.

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That's because the Borel transform is a (refinement of) the Fourier transform [9], the spectral measure unitary transformation for ∂, $f' \mapsto -p\hat{f}$.

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The all important Borel plane

For a function f or a formal series, this is the space where $\mathcal{L}^{-1}f$ or $\mathcal{B}\tilde{f}$ lives. For time-dependent Schrödinger, this coincides with the *energy space*. Due to the fact that the Borel transform is regularizing, $\mathcal{B}\tilde{f}$ has only regular singularities. These singularities contain most of the information about f, qualitative and quantitative alike. E.g., for $e^{-x}\mathrm{Ei}(x) \sim \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$, it is the p plane, where $\mathcal{B}\tilde{f} = \frac{1}{1-p}$

Figure: The Borel plane for $e^{-x}\mathrm{Ei}(x)$ (left); (right): typical Borel plane (Painlevé transcendents, or anharmonic oscillators): p-plane singularities of resurgent functions are *always spaced in periodic arrays, and are regular singularities!* This is instrumental in recovering global information from divergent resurgent series.

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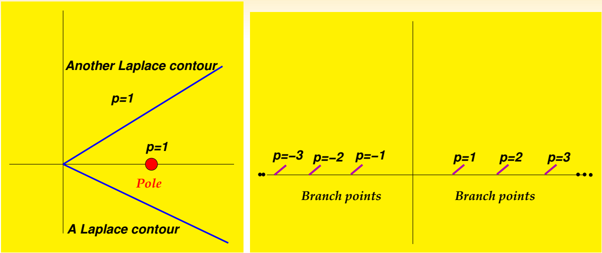

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What if we only have a finite # of asymptotic terms?

We are developing methods to approximately obtain the Borel plane structure of resurgent functions/series $\tilde{f}$, from a limited number of asymptotic coefficients, with limited accuracy. Provably exact in the limit when all the information exists, in practice we get very good accuracy with 10-15 terms and 10 digit accuracy. We still need the Borel plane. Here we took 200 terms in soln. of P1.

The first step is standard in numerical resummation: Padé of $\mathcal{B}\tilde{f}$.

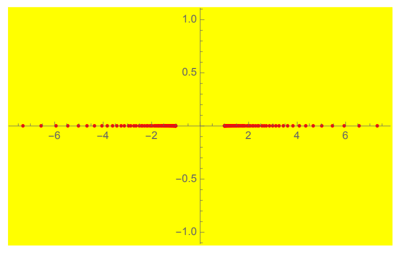

Figure: Borel-Padé for Painlevé. After Borel transform $\mathcal{B}\tilde{f} = F$, F is convergent in the unit disk $\mathbb{D}$. Padé of F gives the position of the singularity lines and of the first singularities: poles at ± 1. It **misses** however all other poles; this will be fixed in the next step. Let $\mathcal{D} = \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, the yellow region.

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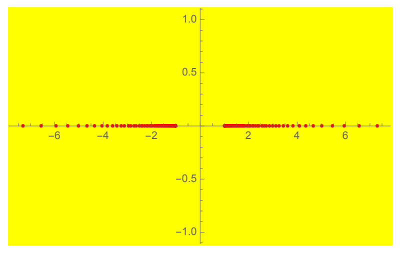

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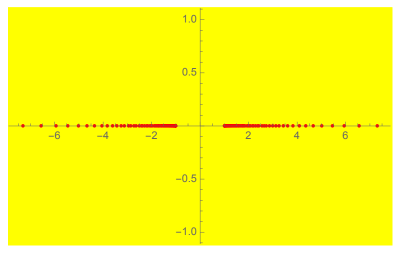

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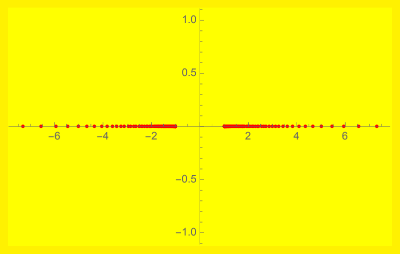

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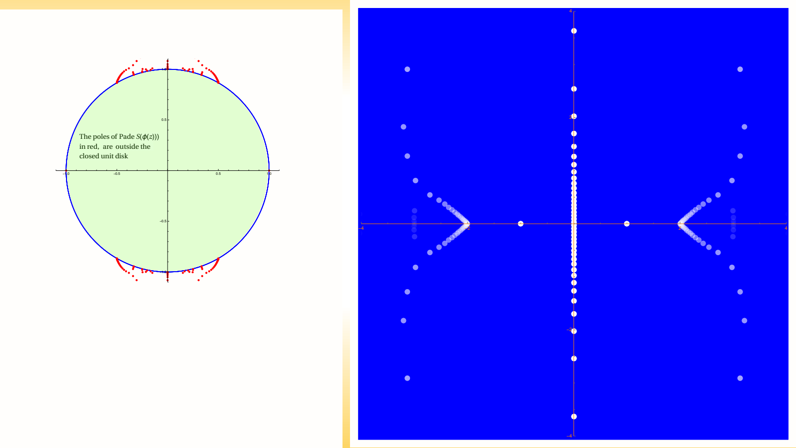
Global representation: resurgent-conformal Padé

The domain of analyticity of F is $\mathcal{D}$. Let $p = \varphi(q)$ be the conformal map of the unit disk $\mathbb{D}$ onto $\mathcal{D}$. Expand $F(\varphi(q))$ in series for small q, $S(q)$. $S(q)$ *must converge in* $\mathbb{D}$. Padé $S(q) \mapsto P(q)$. Conformal-Padé: $F(p) = P(\varphi^{-1}(p))$ in the whole of $\mathcal{D}$!



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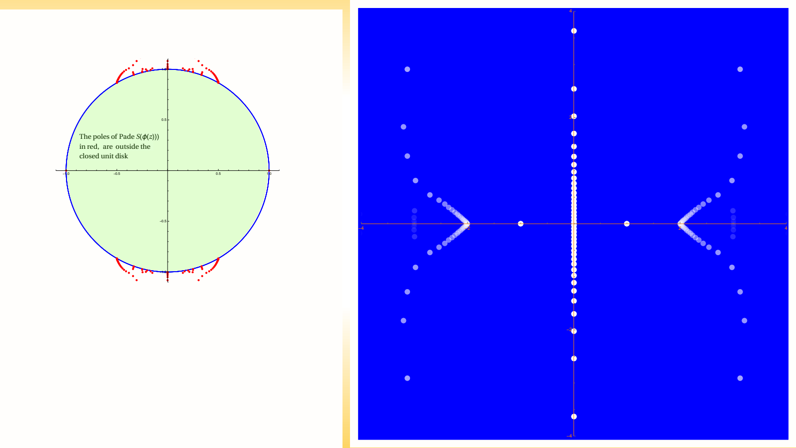

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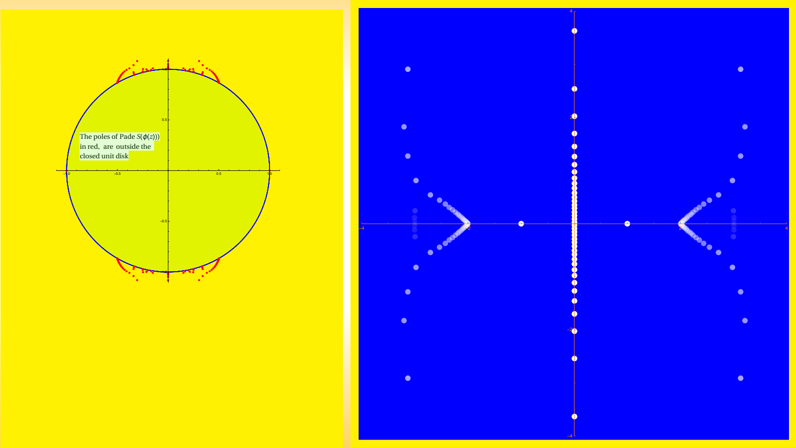

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Large to small coupling connection

Calculate a resurgent fcn. in the whole physical domain using asypt. info only

Resurgent-Padé gives 60 digits of accuracy up to $|p| = 50$ in Borel plane (using 200 asympt terms; divide by 10 for 15 terms).

Therefore, $\mathcal{L}\mathcal{B}\tilde{f}$ has accuracy e^{-50} down to $x = 1$. Re-expanding at 1, we get at least 40 digits of accuracy in $\mathbb{C}$, all the way down to 0. Similarly for the

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does not rule out rational function expansions. Some expansions in Pochhammer symbols $1/(x)_m$ (here $(x)_m = x(x+1)\cdots(x+m-1)$) were known–in fact since Stirling.

$$\ln\Gamma(x) = x\ln x - x + \tfrac{1}{2}\ln\frac{2\pi}{x} + \frac{1}{12(x+1)} + \frac{1}{12(x+1)(x+2)}$$
$$+ \frac{59}{360(x+1)(x+2)(x+3)} + \cdots + \frac{c_m}{(x)_m} + \cdots; \quad c_m \sim m^{-a} m!$$

We have, with $\theta = \arg x$,

$$\frac{1}{(x)_m} \sim \frac{\rho_\theta^m}{m!}; \quad \rho_\theta \in \begin{cases} \{1\}; \ |\theta| \le \frac{\pi}{2} \\ (1,2); \ |\theta| \in (\frac{\pi}{2}, \pi) \end{cases}$$

thus the series converges (albeit slowly) and **only** for $\mathrm{Re}\, x > 0$. All special functions admit similar representations, but these drawbacks make them of little use and low popularity. Why is the domain of convergence limited to a half-plane? The singularity type of the expansion and of the function are of different type. Dyson's argument, in spirit, still applies, and, as we see, still needs further care.

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And if we address all issues in Dyson's argument?

We remove the last obstruction by allowing poles to accumulate on one ray.

Remarkably, this is enough to obtain the needed 2^{-m} improvement in the decay of the coefficients.

Remarkably too, all resurgent functions can be represented uniformly in their domain, by these enhanced rational expansions.

$$e^{-x}\mathrm{Ei}(x) = i\pi e^{-x} - \sum_{m=1}^{\infty} \frac{\Gamma(m)}{2^m (y)_m} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Gamma(m) e^{-\frac{i\pi}{2^k}}}{(1+e^{-\frac{i\pi}{2^k}})^m} \frac{1}{(2^k y)_m}; \; x \notin i\mathbb{R}^-$$

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Binary rational expansions as global representations

Theorem (OC, RD Costin, (2018))

The following are equivalent:

(i) f *is represented, for any* $\delta \in (-a, a), R > 0$ *by some Pochhammer symbol-rational expansion, uniformly convergent for* $|z| > R$ *in* $\mathbb{C} \setminus \mathbb{R}^- e^{i\delta}$.

(ii) f *has a Cauchy-Stieltjes representation*

$$f(z) = \int_{-\infty}^{0} \frac{F(s)}{s-z} ds \text{ for } z \in \mathbb{C} \setminus \mathbb{R}^-$$

with F *analytic in* $\{z : |z| > 0, \arg z \in (-\pi - a, -\pi + a)\}$ *and* $O(1/z)$ *for large* z*;*

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Resurgent functions (can be decomposed to) satisfy (iii) and thus can be represented convergently in terms of binary rational expansions

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Applications

Proof of the Dubrovin conjecture, which states that the tritronquée solutions of Painlevé P1,

$$y'' = y^2 + z$$

i.e., the solutions that have a maximal asymptotic sector S of analyticity ($\frac{4}{5}(2\pi)$) are analytic in the closed sector above, down to the origin (OC, Huang, Tanveer, Duke Math J 2014).

P1 is an ODE with meromorphic coefficients, and thus tritronquée is resurgent. Its binary rational representation is convergent if $\{z \in S, |z| > 0\}$ and has the behavior $O(1/z)$ as $z \to 0$. On the other hand all singularities of P1 are double poles. Thus the tritronquée is analytic in a neighborhood of $\{z \in S, |z| \geq 0\}$. $\square$

Convergent representations for entropy, partition functions, effective action a.s.o. in QFT and string theory. In (OC, Dunne, J.Phys A (2018)) we give many examples, such as strong-coupling expansions of one-loop corrections for Wilson loop minimal surfaces in $AdS_5 \times S^5$.

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New proof, OC, RD Costin, G Dunne.

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Plot of Y_0 on the imaginary axis

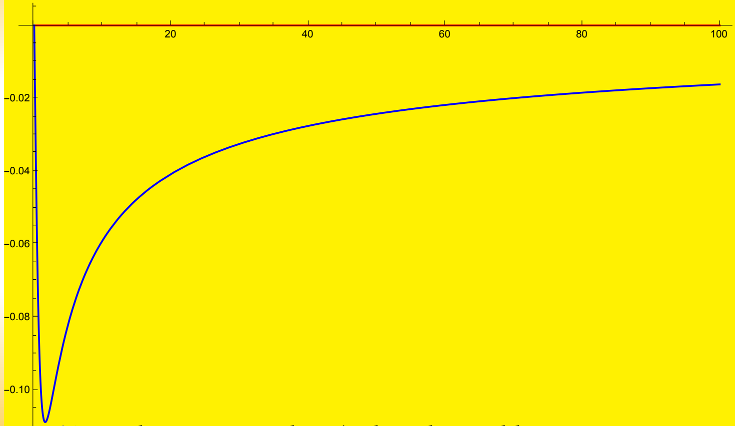


Figure: Y_0 on the imaginary line (red=real part, blue=imaginary part; it looks essentially the same in all directions except for the Stokes ray.

Plot of Y_0 on the singularity line

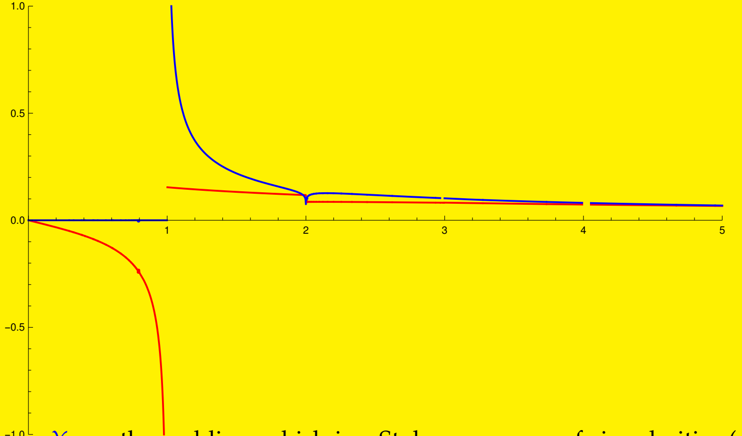


Figure: Y_0 on the real line, which is a Stokes ray, a ray of singularities (red=real part, blue=imaginary part. Note: conformal-Padé is calculated on the very singular line.

Plot of y_0 in the domain of analyticity

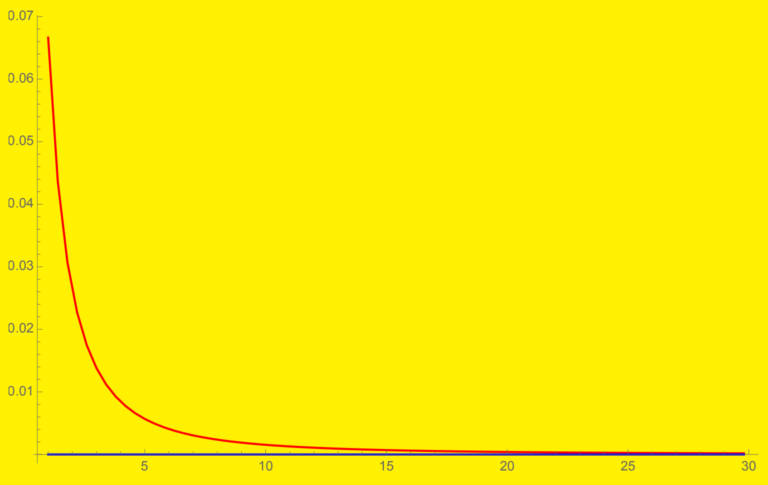


Figure: y_0 on $-i\mathbb{R}^+$; it looks essentially the same inside the analyticity sector.

The Stokes transition: $\operatorname{Im} e^x \sqrt{x}(y_0^+ - y_0^-)$ on the Stokes line $\mathbb{R}^+$

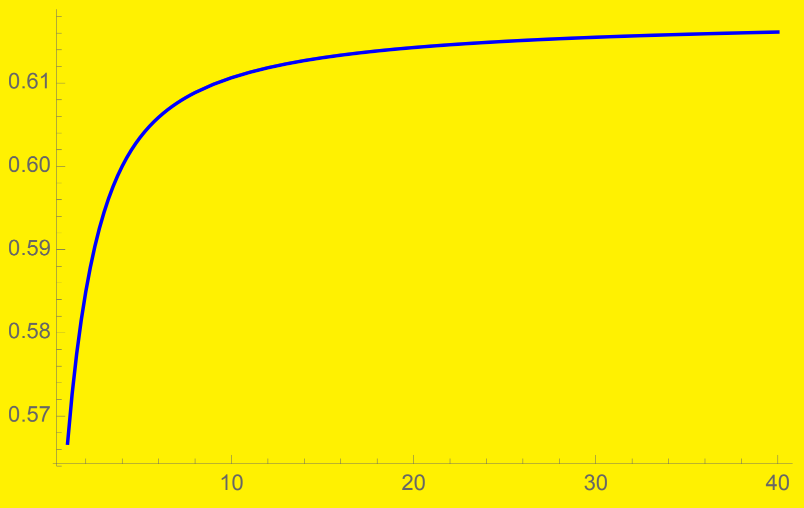
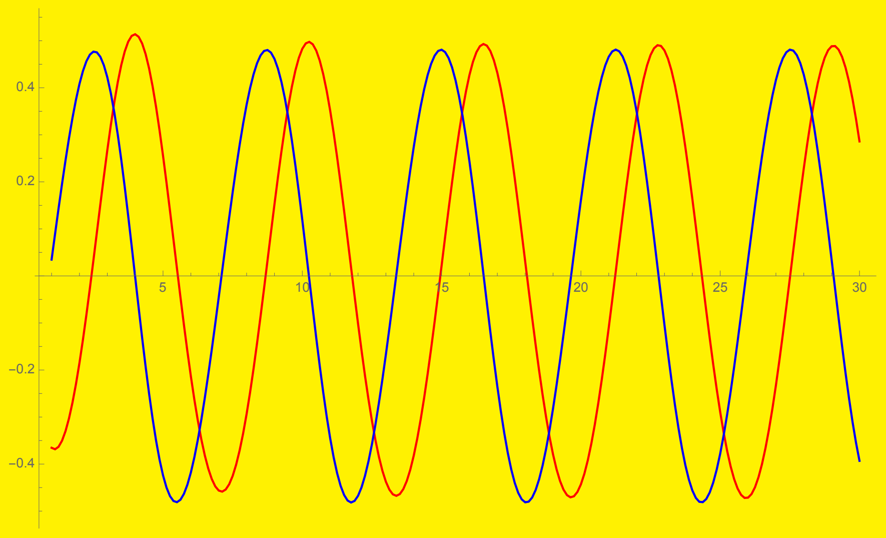


Figure: At $x = 110$ one gets S with 3 digits, where the real part is about 10^{50}.

Plot of y_0 on the edge of the sector of analyticity, an antistokes line



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