# Order in Chaotic Quantum Systems: An Emergent Phenomenon

Saclay, January 2017

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# Contents

# 1. Quantum Chaos

**Deterministic classical chaos:** Trajectories in phase space are instable. Exponential divergence of trajectories that start from neighboring points in phase space. Long-term behavior in time not predictable either analytically or numerically. Only probabilistic statements possible. (Maxwell, Poincare, Kolmogorov).

Quantum chaos ("Quantum Manifestations of Classical chaos"): Theory less completely developed. Investigations on
• Spectra (eigenvalues and eigenfunctions)
or
• Time-evolution of wave packets (Chirikov).

Here: Discuss only spectra of completely chaotic systems.



A. Lichtenberg M. Lieberman, Regular and Stochastic Motion
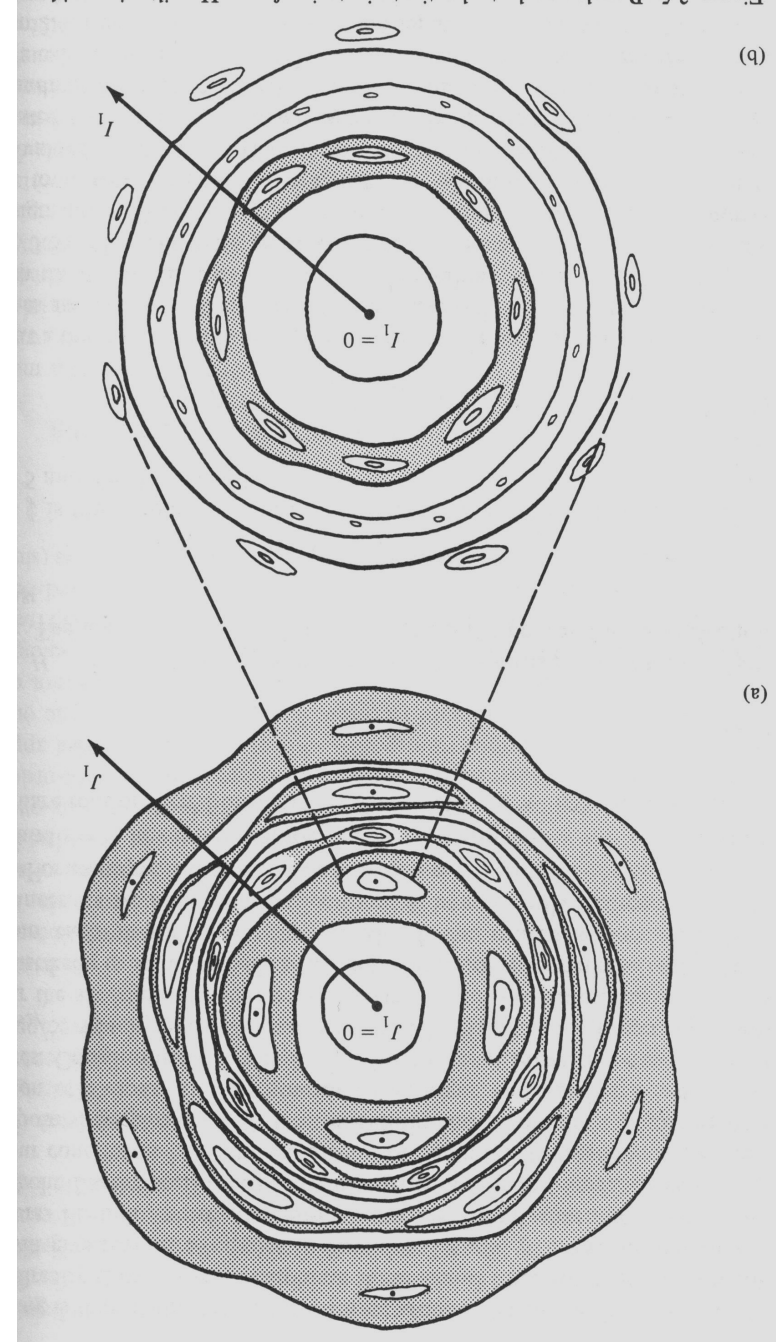
Can we make general statements on systems that are not integrable and do not possess any symmetries? Yes, this is possible using

# Random Matrices (Wigner).

Matrix representation $H_{\mu\nu}$ of Hamiltonian in Hilbert space. Indices $\mu,\nu = 1, ... N$ and N large. In systems with time-reversal invariance we can always choose $H_{\mu\nu} = H_{\nu\mu}$ real. Further symmetries shall not exist.

Essential Point: Consider ensemble of such Hamiltonian matrices. How to choose ensemble? No preferred direction in Hilbert space: Invariance under orthogonal transformations: "Gaussian Orthogonal Ensemble" (GOE). All results derived by averaging (integrating) over ensemble: Valid for "almost all" Hamiltonian systems.

$$N \ \exp [ - \text{trace} (H^2) / \lambda^2 ] \ \Pi \, dH_{\mu\nu}$$

All states are coupled to each other. Only free parameter $\lambda$ determines mean level spacing. Gaussian cutoff factor arbitrary but convenient. Quantitative and parameter-free predictions are possible. These are universal and ergodic. Also other symmetry classes (GUE, GSE). F. J. Dyson, J. Math. Phys. 3 (1962) 1199.

# Quantitative predictions:

**(a)** Distribution of spacings of neighboring eigenvalues ("nearest-neighbor spacing distribution").

s is the level distance in units of the mean level spacing. Result is parameter-free. Level repulsion at small distances.
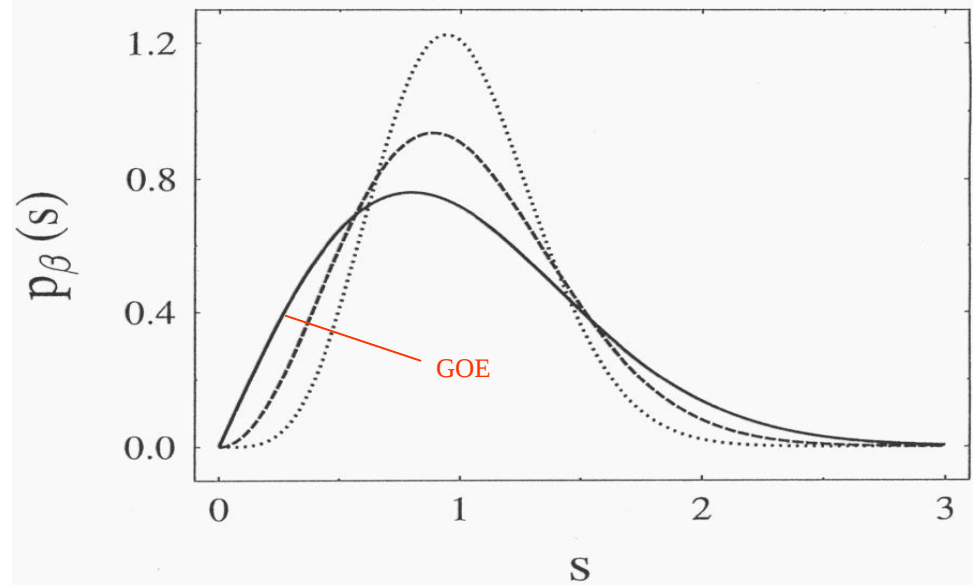
**(b)** Variance of the number of levels in interval of length L ("level variance").

L is measured in units of the mean level spacing. Variance grows only logarithmically with L! Dyson-Mehta or Delta 3 statistics used below is directly related to level variance.
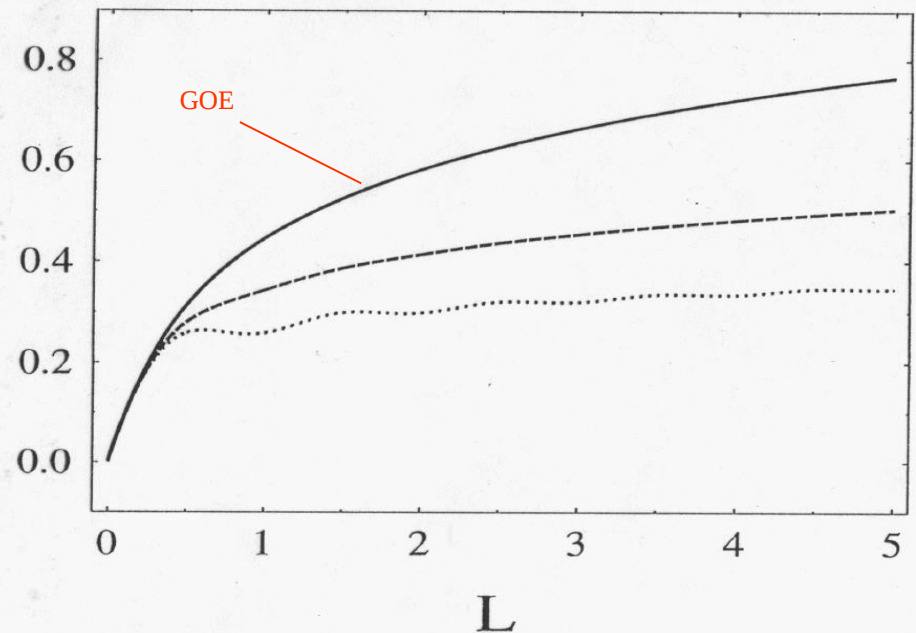
These are statistical measures. Tests require large data sets.

**Studies of chaotic quantum systems have long history. Two cases that have been studied intensely:**

(a) Hydrogen atom in strong magnetic field. The field breaks the rotational symmetry of the Coulomb potential. Only cylindrical symmetry about field direction remains. For Rydberg states, that causes classically chaotic motion. Large number of states measured.

More recently: Chaos in ultracold gas of Er atoms. A. Frisch et al., Nature 507 (2014) 475.

Numerical investigations of several chaotic systems culminated in

(b) Sinai billard. A "toy model". Has mirror symmetry in regard to 4 axes. Generate about 1000 lowest eigenvalues of states with fixed symmetry numerically.

O. Bohigas, E. Giannoni, C. Schmit, Phys. Rev. Lett. 52 (1984) 1.
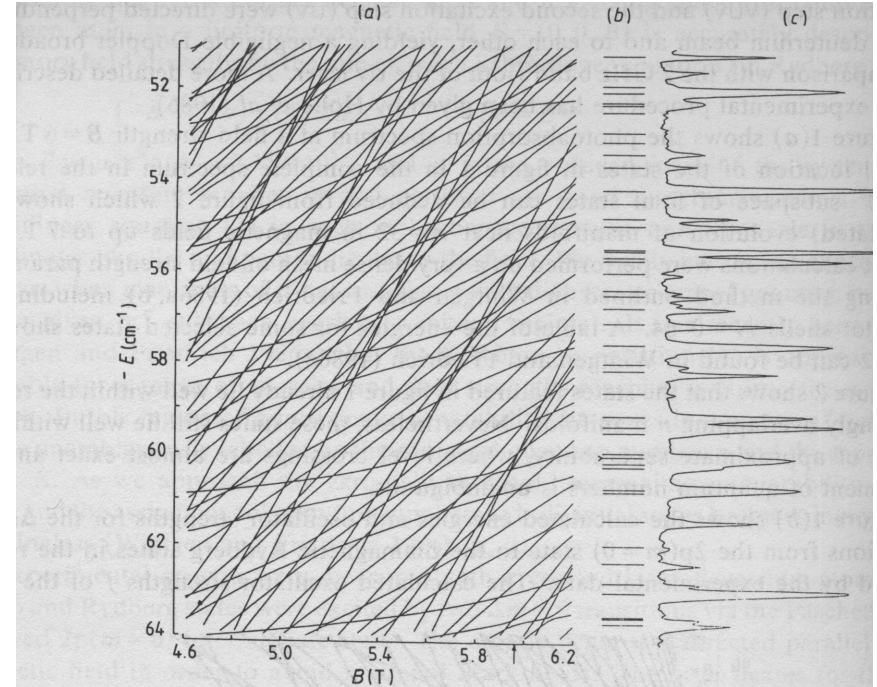


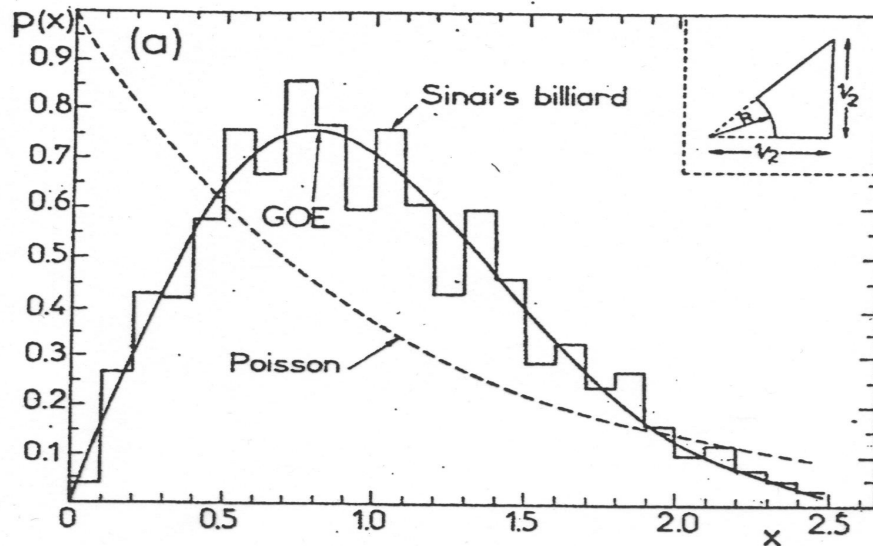D. Wintgen, A. Holle, G. Wiebusch, J. Main, H. Friedrich, K. H. Welge, J. Phys. B 19 (1986) L 557
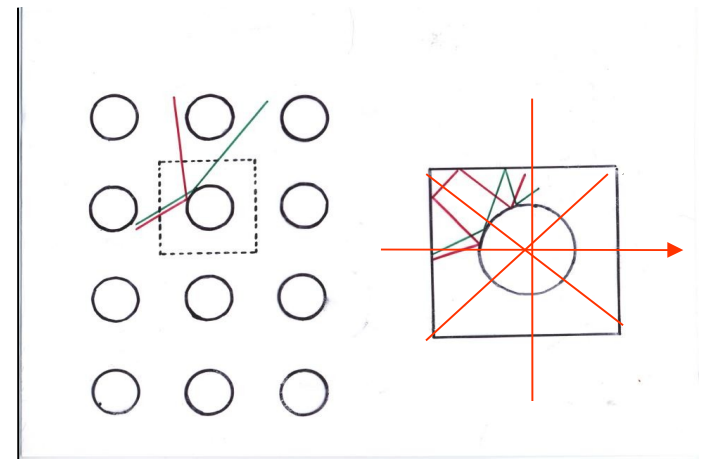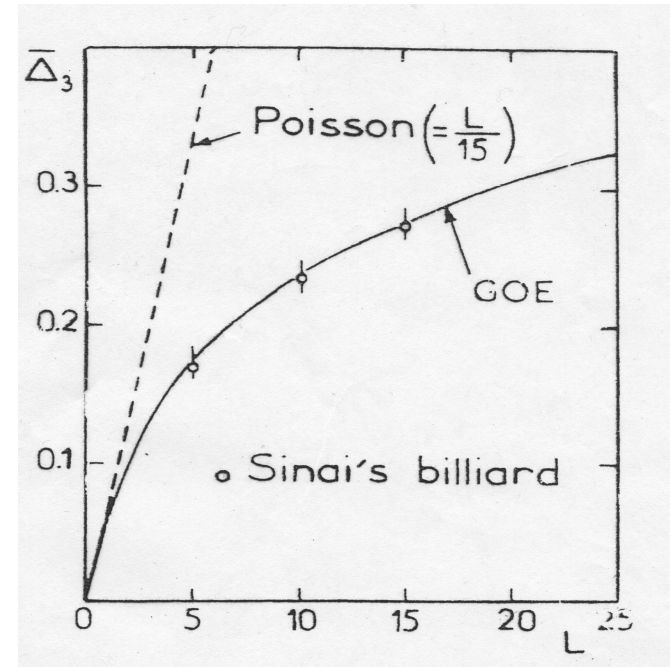
Investigate spectral fluctuations (distribution of eigenvalues and eigenfunctions) with the measures provided by the theory of random matrices.



O. Bohigas, E. M Giannoni and C. Schmit, Phys. Rev. Lett. 52 (1984) 1.

"Bohigas-Giannoni-Schmit conjecture": The spectral fluctuation properties of fully chaotic quantum systems coincide with those of the random-matrix ensemble in the same symmetry class. Many numerical studies confirm conjecture. Analytical proof uses semiclassical approximation for level-level correlator. Random-matrix predictions apply in energy interval of length $\Delta E = h / \tau$ with $\tau$ = period of shortest periodic classical trajectory. S. Heusler, S. Müller, A. Altland, P. Braun, F. Haake, New J. Phys. 11 (2009) 103025..

Proof for quantum graphs and for all correlators of both levels and scattering matrix elements. Z. Pluhar and H. A. Weidenmueller, Phzs. Rev. Lett. 112 (2014) 144102.

# Many-body systems

Levels with fixed quantum numbers: Long sequences of data (neutron scattering near threshold or proton scattering near Coulomb barrier)



Poisson

NDE
1726 spacings

GOE

1.0

0.5

0     1     2     3    S
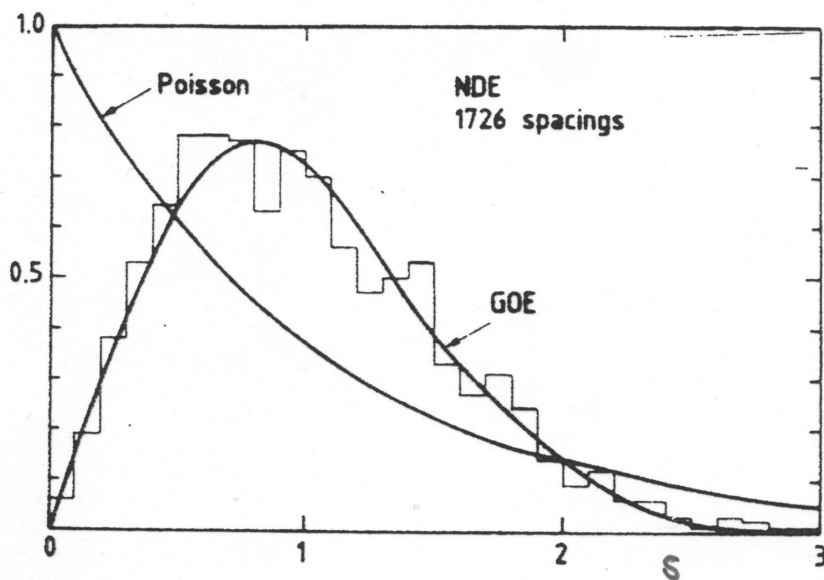
R. Haq, A. Pandey, O. Bohigas, Phys. Rev. Lett. 43 (1982) 1026
and Nuclear Data for Science and Technology, Riedel (1983) 209.

$\Delta_3$

Poisson

GOE

0.3

0.2

0.1

● Experiment (NDE)

0    5    10    15    20    25   L
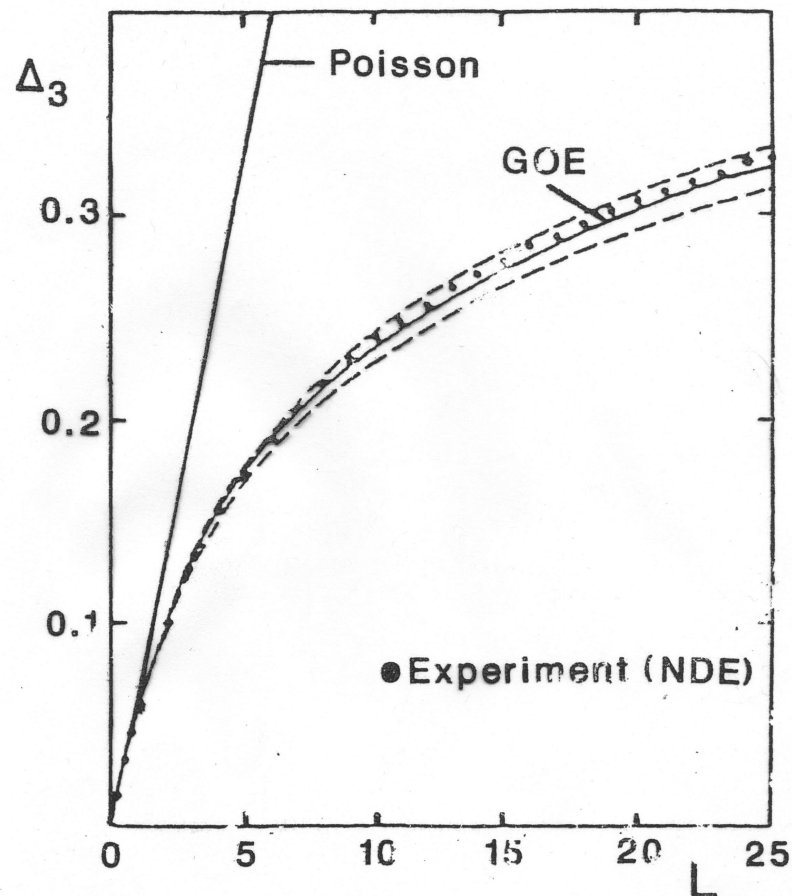
More but weaker evidence from other types of nuclear data. Similar but weaker evidence for complex atoms and molecules.

# 2. Emergence of Order

So far focus on sequence of states with identical quantum numbers. In nuclei: Spin, parity, isospin. Agreement of such a spectrum with RMT prediction implies zero information content. But a (piece of a) single spectrum (fixed quantum numbers) does not yield complete knowledge of the system. Relative order of spectra with different quantum numbers? Defined for every dynamical Hamiltonian but no prediction from RMT. Consider

Ensemble of single-particle Hamiltonians with random two-body interaction.
C. W. Johnson, G. F. Bertsch, D. J. Dean, Phys. Rev. Lett. 80 (1998) 2749.

Nucleons in degenerate single-particle states with spins j. Two-particle states with spins (j, j') coupled to spin I. States (I, $\alpha$). For fixed I, matrix elements $V_{\alpha\alpha'}$ of two-body interaction are Gaussian random variables with zero mean value and unit variance. Choose

$$j = \left\{ \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2} \right\} \quad \text{(dimension 12)} \quad \text{and} \quad j = \left\{ \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2} \right\} \quad \text{(dimension 20)}.$$

Total spin J and isospin T (reflects neutron-proton symmetry) are good quantum numbers.

TABLE I. Percentage of ground states (g.s.) of the RQE that have $J = 0, T = T_z$ for our target nuclides, as compared to the percentage of all states in the model spaces that have these quantum numbers.

| $N$ | $\Omega$ | Nucleus | $J = 0, T = T_z$ g.s. | $J = 0, T = T_z$ Total space |
|---|---|---|---|---|
| 6 | 12 | $^{22}O$ | 76% | 9.8% |
| 6 | 20 | $^{46}Ca$ | 75% | 3.5% |
| $N = 4, Z = 4$ | 12 | $^{24}Mg$ | 66% | 1.1% |

C. W. Johnson, G. F. Bertsch, D. J. Dean, Phys. Rev. Lett. 80 (1998) 2749.

## Preponderance of ground states with spin zero!

Also found: Large gap between spin zero ground state and first excited state. Large enhancement of transition strength between ground state and first excited state compared to statistical estimate.

A flurry of activity uncovers similar regularities in bosonic and electronic many-body systems with random two-body interactions.

R. Bijker, A. Frank, Phys. Rev. Lett. 84 (2000) 420;
P. Jaquod, A. D. Stone, Phys. Rev. Lett. 84 (2000) 3938;
V. Zelevinsky, A. Volya, Phys. Rep. 391 (2004) 311.

# 3. Explanations?

Use single index $(\alpha\alpha') \to (a)$ and write for the random two-body matrix elements $V_{\alpha\alpha'} \to v_a$ . For fixed total spin J Hamiltonian H(J) is linear in the two-body matrix elements,

$$H(J) = \sum_a v_a C_a(J)$$

The matrices $C_a(J)$ transport the random two-body interaction into the many-body Hilbert space of states with spin J. These matrices depend only on the geometry of the shell model. For $J \neq J'$ , H(J) and H(J') depend on same random variables and are, therefore, correlated. Linear transformation among pairs of states {a} leads to

$$d^{-1}(J)\mathrm{Trace}[C_a(J)C_b(J)] = \delta_{ab}s_a^2(J) \;\;.$$

Key to understanding: Spectral widths (= r.m.s. widths of  level densities for different spin values) defined by

$$\sigma^2(J) = d^{-1}(J)\mathrm{Trace}[H^2(J)] = \sum_a v_a^2 s_a^2(J) \;\;.$$

The matrix elements $v_\alpha$ are zero-centered independent Gaussian random variables with unit variance. Spectral widths are correlated!
Preponderance of spin zero ground states depends on values of $s_a^2(J)$ . These are determined  by the model space!

Spectral radius $R_J = r_J \sigma(J)$ where $r_J$ is not random and decreases strongly with J. Correlations between spectral widths increase probability for ground states with spin zero.   T. Papenbrock , H. A. Weidenmüller, Phys. Rev. Lett. 93 (2004) 132503.

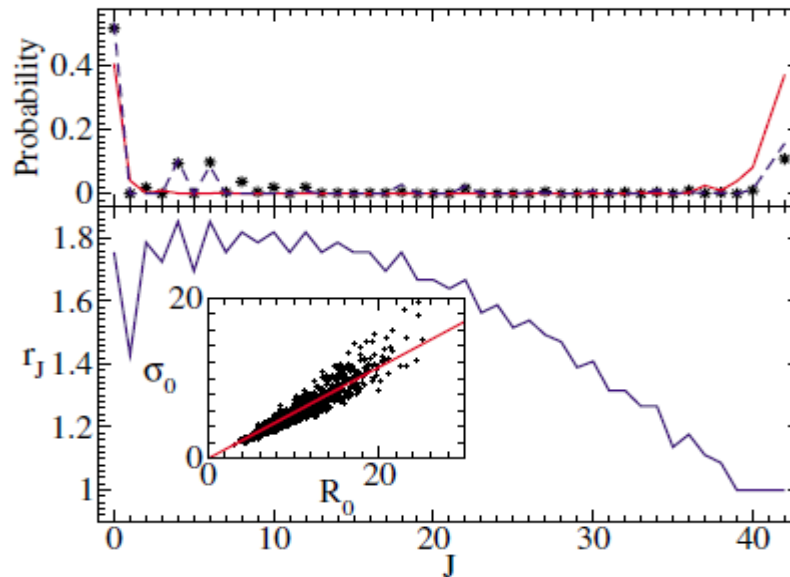

FIG. 1 (color online). Six fermions in a shell with spin $j = 19/2$. Top: Probability that the ground-state has spin $J$ (data points); probability that spin $J$ has the largest spectral width (solid line); probability that the product $r_J \sigma_J$ is maximal (dashed line). Bottom: Scaling factor $r_J$ between the widths and spectral radii. Inset: Spectral radius $R_0$ versus width $\sigma_0$ (data points) and the linear fit (line) for total spin $J = 0$. (Results from 900 random realizations).

Preponderance of ground states with spin zero is caused by geometry of the shell model.

# 4. Summary

In chaotic quantum systems, (pieces of spectra) of states with fixed quantum numbers follow RMT predictions and carry zero information content.

Ensemble of random Hamiltonians displays preponderance of spin zero ground states and other regularities.

Hamiltonians for different spin states are correlated. Correlations lead to a quantitative understanding of preponderance of spin zero ground states.

Correlations reflect the geometry of the underlying dynamical system (here: the shell model). Statistically, the geometry dominates the action of the random two-body matrix elements. Not all, but unexpectedly many ground states have spin zero.

# 1. Why random matrices? What are random matrices?

Below the first threshold for particle emission (and aside from gamma decay), the spectra of atoms, molecules, and atomic nuclei are discrete. The states are characterized by quantum numbers that relate to symmetries: spin ↔ rotational symmetry, parity ↔ reflection symmetry, isospin ↔ neutron-proton-symmetry.
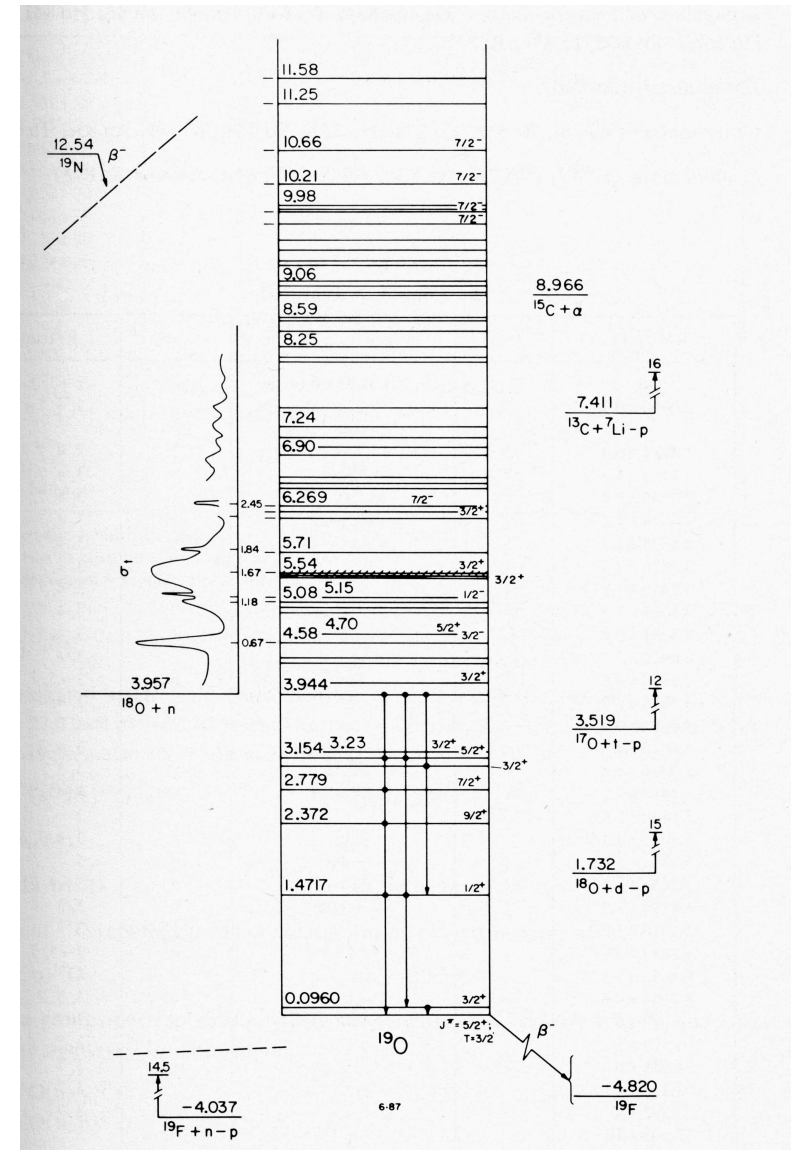
Such spectra can frequently be reproduced using simple, integrable models: regular dynamics.

Regular: Rotational bands with spin / parity 0+, 2+, 4+, ...and excitation energies proportional to J(J+1). In molecules and in atomic nuclei.
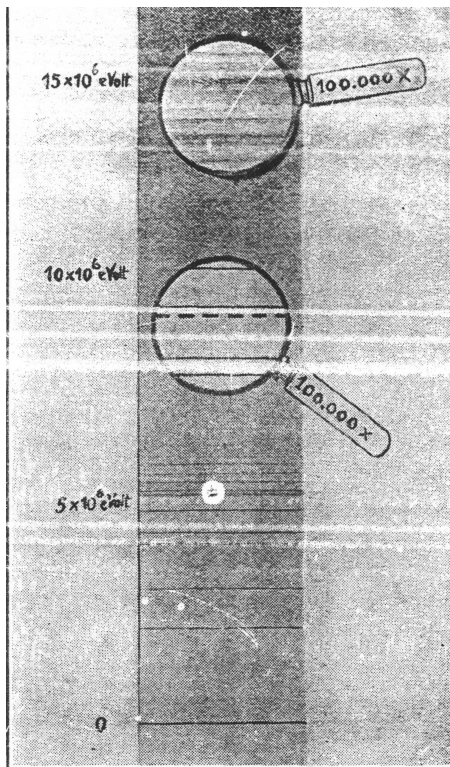
Regular: Motion of independent particles in the mean field. In atoms and in atomic nuclei ("nuclear shell model").

Strong evidence for the validity of both models for regular motion in atoms, molecules, and atomic nuclei. Applies typically to low-lying states with a variety of quantum numbers.



A. Bohr and B. Mottelson, Nuclear Structure
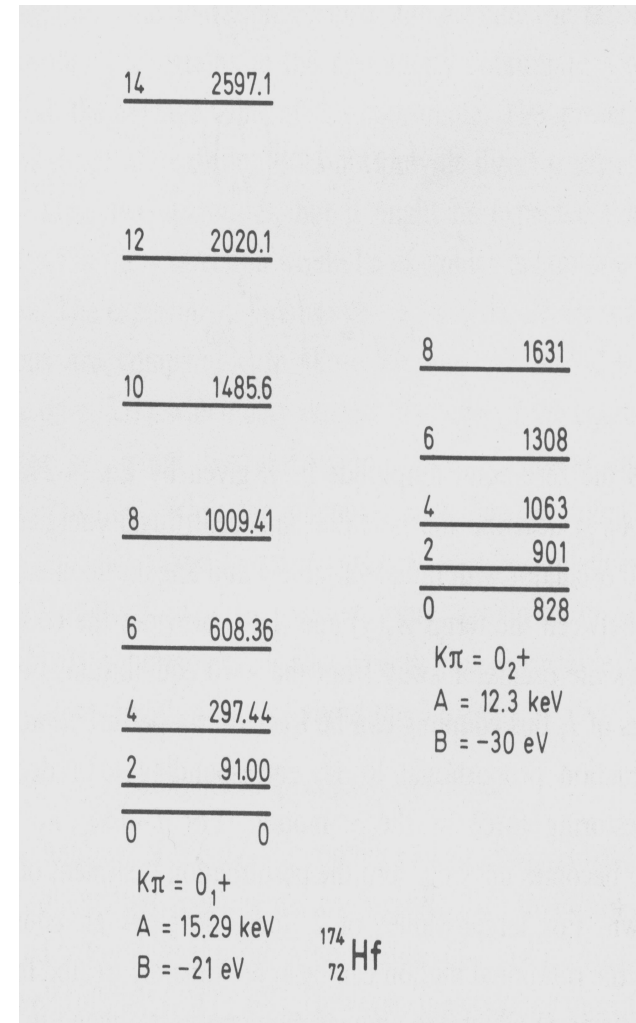
# But there is also strong evidence for non-regular behavior!

In atomic nuclei there exist long sequences (about 150 to 200 elements) of states with identical quantum numbers.



Niels Bohr: The narrow and narrowly spaced resonances are not compatible with the motion of independent particles in the nucleus. The "compound nucleus" is a system of strongly interacting nucleons.

J. B. Garg et al., Phys. Rev. 134 (1964) B 985.

N. Bohr, Nature 137 (1936) 344.
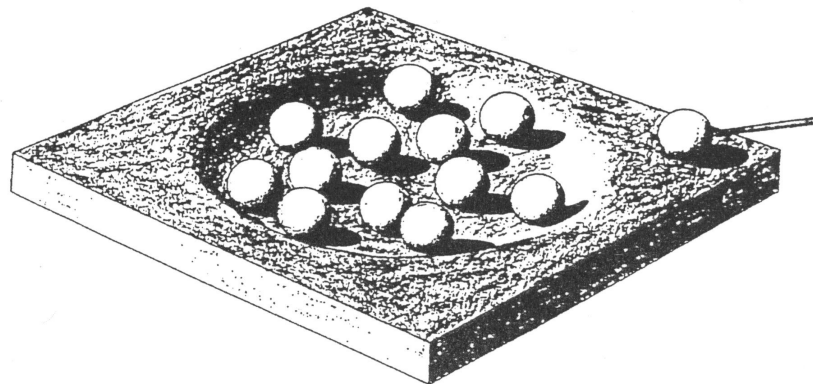
Conclusion:

The spectral fluctuations of sufficiently complex many-body systems follow predictions of random-matrix theory. Reason not completely understood.

Questions:

How can random-matrix theory be reconciled with the nuclear shell model (integrable)?

That is the status of about 1990. Since then an explosion of random-matrix theory and its applications.

# 4. Chaotic Scattering

Resonances seen in neutron scattering cross section follow random-matrix predictions. Is it possible to develop a theory of resonance scattering based upon random-matrix theory?
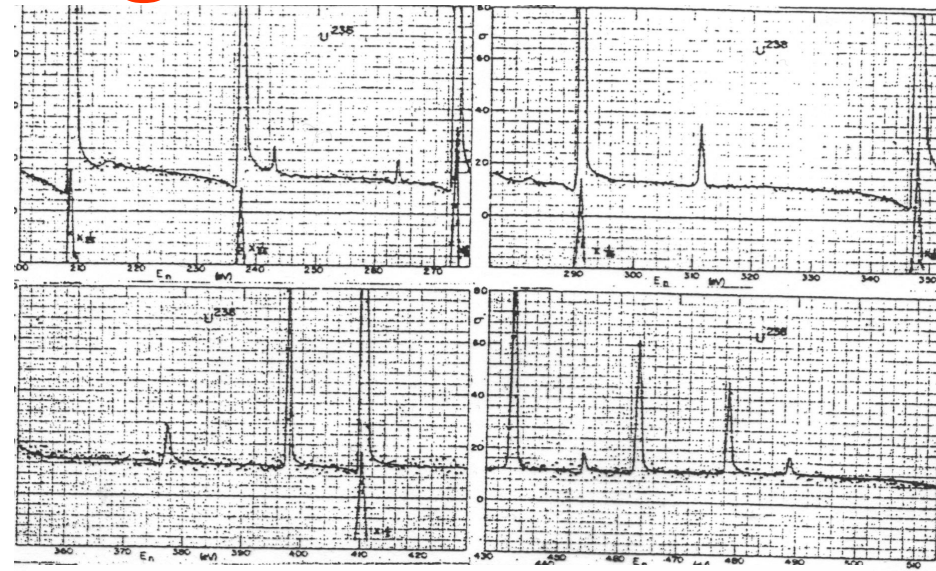
Stochastic behavior of cross sections not confined to isolated resonances. Also for strongly overlapping resonances ("Ericson fluctuations"). Can both cases be covered in terms of a random-matrix approach?





O. Häusser et al., Nucl. Phys. A 109 (1968) 329.
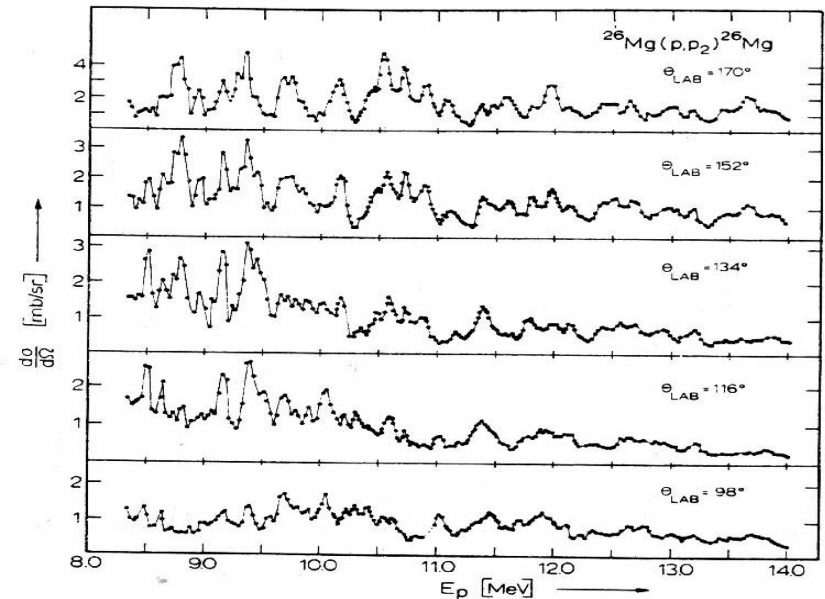
Closely related scattering problems arise in many parts of physics: nuclear physics, transport of electrons through disordered solids, transmission of light through a medium with a disordered index of refraction, transmission of radio waves through the turbulent atmosphere, transmission of electromagnetic waves through cavities in the form of chaotic billiards, …

A unified approach to all these scattering problems in terms of a random-matrix description has emerged in the last 20 years.

How to build such a theory? A nutshell description follows.

The amplitude for quantum-mechanical resonance scattering from channel a to channel b depends on energy E, on the Hamiltonian H describing the resonances, and on matrix elements that couple channels and resonances. The universal form is (generalized Breit-Wigner)

$$S_{ab}(E) = \delta_{ab} - i\pi \sum_{\mu\nu} W_{a\mu} \left[ (E - H + 2i\pi W^{\dagger}W)^{-1} \right]_{\mu\nu} W_{\nu b} .$$

Replace in that S matrix the actual Hamiltonian H by a random-matrix ensemble of proper symmetry. This generates ensemble of scattering matrices which is used to calculate properties of chaotic scattering. The cross section is proportional to the square of the scattering amplitude.

The parameters are $\lambda$ (given by the mean spacing of the resonances) and the quantities $\sum_{\mu} W_{a\mu} W_{\mu b} / \lambda$. The number of the latter exactly corresponds to the number of average S-matrix elements. Therefore it is possible to predict cross-section averages and fluctuations in terms of average S-matrix elements.

The average cross section is known analytically for all parameter values.

J. J. M. Verbaarschot, H. A. Weidenmüller and M. R. Zirnbauer, Phys. Rep. 129 (1985) 367.

# Example: Microwave resonator as chaotic billiard

A flat microwave resonator (height d = 0.84 cm) admits only a single vertical mode of the electric field up to a frequency of 18.75 GHz. In that frequency domain, the Helmholtz equation is equivalent to the Schrödinger equation for a two-dimensional billiard. For a proper choice of the shape, the billiard is chaotic. Measurements of the output amplitudes versus input amplitudes allow for a precise test of chaotic scattering theory.
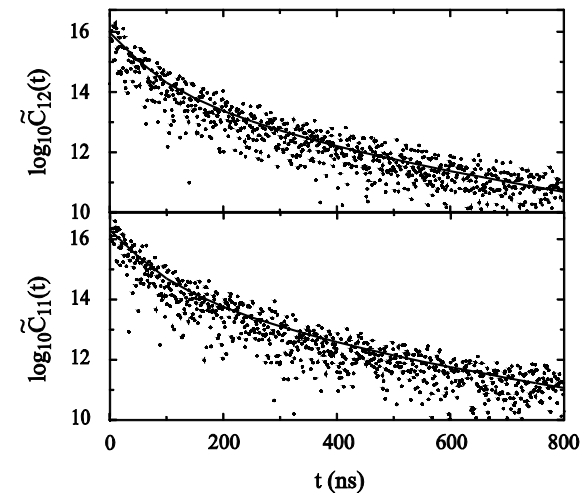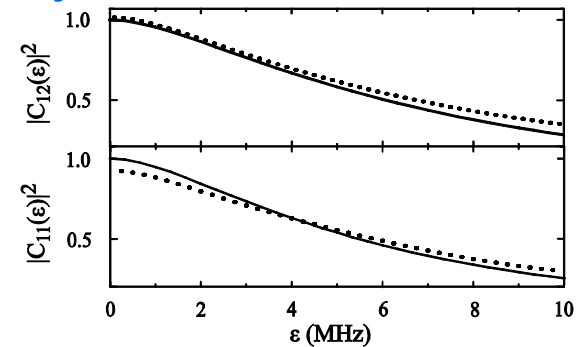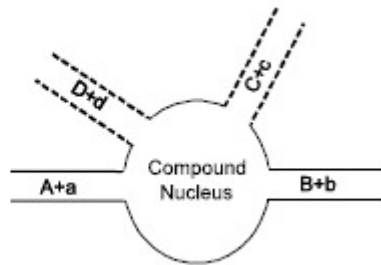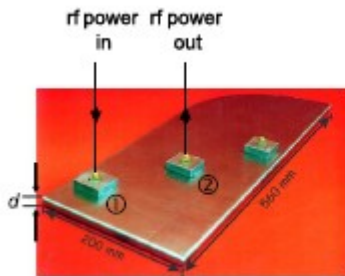


Flat microwave resonator (left) as a model for compound-nucleus scattering (right) or any other stochastic scattering process.

Autocorrelation function and log of its Fourier transform for weakly overlapping resonances. Notice the non—exponential decay in time.

B. Dietz et al., Phys. Rev. E 78 (2008) 055204R.

# 5. Random matrices in condensed-matter physics and in QCD

Condensed matter: Transmission of electrons through quantum dots with disorder. Representation of the insulator - conductor transition (Anderson) with random matrices. New classes of random matrix

ensembles in Andreev scattering. Topological insulators.

Andreev scattering: Interphase of superconductor and disordered normal conductor. Electron in normal conductor cannot penetrate into superconductor (pairing gap). Picks up second electron and leaves a hole. Near Fermi energy that process creates several new classes of random-matrix ensembles.  A. Altland and M. R. Zirnbauer, Phys. Rev. B 55 (1997) 1142

QCD: In the low-energy domain, QCD is equivalent to a random-matrix ensemble with chiral symmetry. This generates additional classes of random-matrix ensembles. These are used, for instance, for extrapolating lattice-gauge calculations to infinite system size. E. Shuryak and J. Verbaarschot, Nucl. Phys. A 560 (1993) 306.

There exists a total of ten random-matrix ensembles.

# 6. Mathematical aspects

The need to work out answers from random-matrix theory has triggered important developments in mathematical physics.
Examples: Supersymmetry (combination of commuting and anticommuting integration variables). Exploration of symmetric Riemannian spaces.

There is a very curious connection between GUE and number theory. According to the Riemann hypothesis, all non-trivial zeros of the Riemann zeta function,

$$\zeta(s) = \prod_p (1 - 1/p^s)^{-1}$$

in the complex s-plane lie on a straight line parallel to imaginary axis. Numerical results up to millions of zeros show that distribution of spacings follows those of GUE. Use this fact and known properties of GUE to conjecture properties of that distribution in analytical form. What have prime numbers to do with randomness?

# 7. Summary

- Random matrices find broad applications in quantum physics and beyond (quantum chaos, many-body systems, disordered systems, QCD, number theory).
- There are three canonical ensembles. Another seven ensembles play a role near a special energy of the system (f.i., Fermi energy).
- Random-matrix simulations help in understanding system properties.
- Theory has stimulated important developments in mathematical physics.
- Typical input: Mean values (must be determined dynamically). Prediction: Fluctuations.
- Usual approach in physics: Calculate system properties from Hamiltonian (bottom up). Here different (top down): Test whether given data set agrees with random-matrix prediction. If so, data carry no information content beyond mean values.

A rich and quickly growing field.