

Counting pairs in the shell model

P. Van Isacker, GANIL, France

Why and how to count pairs

Isoscalar vs isovector pairs

jj vs LS pairs

Aligned $T = 0$ np pairs

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Why?

Pairing and isospin symmetry in proton-rich nuclei

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$$\langle \mathcal{N}_{np} \rangle = \frac{(\mathcal{N} - T) \left(1 - \frac{\mathcal{N} - T - 3}{2\Omega}\right)}{2T + 3}$$

$$\langle \mathcal{N}_{pp} \rangle = (T + 1) \langle \mathcal{N}_{np} \rangle ,$$

$$\langle \mathcal{N}_{nn} \rangle = \langle \mathcal{N}_{pp} \rangle + T \left(1 - \frac{\mathcal{N} - 1}{\Omega}\right)$$

These expressions are striking. Not surprisingly, when $T = 0$, i.e. $N = Z$, all three expectation values are equal. But adding just a single pair of neutrons so that $T = T_z = 1$ causes the number of np pairs to drop by about 40%, while the number of pp pairs actually increases by about 7% (the number of nn pairs, naturally, increases even more). This result, which in

Pair-number operators

Two-body operators that count pairs, defined by diagonal two-body matrix elements.

jj coupling: pairs coupled to J_2

$$\langle j_{1\rho}j_{2\rho'}; J'_2 | \hat{N}_{\rho\rho'}(J_2) | j_{1\rho}j_{2\rho'}; J'_2 \rangle = \delta_{J_2 J'_2}$$

LS coupling: pairs coupled to L_2 and S_2

$$\langle l_{1\rho}l_{2\rho'}; L'_2 S'_2 | \hat{N}_{\rho\rho'}(L_2, S_2) | l_{1\rho}l_{2\rho'}; L'_2 S'_2 \rangle = \delta_{L_2 L'_2} \delta_{S_2 S'_2}$$

The pair number is the expectation value in the state of interest.

Counting LS pairs in a jj basis

Time-reversed LS orbitals: $L_2 = 0, l_1 = l_2$.

Anti-symmetry then requires $L_2 + S_2 + T_2$ to be odd

$S_2 = 0 \Rightarrow T_2 = 1$: *isovector pairs*

$S_2 = 1 \Rightarrow T_2 = 0$: *isoscalar pairs*

The pair-number operators must be defined in a jj basis, so the matrix elements become

$$\langle j_{1\nu} j_{2\pi}; J_2 | \hat{N}_{\nu\pi}(L_2, S_2) | j_{3\nu} j_{4\pi}; J_2 \rangle = \begin{bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L_2 & S_2 & J_2 \end{bmatrix} \begin{bmatrix} l_1 & \frac{1}{2} & j_3 \\ l_2 & \frac{1}{2} & j_4 \\ L_2 & S_2 & J_2 \end{bmatrix}$$

Counting jj pairs in a jj basis

$J_2 = 0$ forces $j_1 = j_2$ and $T_2 = 1$: such a pair is genuinely isovector.

$J_2 = 1$ allows $j_1 \neq j_2$ and $T_2 = 0$ or $T_2 = 1$: so $\nu\nu$ and $\pi\pi$ pairs exist with $J_2 = 1$; they are not isoscalar.

A modified operator keeps both nucleons in one orbital:

$$\langle j_\nu j_\pi; J'_2 | \hat{N}'_{\nu\pi}(J_2) | j_\nu j_\pi; J'_2 \rangle = \delta_{J_2 J'_2}, \quad j_\nu = j_\pi$$

Now odd J_2 does imply $T_2 = 0$.

Sum rules

Pair numbers must exhaust the total number of pairs.

jj-coupled pairs:

$$\langle n_\nu n_\pi \alpha J M_J | \sum_{J_2} \hat{N}_{\rho\rho}(J_2) | n_\nu n_\pi \alpha' J M_J \rangle = \frac{1}{2} n_\rho (n_\rho - 1) \delta_{\alpha\alpha'}$$

$$\langle n_\nu n_\pi \alpha J M_J | \sum_{J_2} \hat{N}_{\nu\pi}(J_2) | n_\nu n_\pi \alpha' J M_J \rangle = n_\nu n_\pi \delta_{\alpha\alpha'}$$

LS-coupled pairs:

$$\langle n_\nu n_\pi \alpha J M_J | \sum_{L_2 S_2} \hat{N}_{\rho\rho}(L_2 S_2) | n_\nu n_\pi \alpha' J M_J \rangle = \frac{1}{2} n_\rho (n_\rho - 1) \delta_{\alpha\alpha'}$$

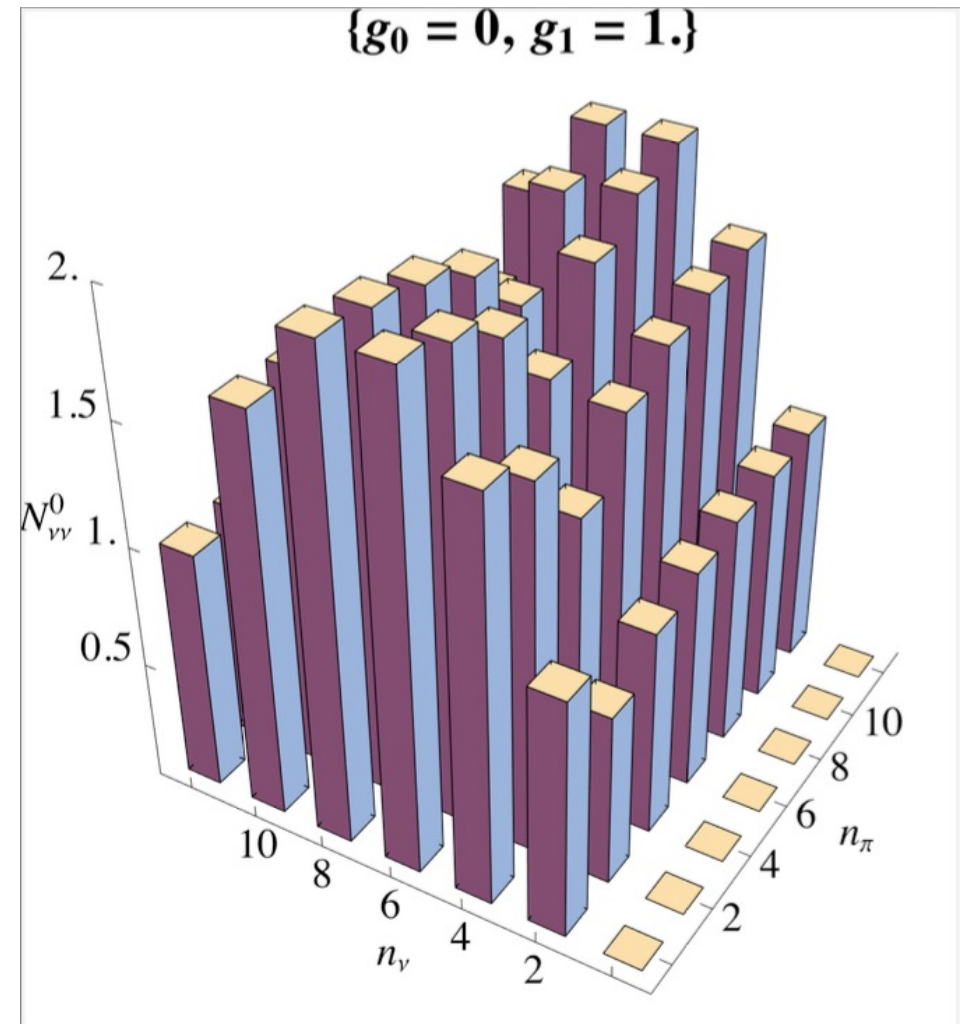
$$\langle n_\nu n_\pi \alpha J M_J | \sum_{L_2 S_2} \hat{N}_{\nu\pi}(L_2 S_2) | n_\nu n_\pi \alpha' J M_J \rangle = n_\nu n_\pi \delta_{\alpha\alpha'}$$

The modified operator of the previous slide does not satisfy these sum rules.

Isovector pairing in $j = 11/2$

The number of $\nu\nu$ pairs with $J_2 = 0$ in the ground state of an isovector pairing interaction.

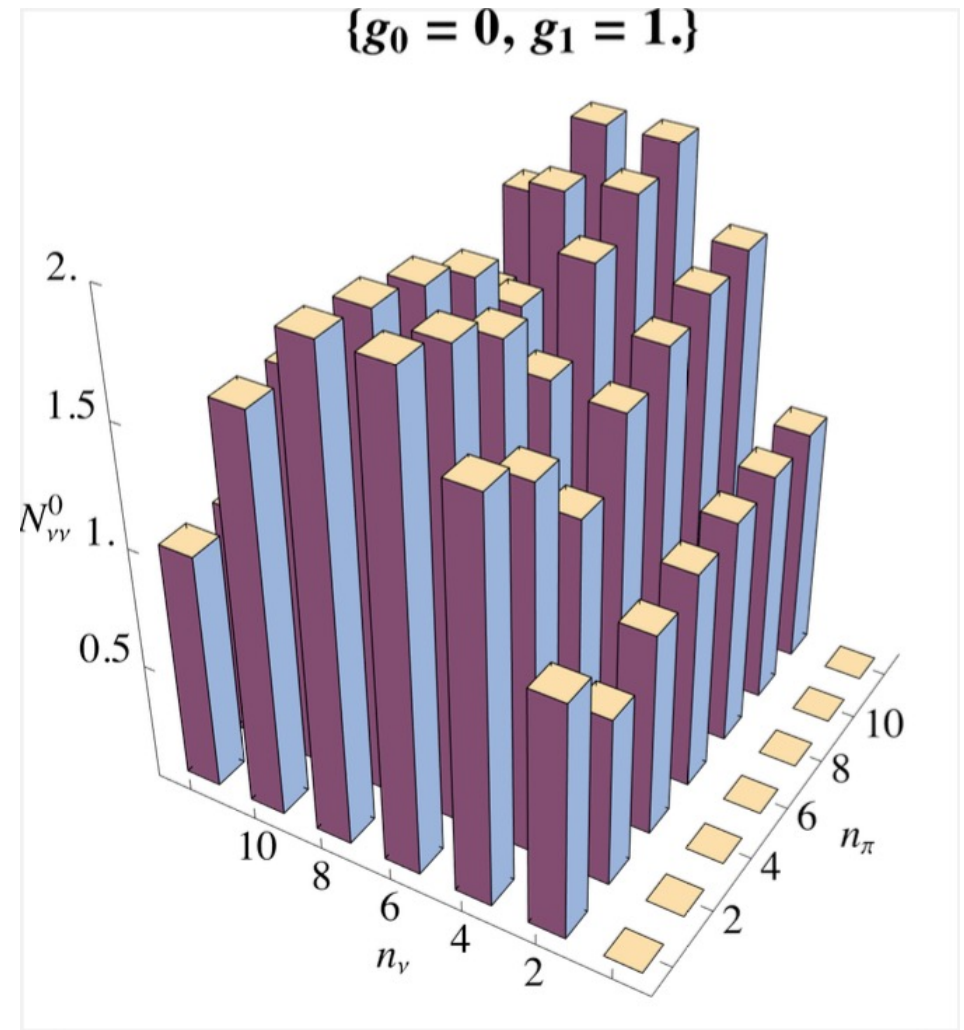
Analytic expression derived from SO(5) symmetry.



Isovector pairing in $j = 11/2$

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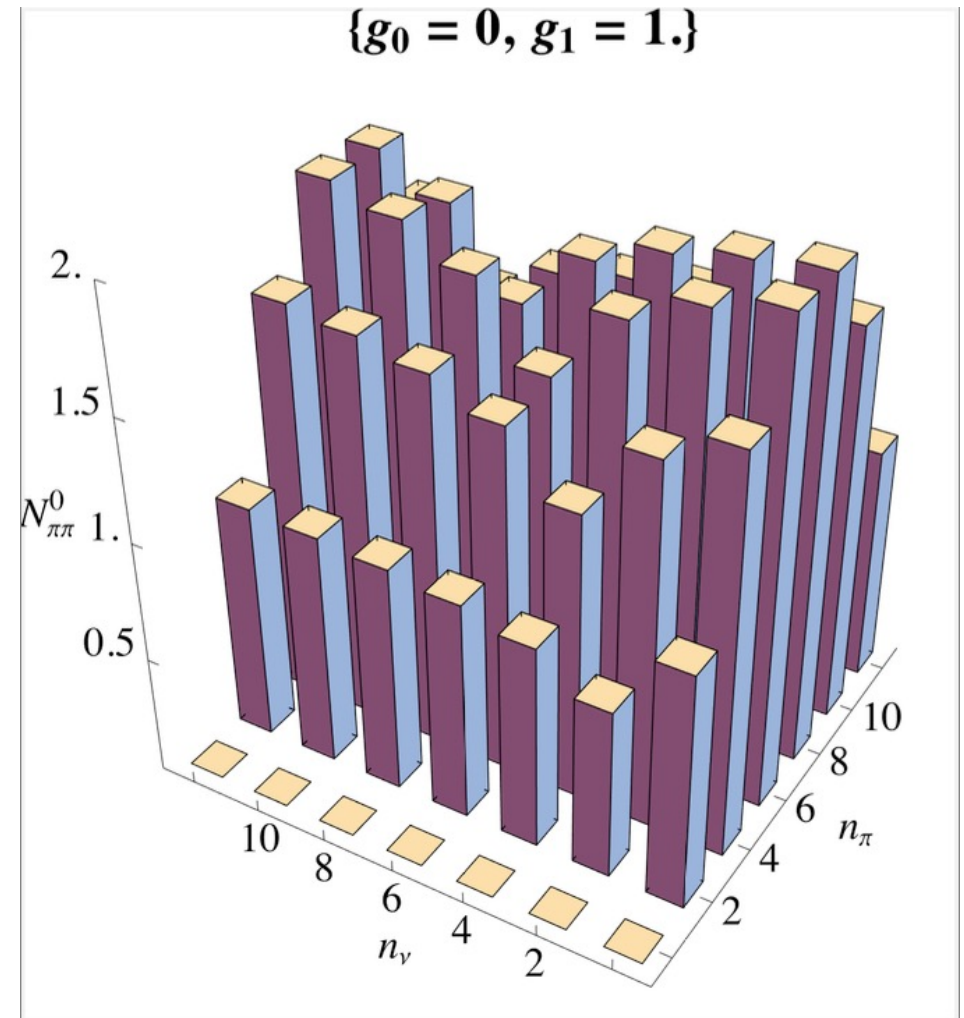
The expectation value of the pair-number operator.



Isovector pairing in $j = 11/2$

The number of $\pi\pi$ pairs with $J_2 = 0$ in the ground state of an isovector pairing interaction.

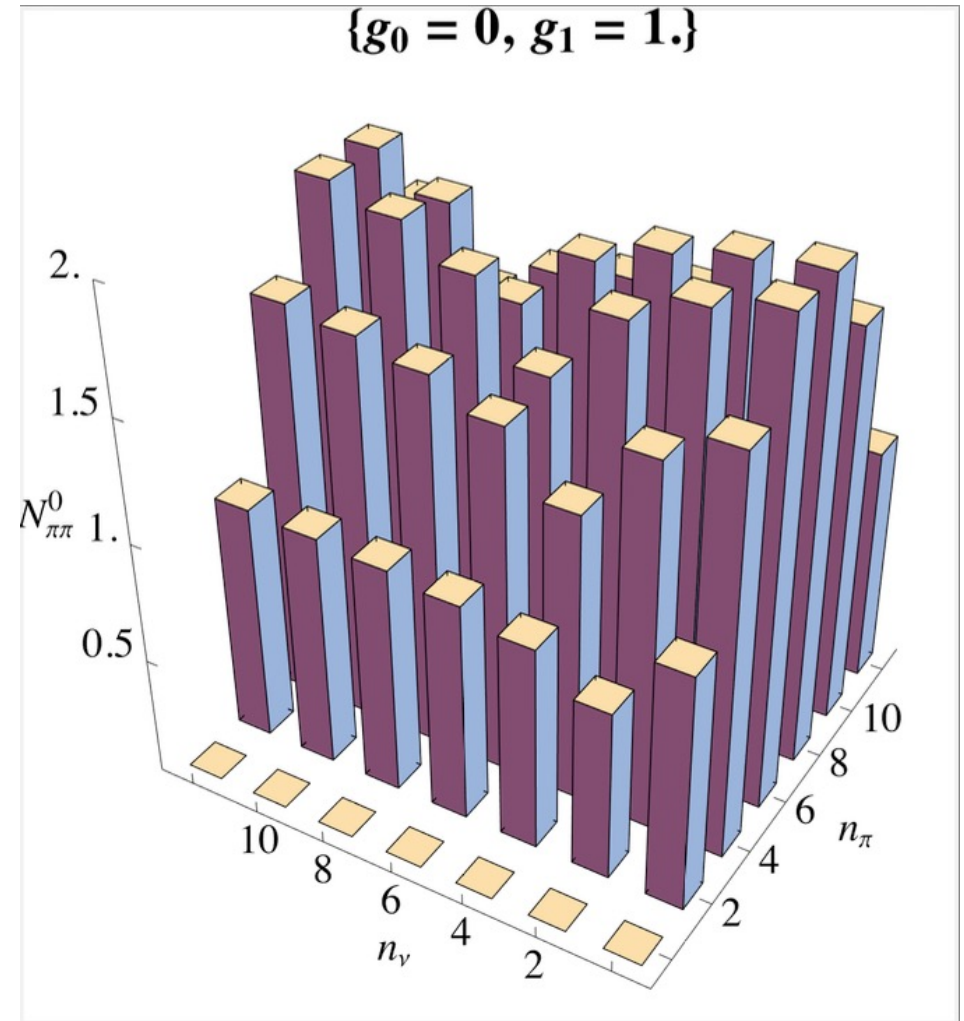
Analytic expression derived from $SO(5)$ symmetry.



Isovector pairing in $j = 11/2$

The number of $\pi\pi$ pairs with $J_2 = 0$ in the ground state of an isovector pairing interaction.

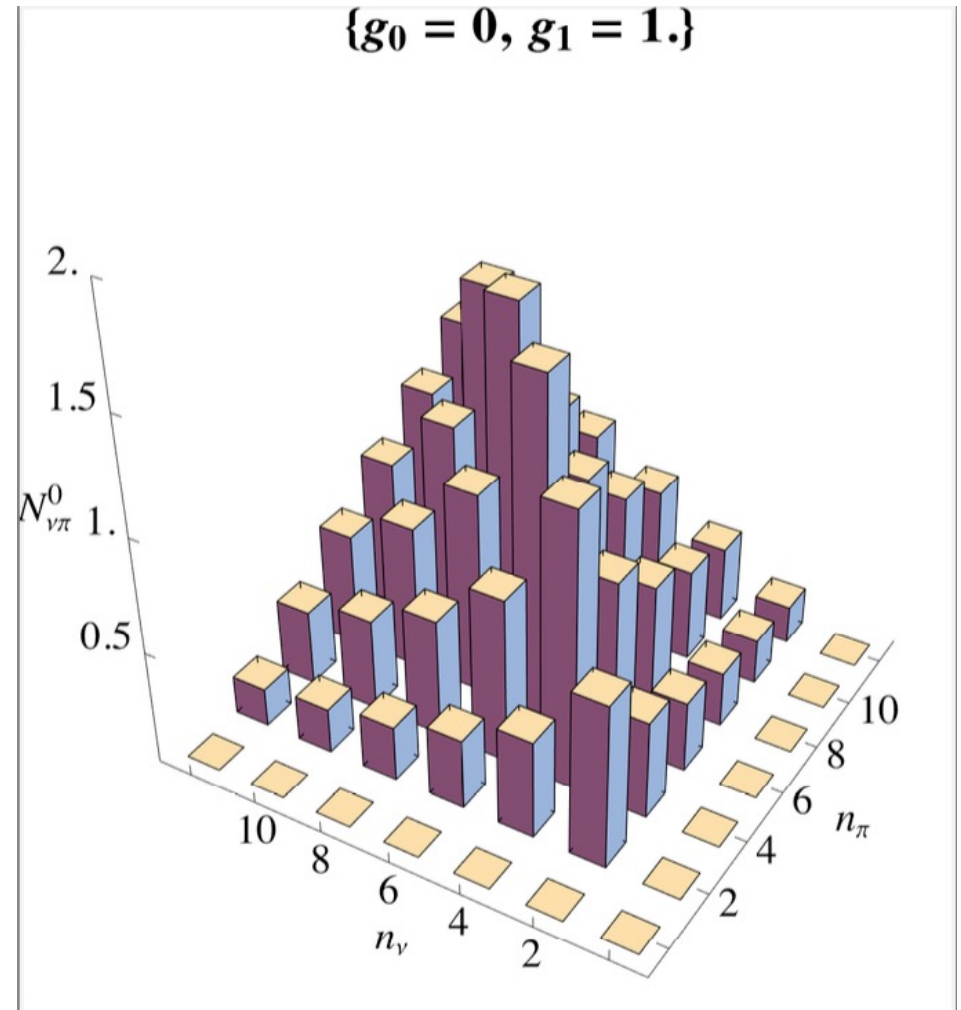
The expectation value of the pair-number operator.



Isovector pairing in $j = 11/2$

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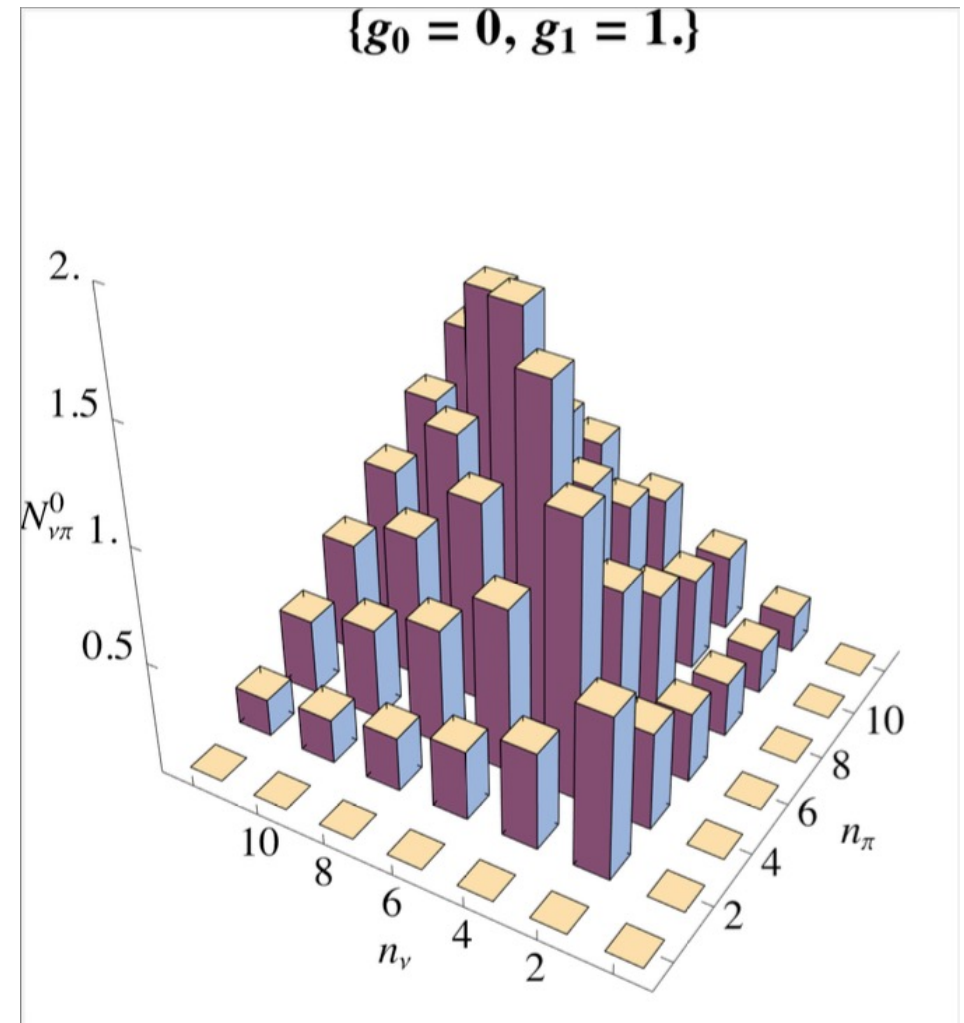
Analytic expression derived from $SO(5)$ symmetry.



Isovector pairing in $j = 11/2$

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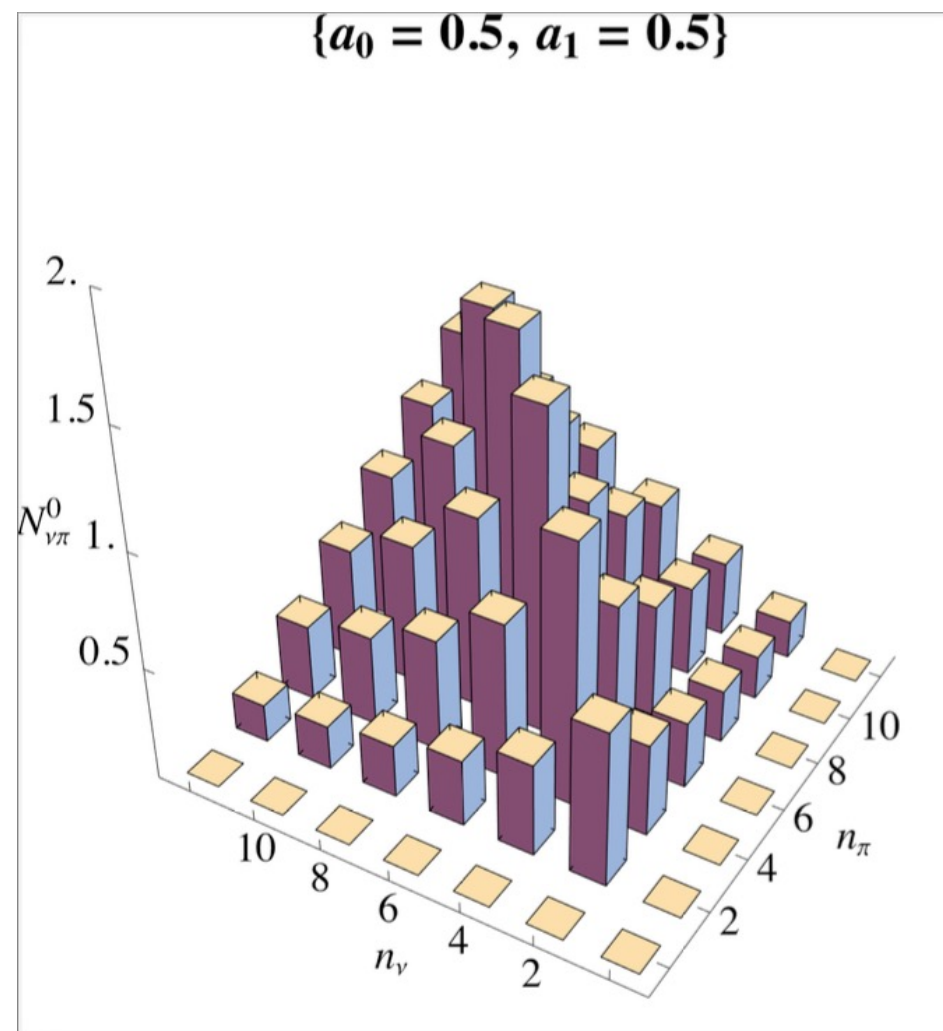
The expectation value of the pair-number operator.



SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
with $a_0 = a_1$.

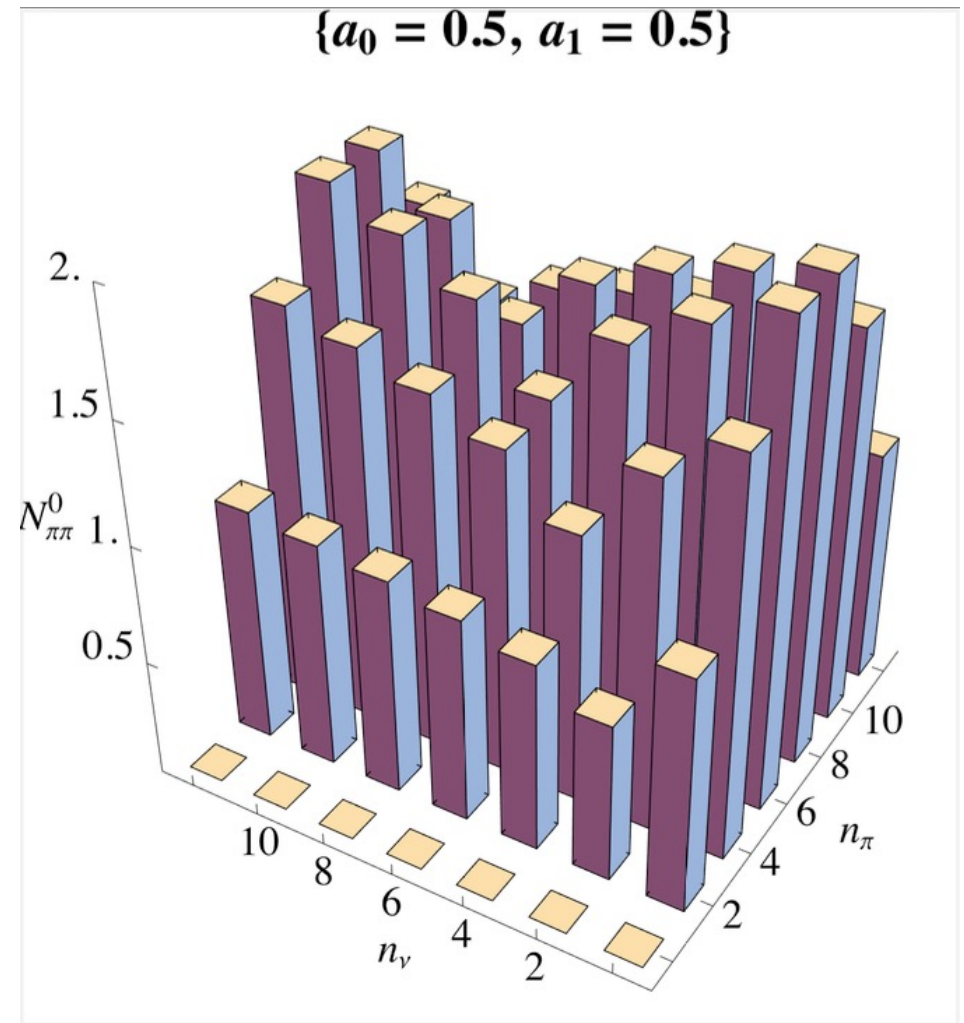
Agrees qualitatively
with isovector pairing.



SDI in a $j = 11/2$ orbital

The number of $\pi\pi$ pairs with $J_2 = 0$ in the ground state of a SDI with $a_0 = a_1$.

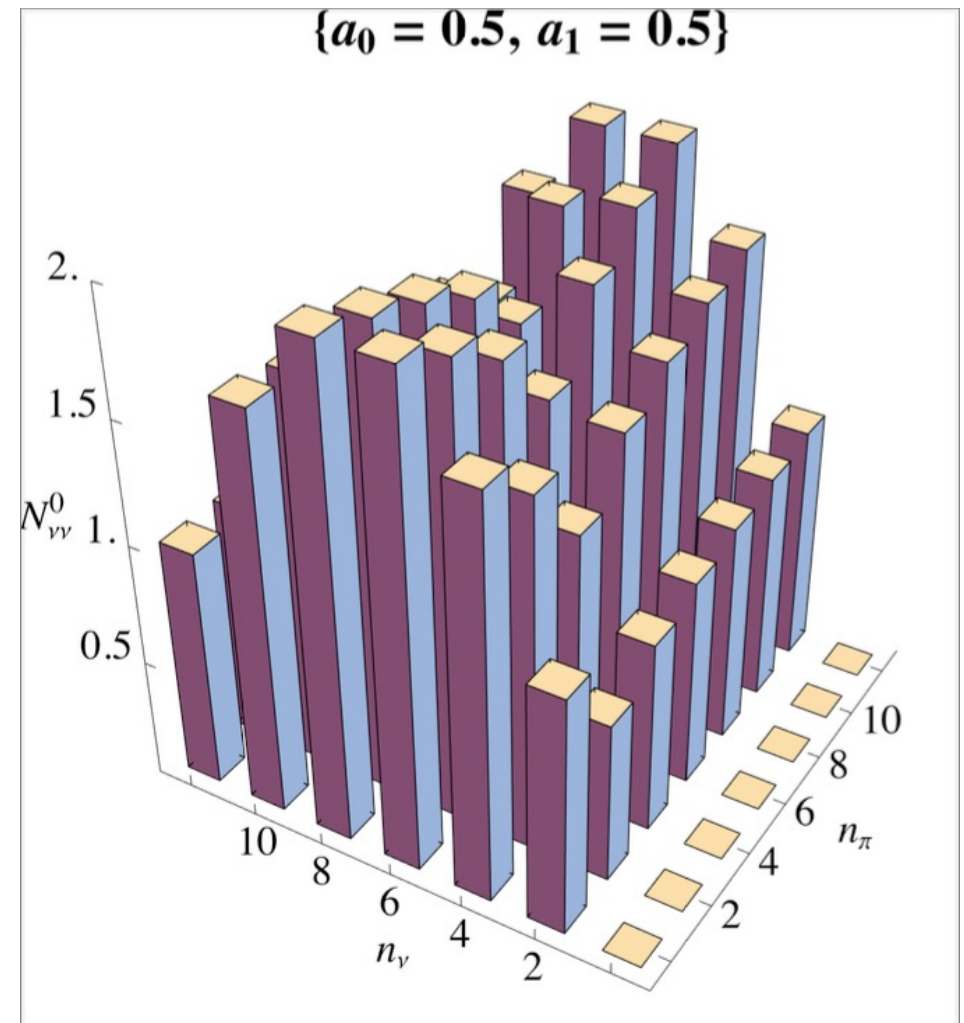
Agrees qualitatively with isovector pairing.



SDI in a $j = 11/2$ orbital

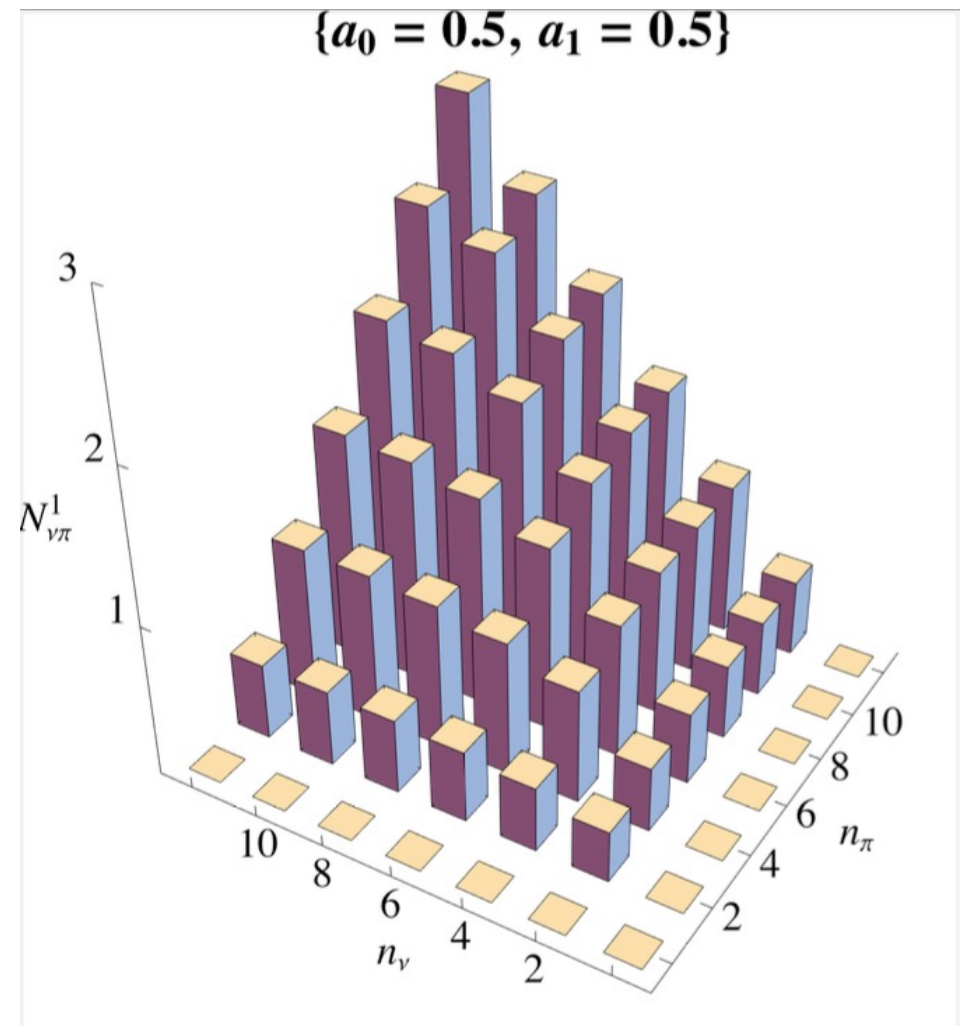
The number of $\nu\nu$ pairs with $J_2 = 0$ in the ground state of a SDI with $a_0 = a_1$.

Agrees qualitatively with isovector pairing.



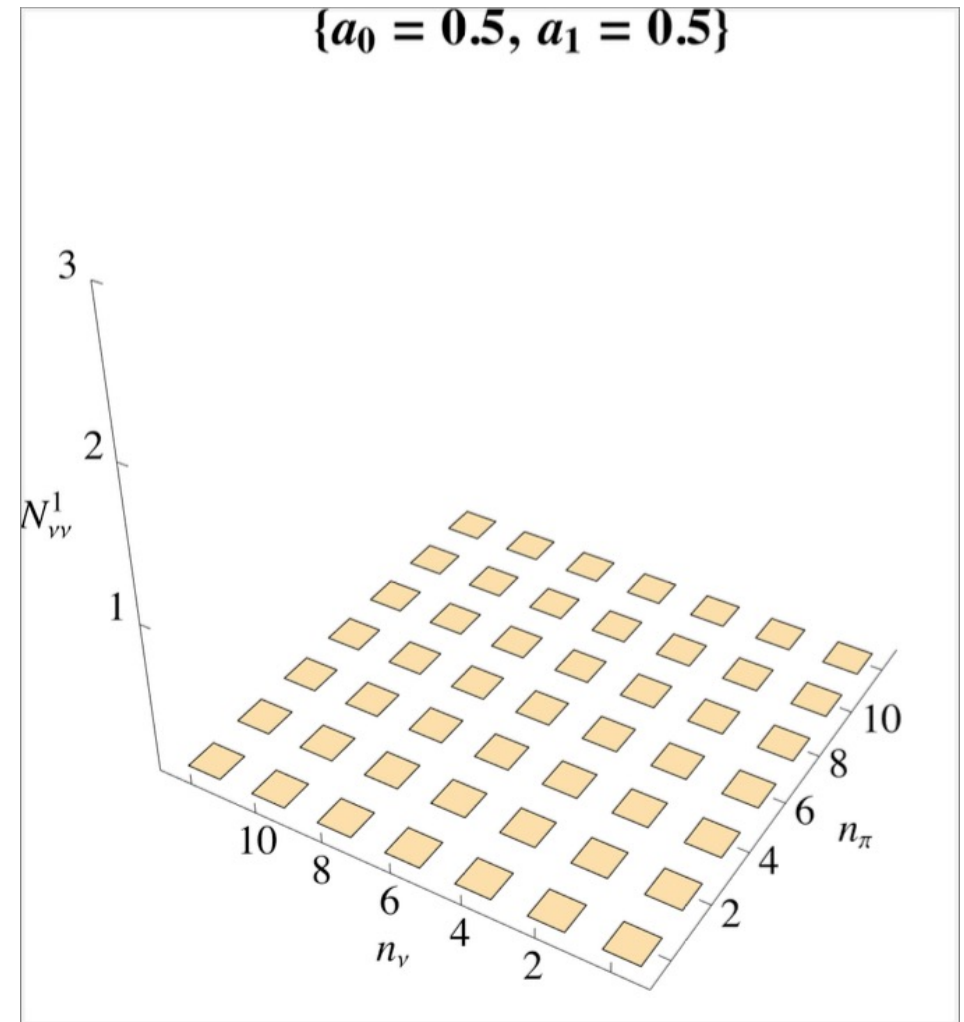
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = a_1$.



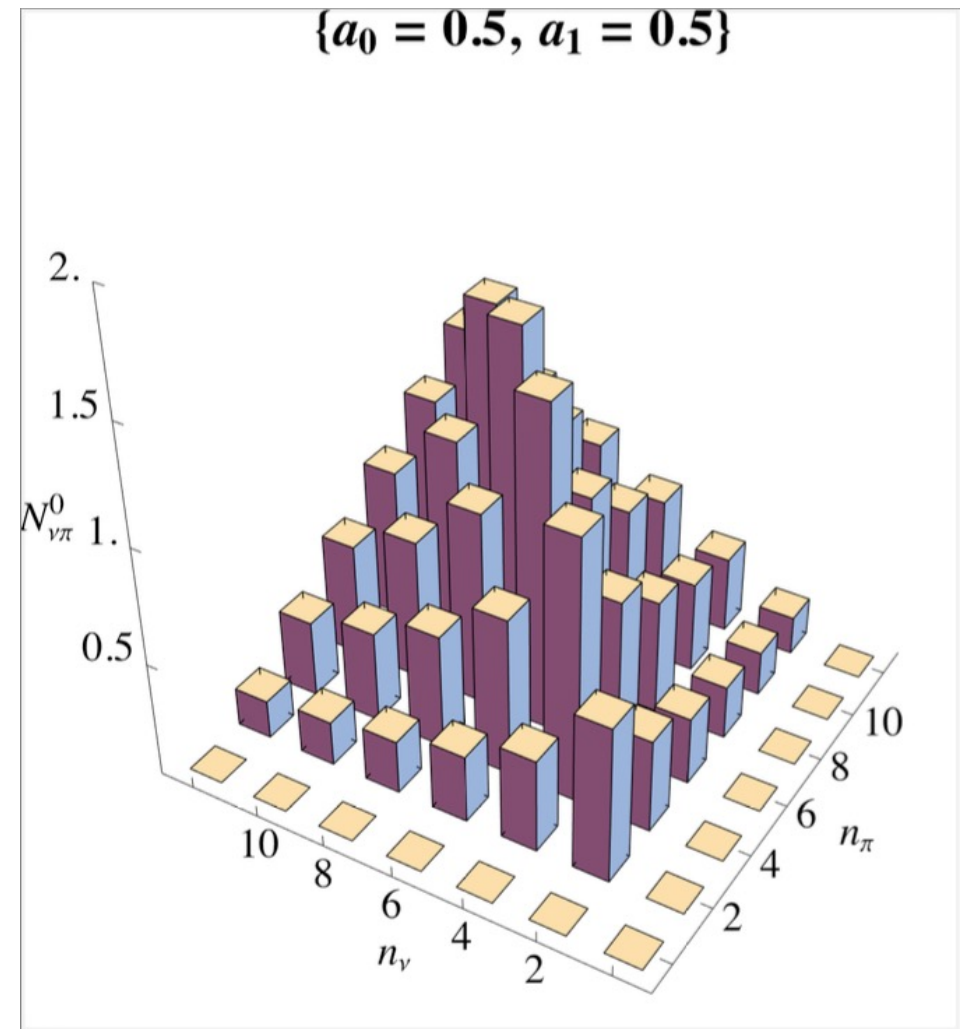
SDI in a $j = 11/2$ orbital

The number of $\nu\nu$ or
 $\pi\pi$ pairs with $J_2 = 1$.
 $J_2 = 1$ pairs have $T_2 = 0$.



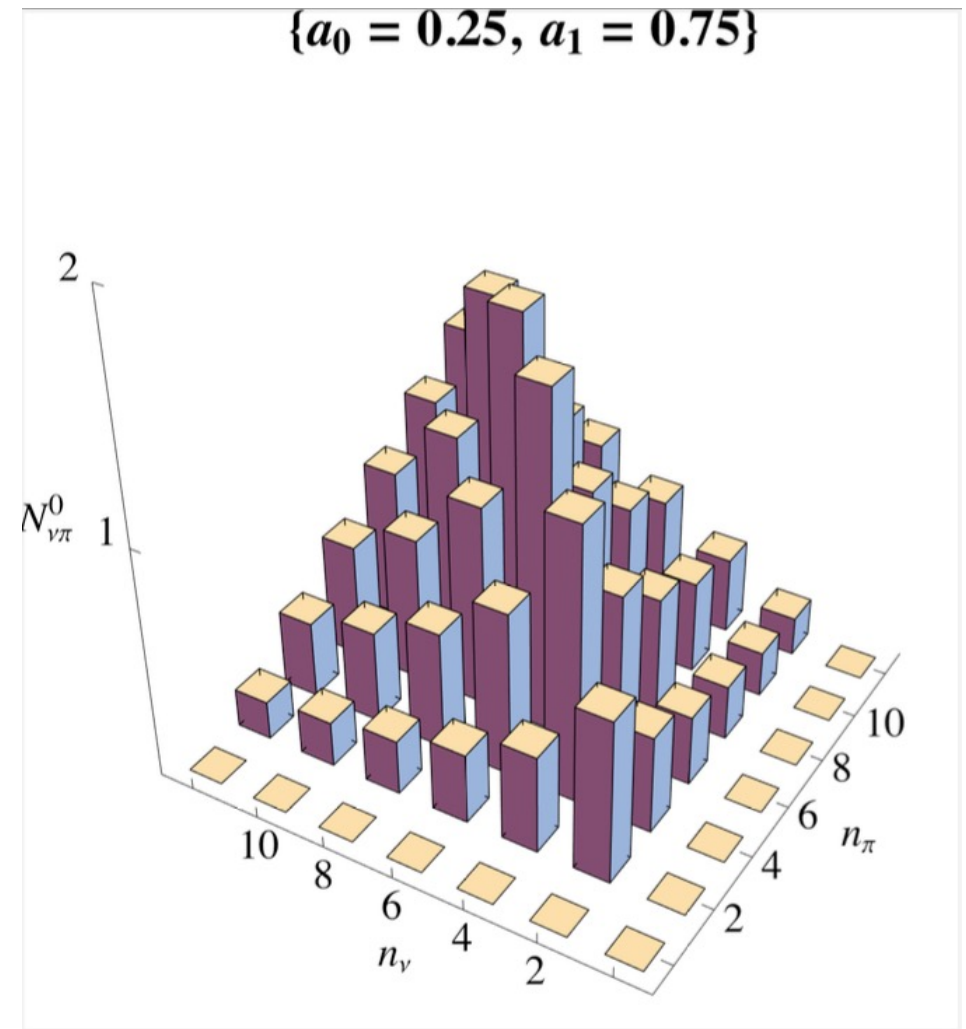
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
with $a_0 = a_1$.



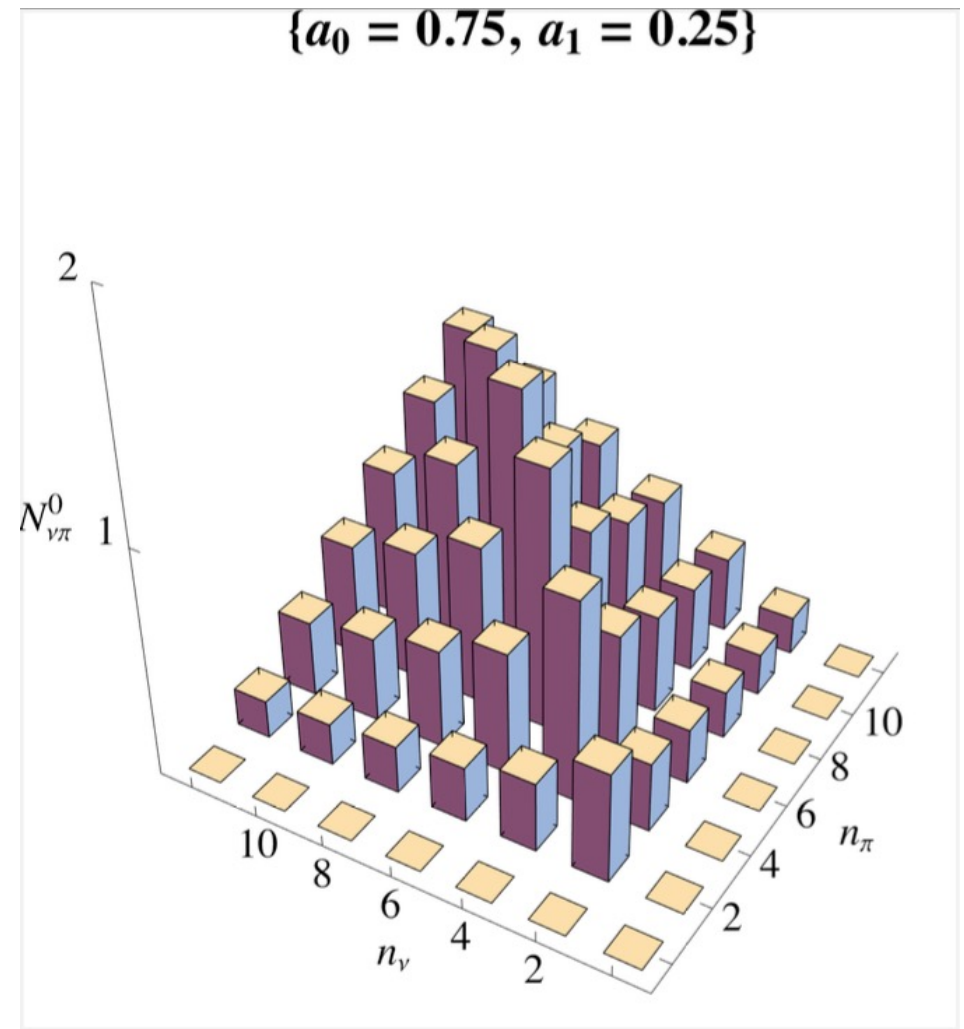
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
with $3a_0 = a_1$.



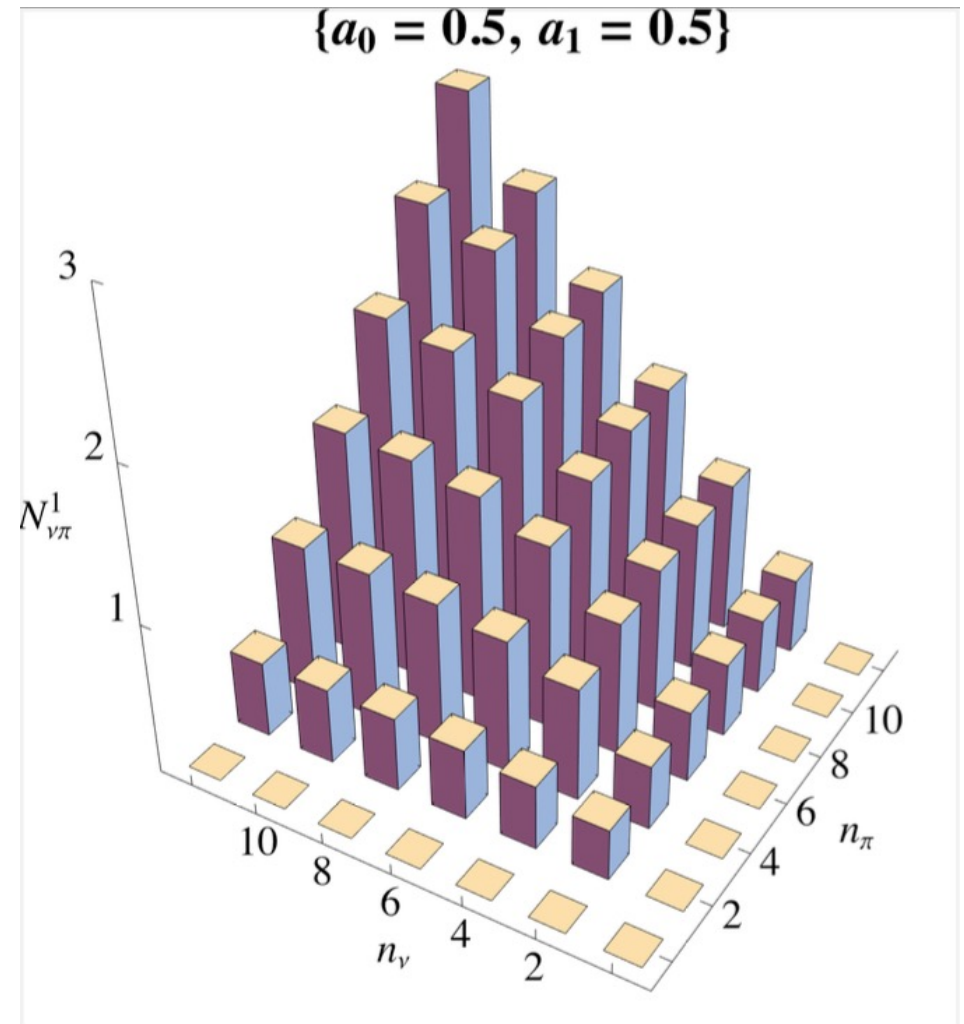
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
with $a_0 = 3a_1$.



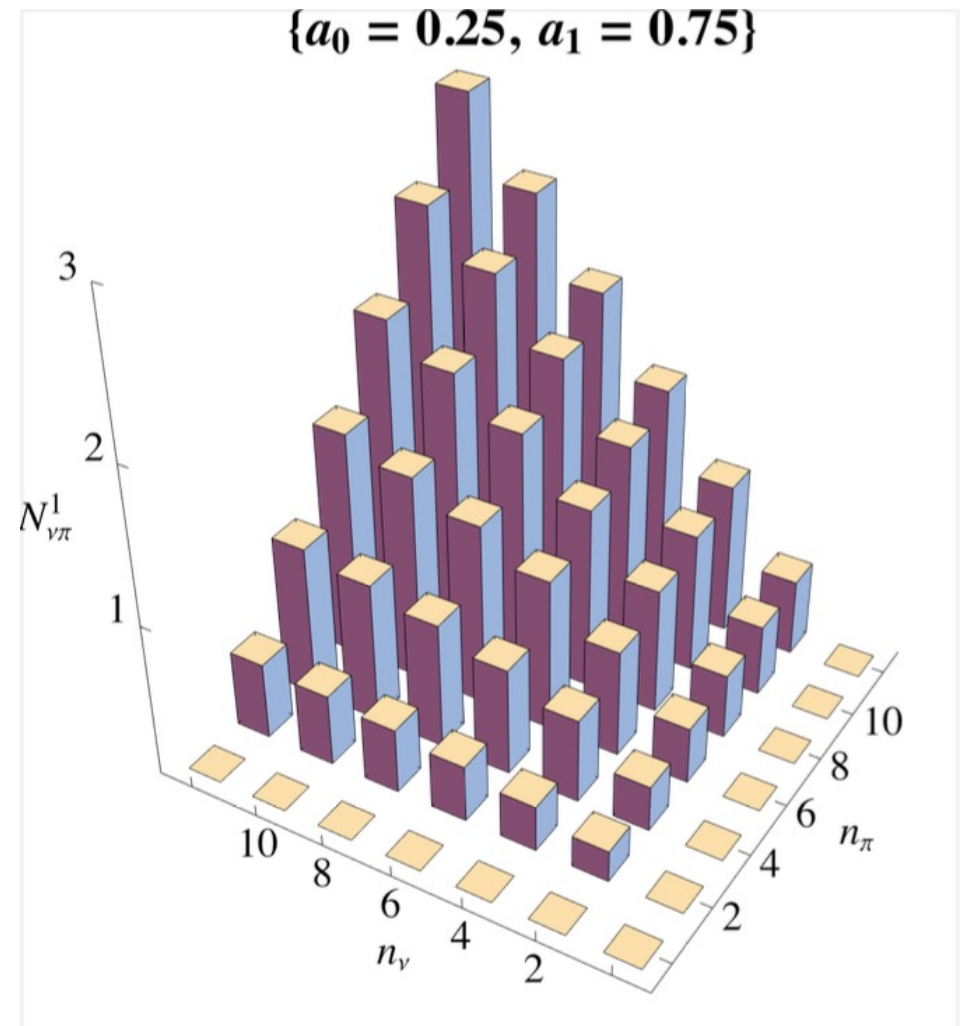
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = a_1$.



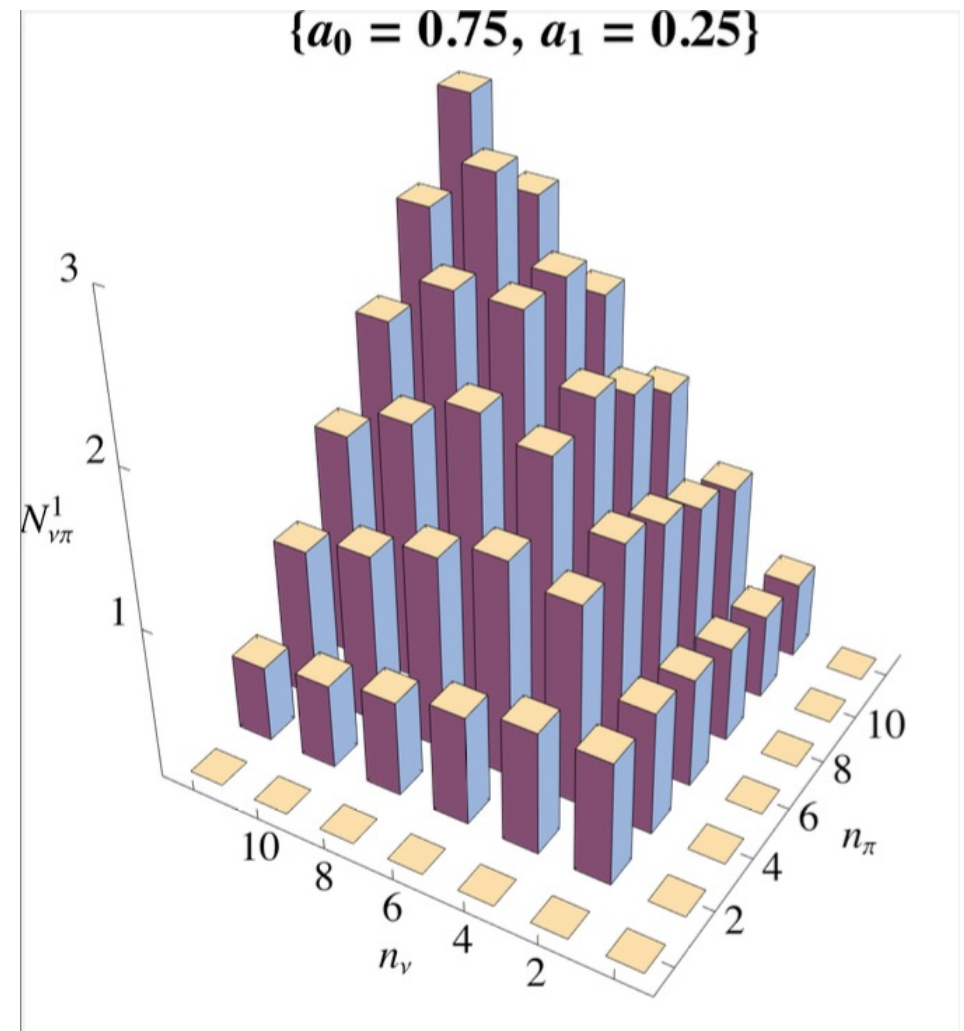
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
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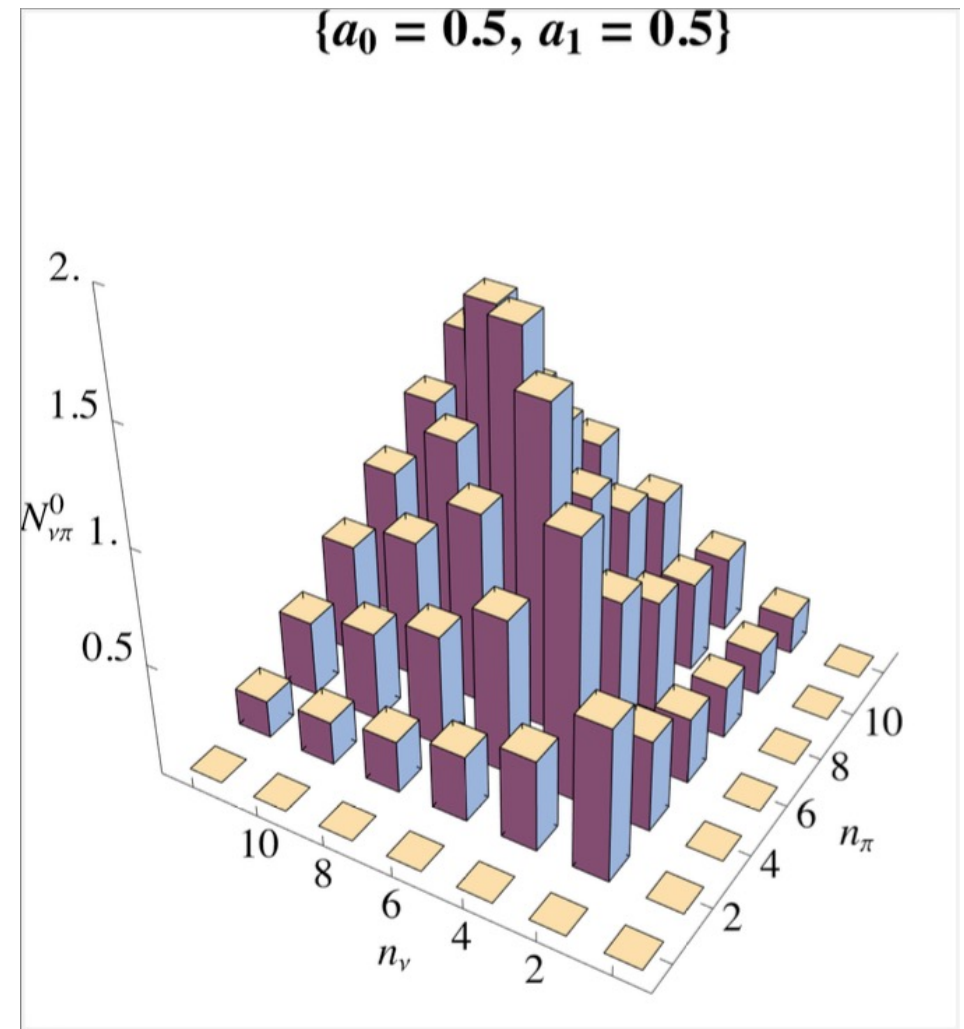
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = 3a_1$.



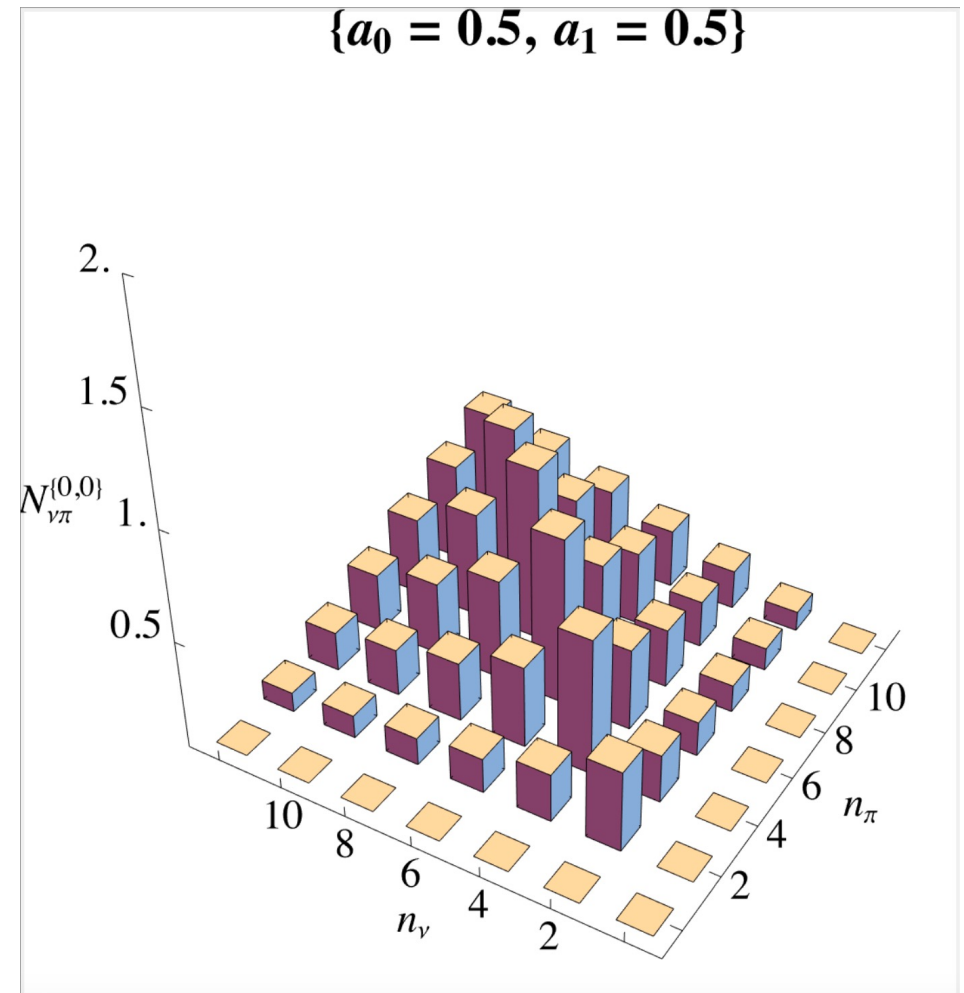
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
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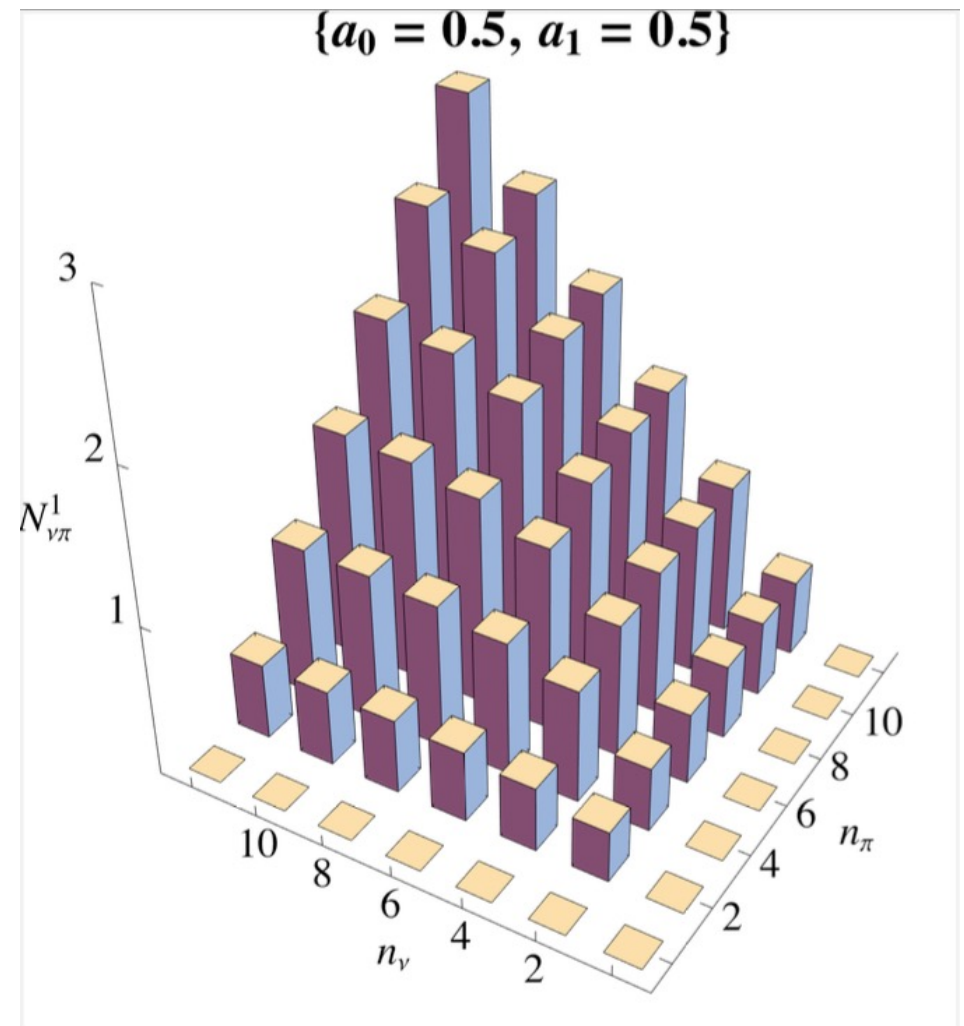
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $L_2 = 0$ and $S_2 = 0$
in the ground state of
a SDI with $a_0 = a_1$.



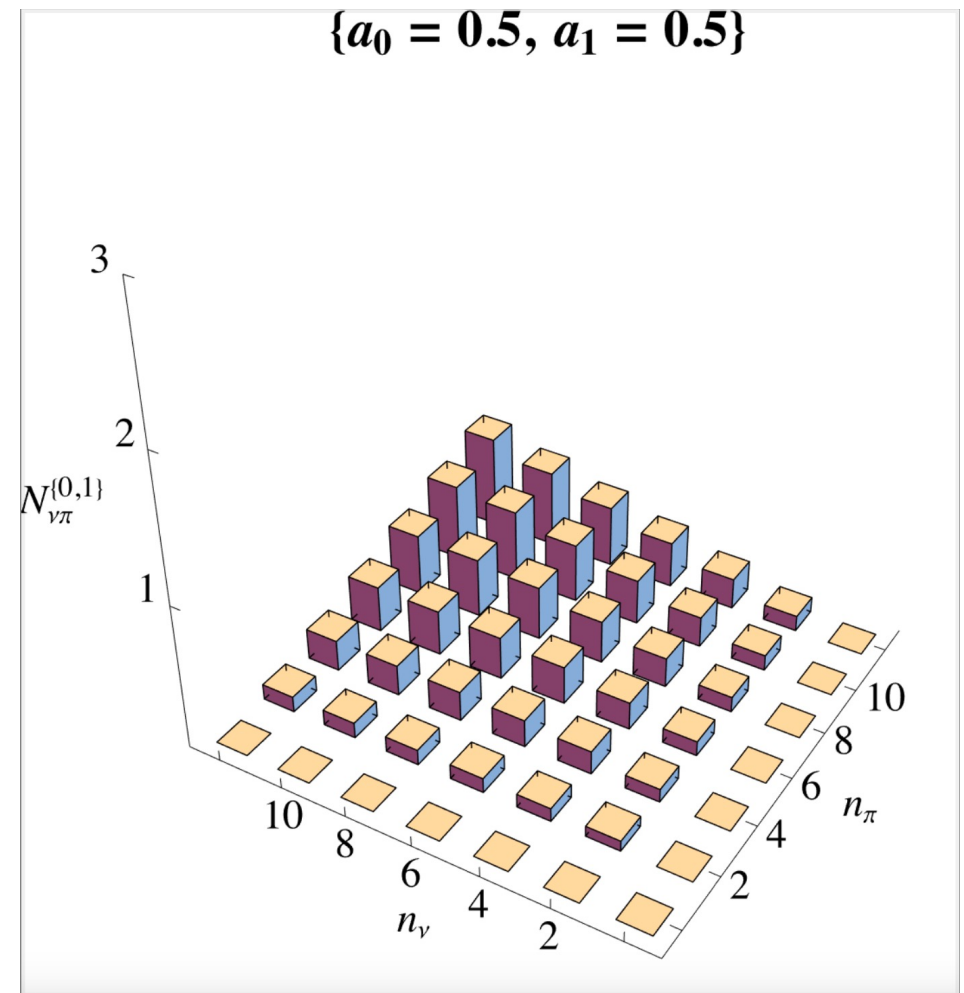
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 1$ in the
ground state of a SDI
with $a_0 = a_1$.



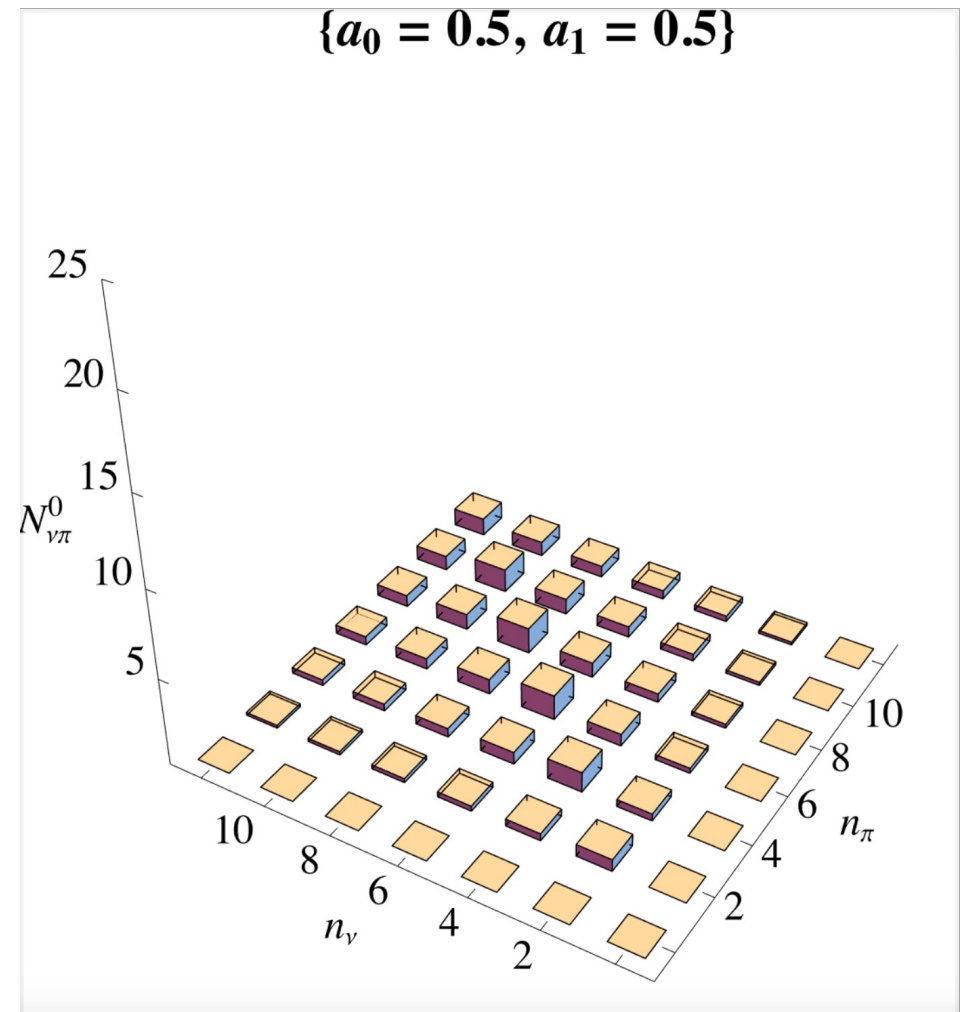
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $L_2 = 0$ and $S_2 = 1$
in the ground state of
a SDI with $a_0 = a_1$.



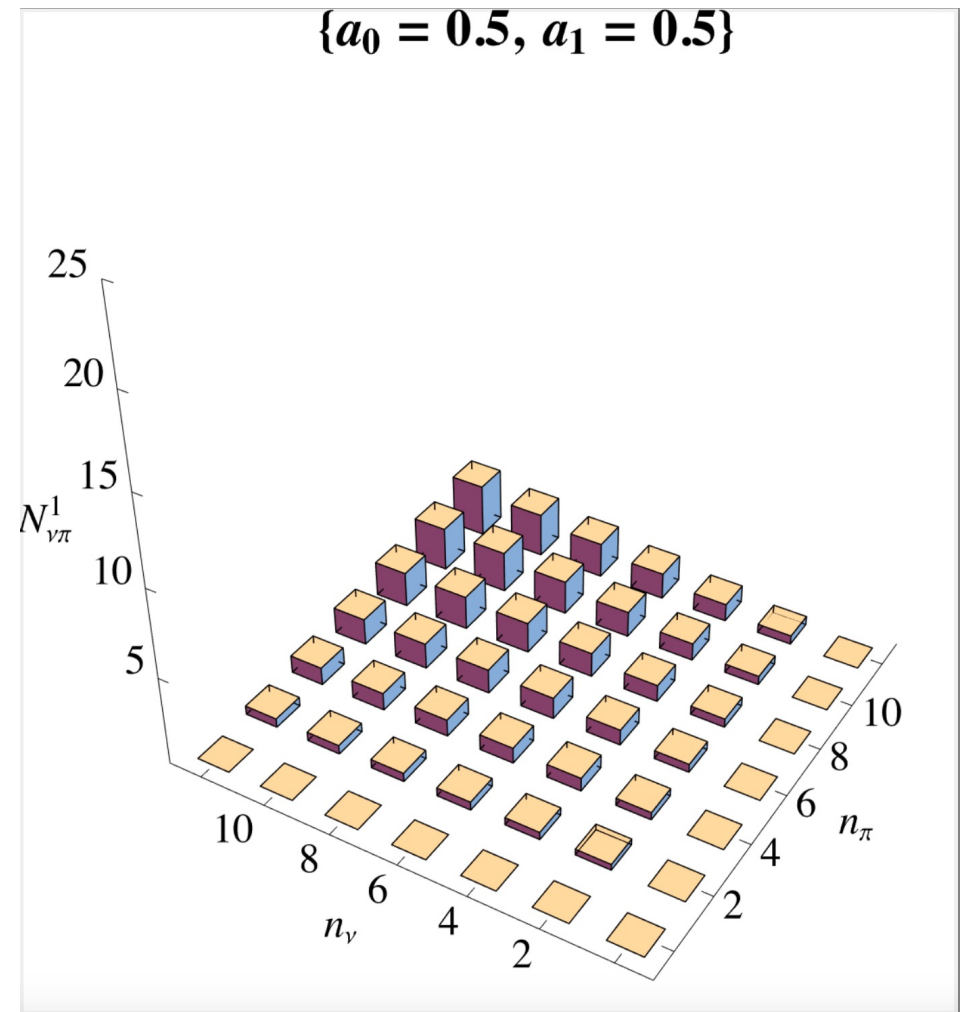
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
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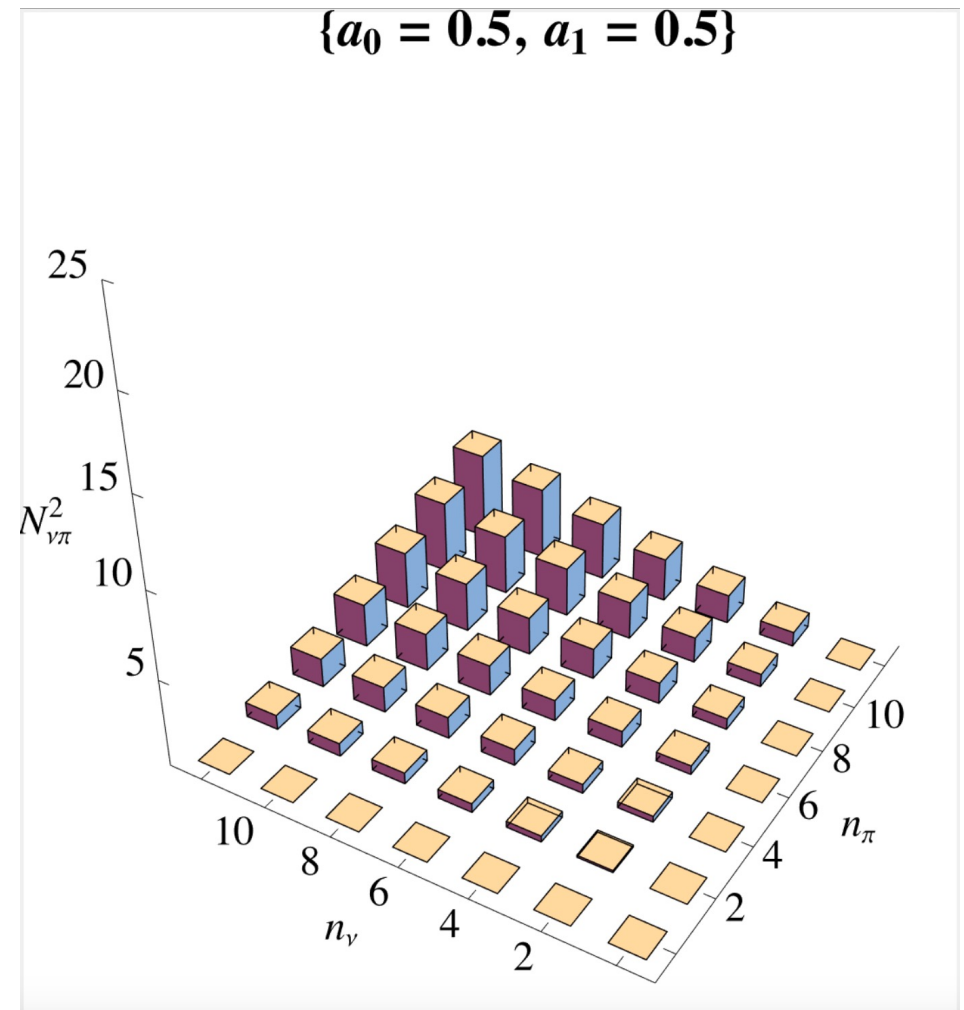
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 1$ in the
ground state of a SDI
with $a_0 = a_1$.



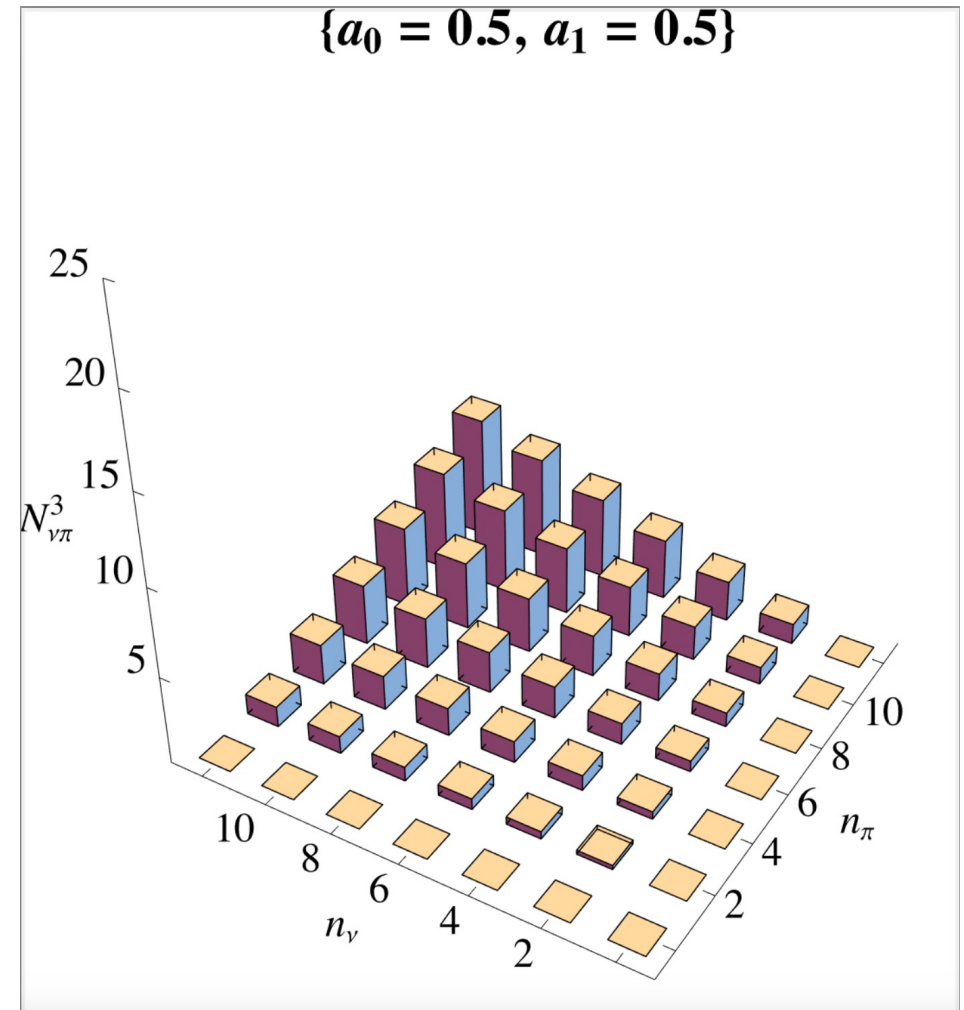
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 2$ in the
ground state of a SDI
with $a_0 = a_1$.



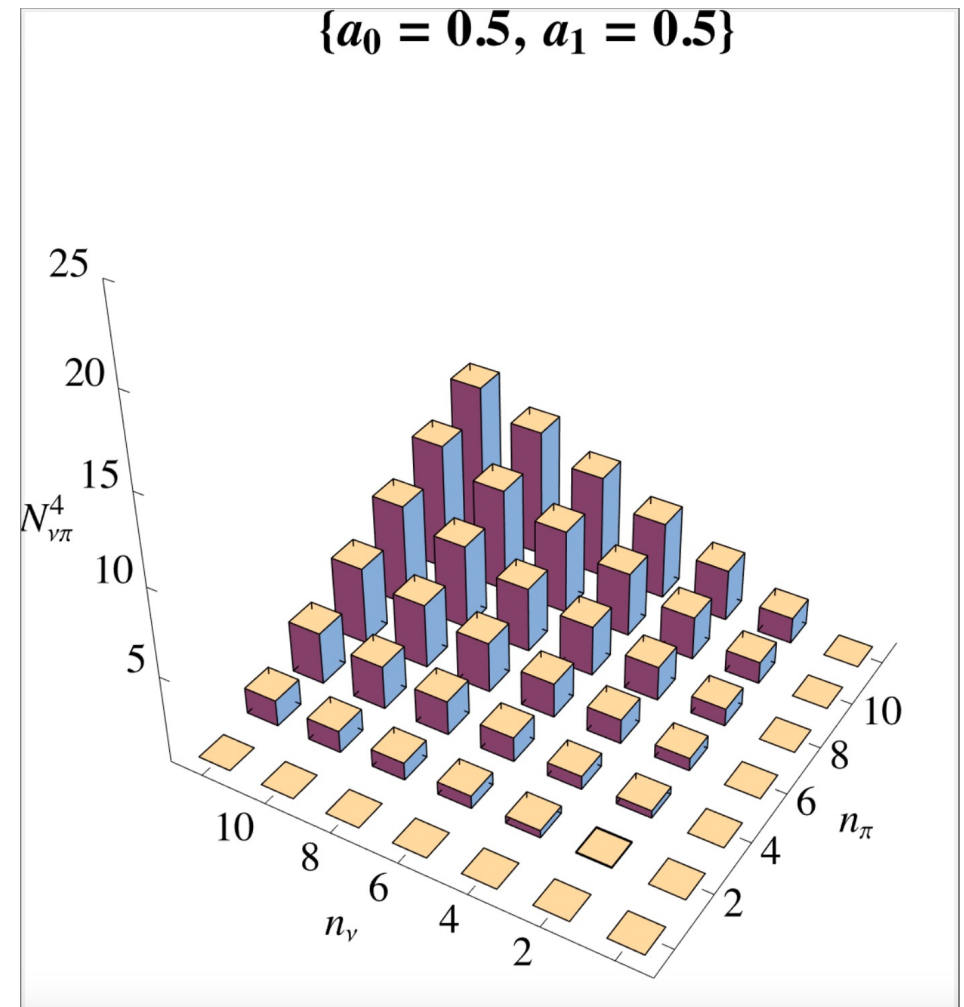
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 3$ in the
ground state of a SDI
with $a_0 = a_1$.



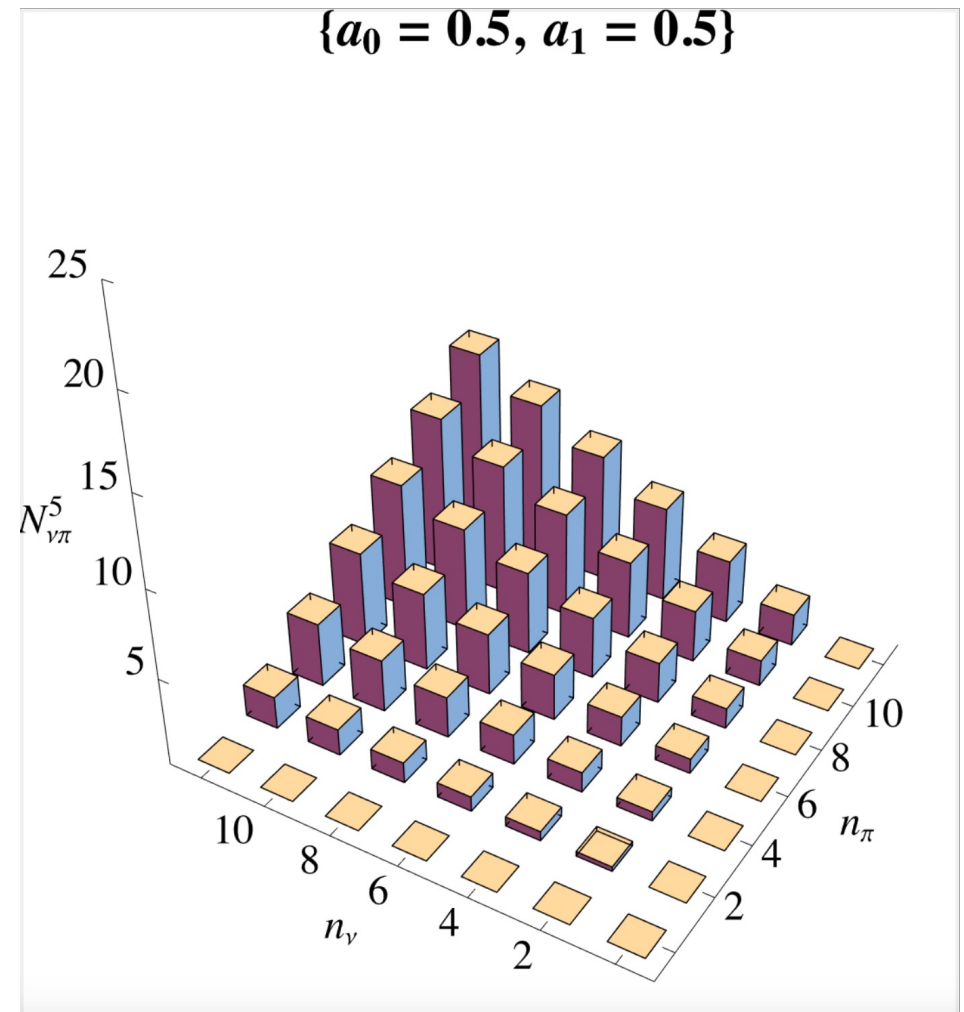
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 4$ in the
ground state of a SDI
with $a_0 = a_1$.



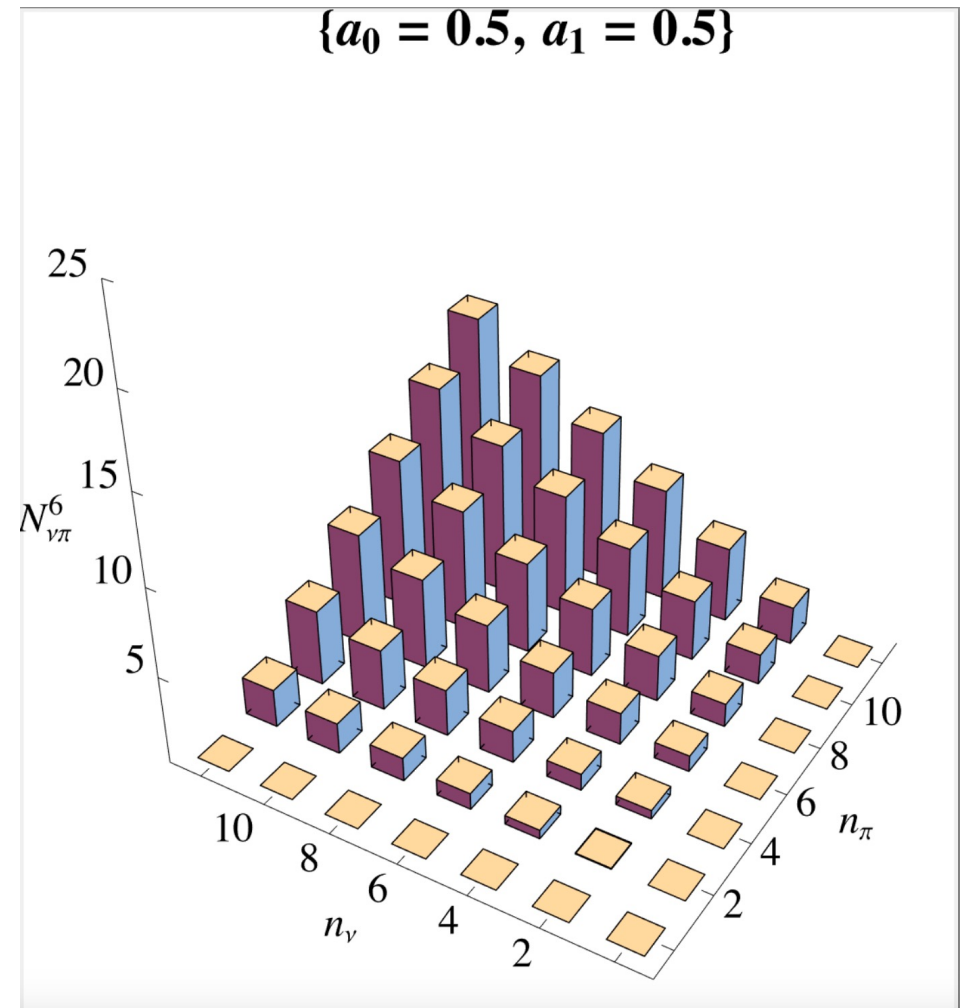
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 5$ in the
ground state of a SDI
with $a_0 = a_1$.



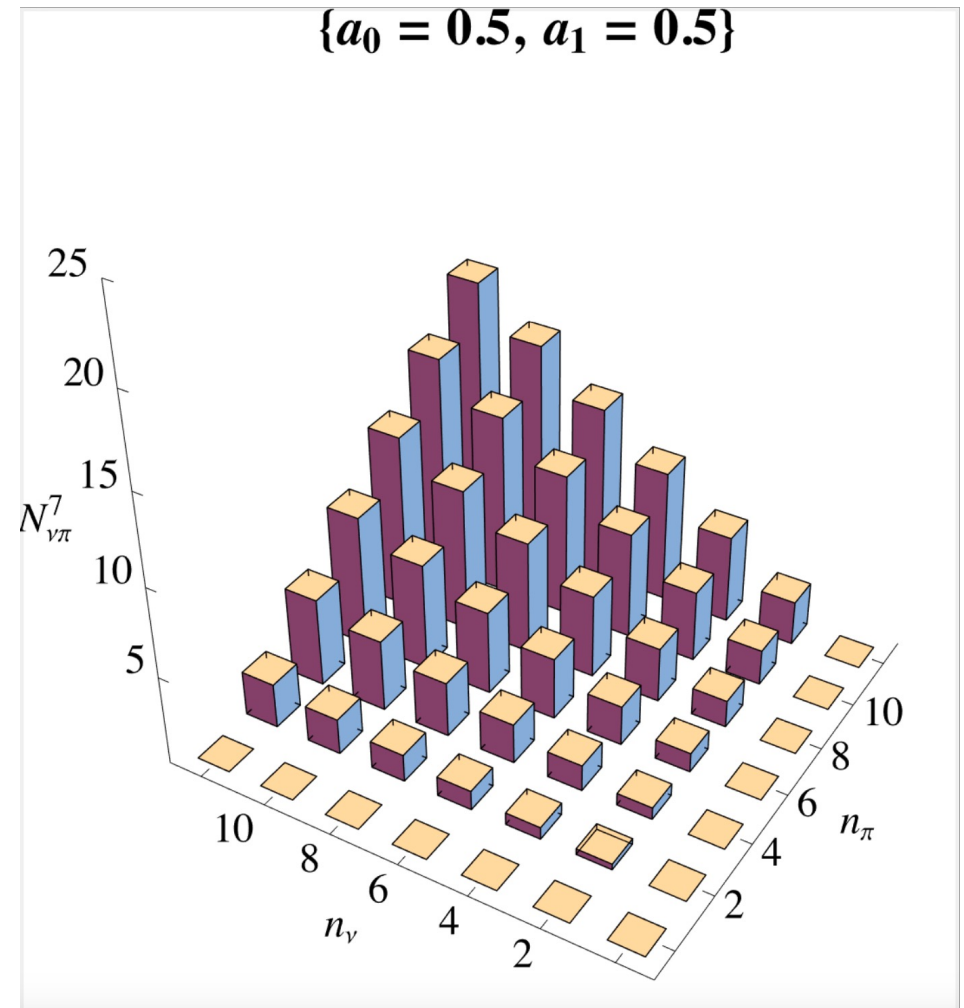
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 6$ in the
ground state of a SDI
with $a_0 = a_1$.



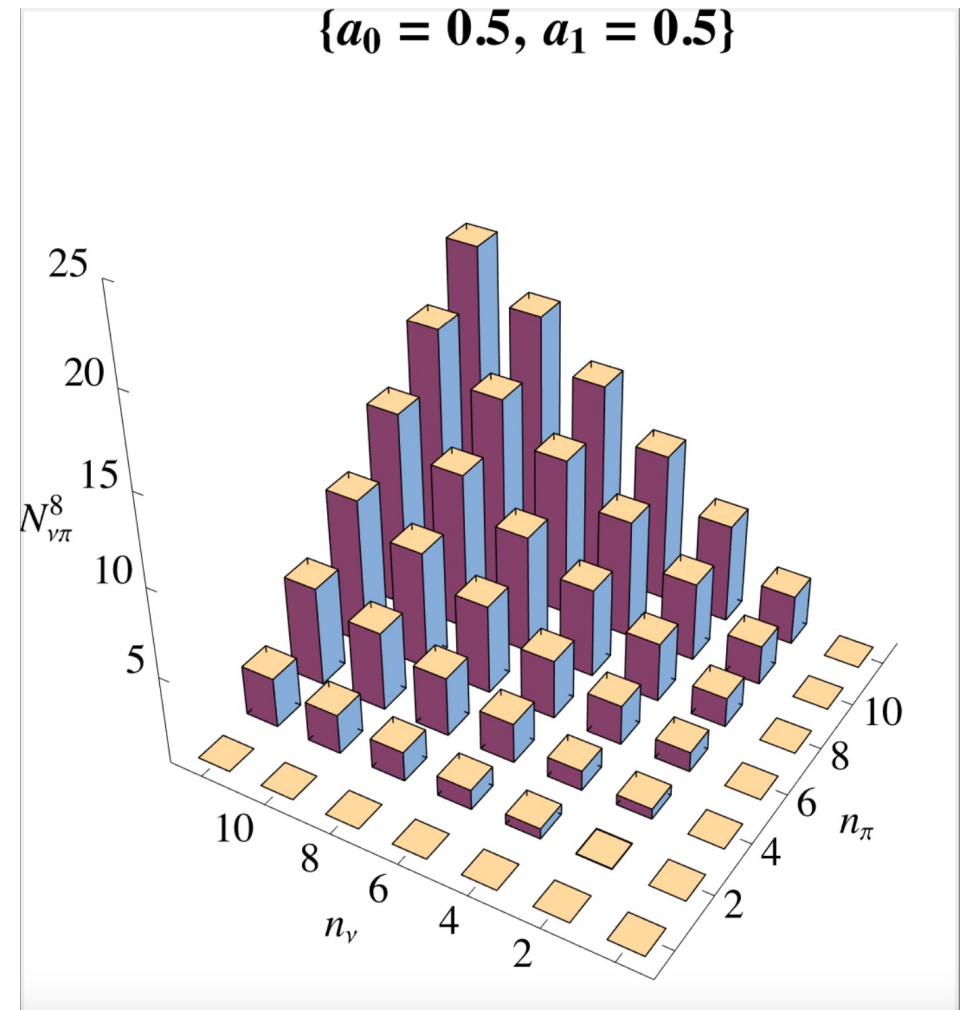
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 7$ in the
ground state of a SDI
with $a_0 = a_1$.



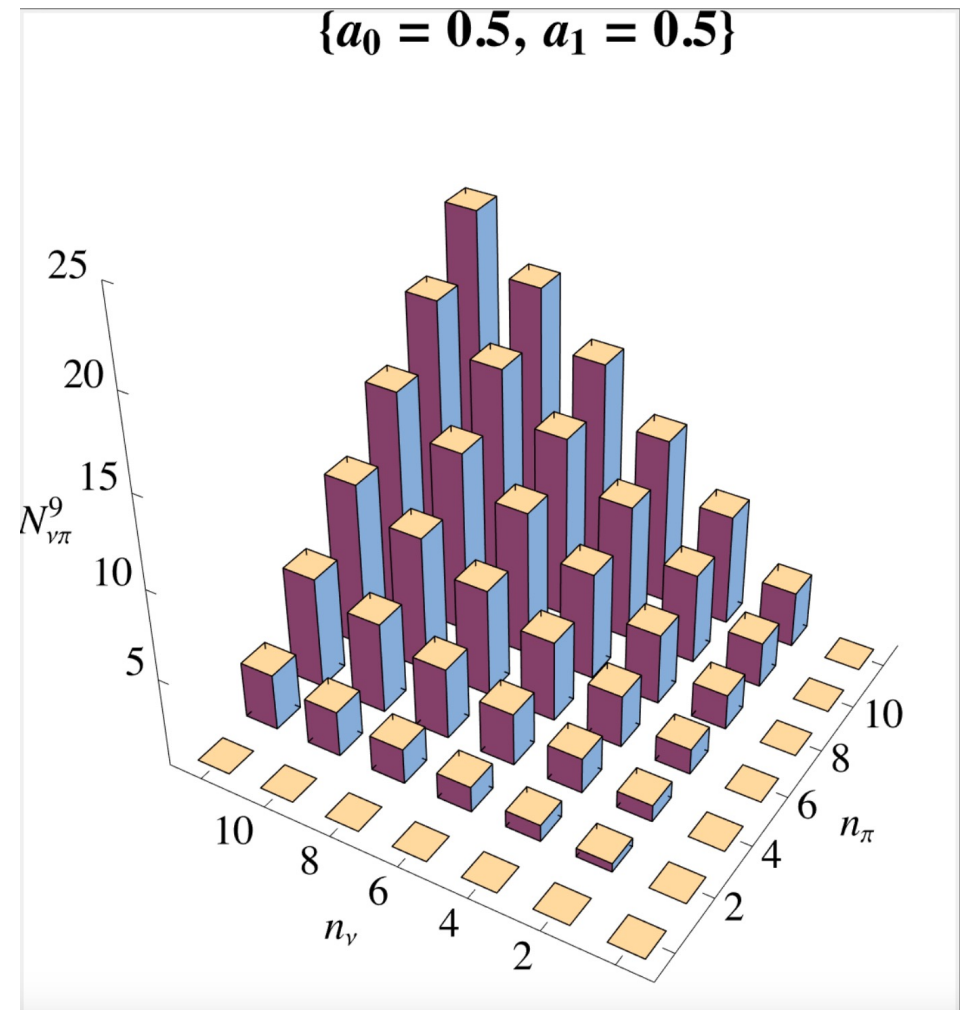
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 8$ in the
ground state of a SDI
with $a_0 = a_1$.



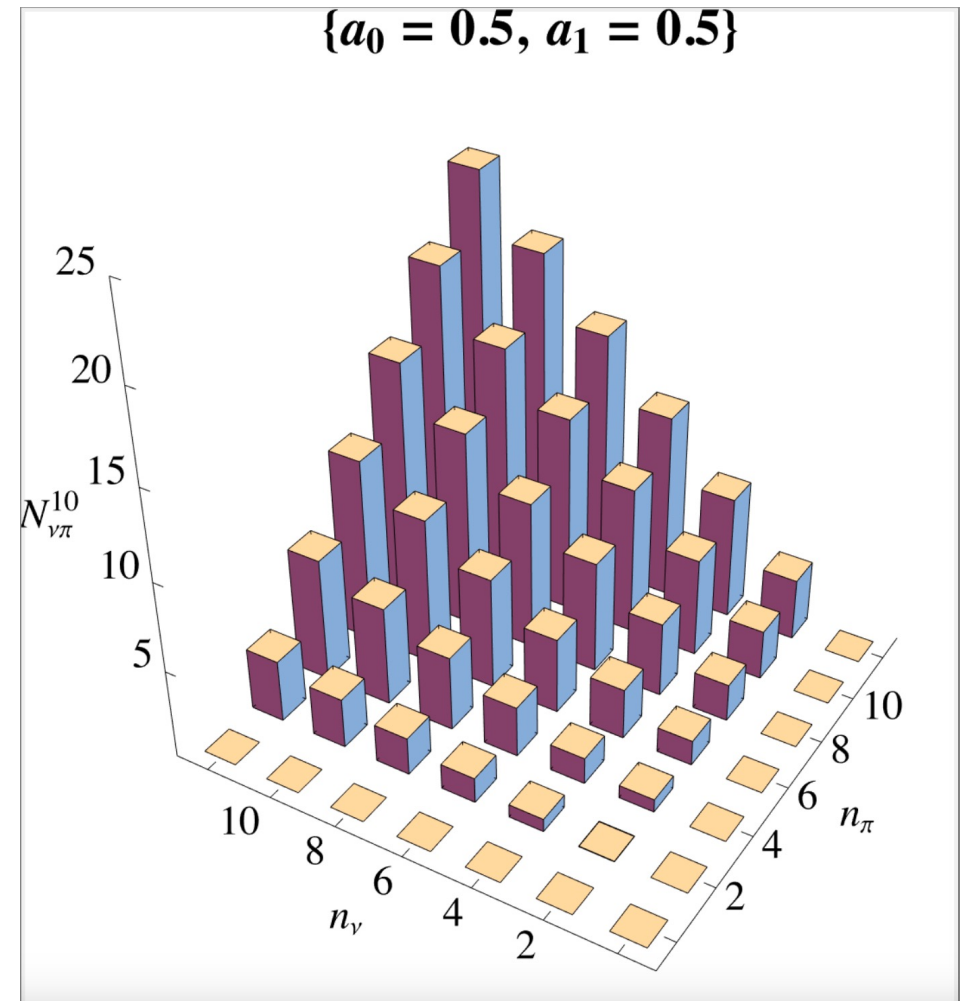
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 9$ in the
ground state of a SDI
with $a_0 = a_1$.



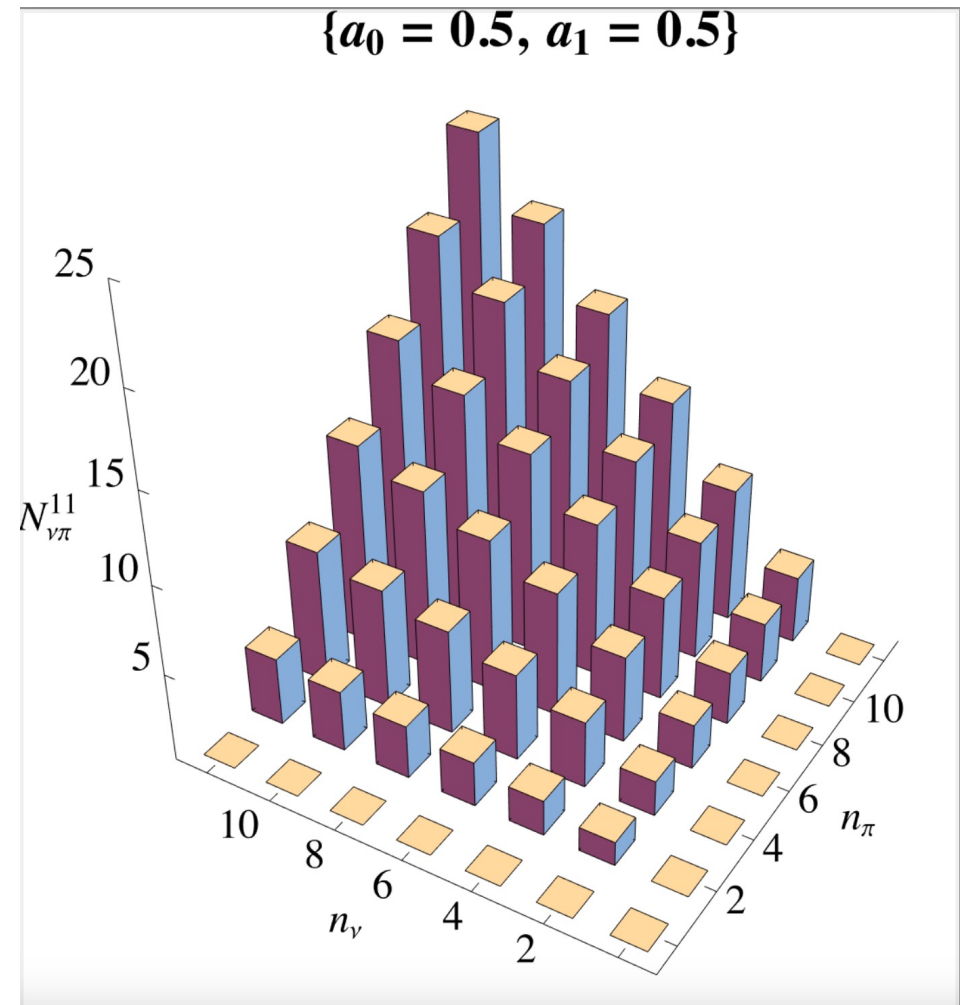
SDI in a $j = 11/2$ orbital

The number of $\nu\pi$ pairs
with $J_2 = 10$ in the
ground state of a SDI
with $a_0 = a_1$.

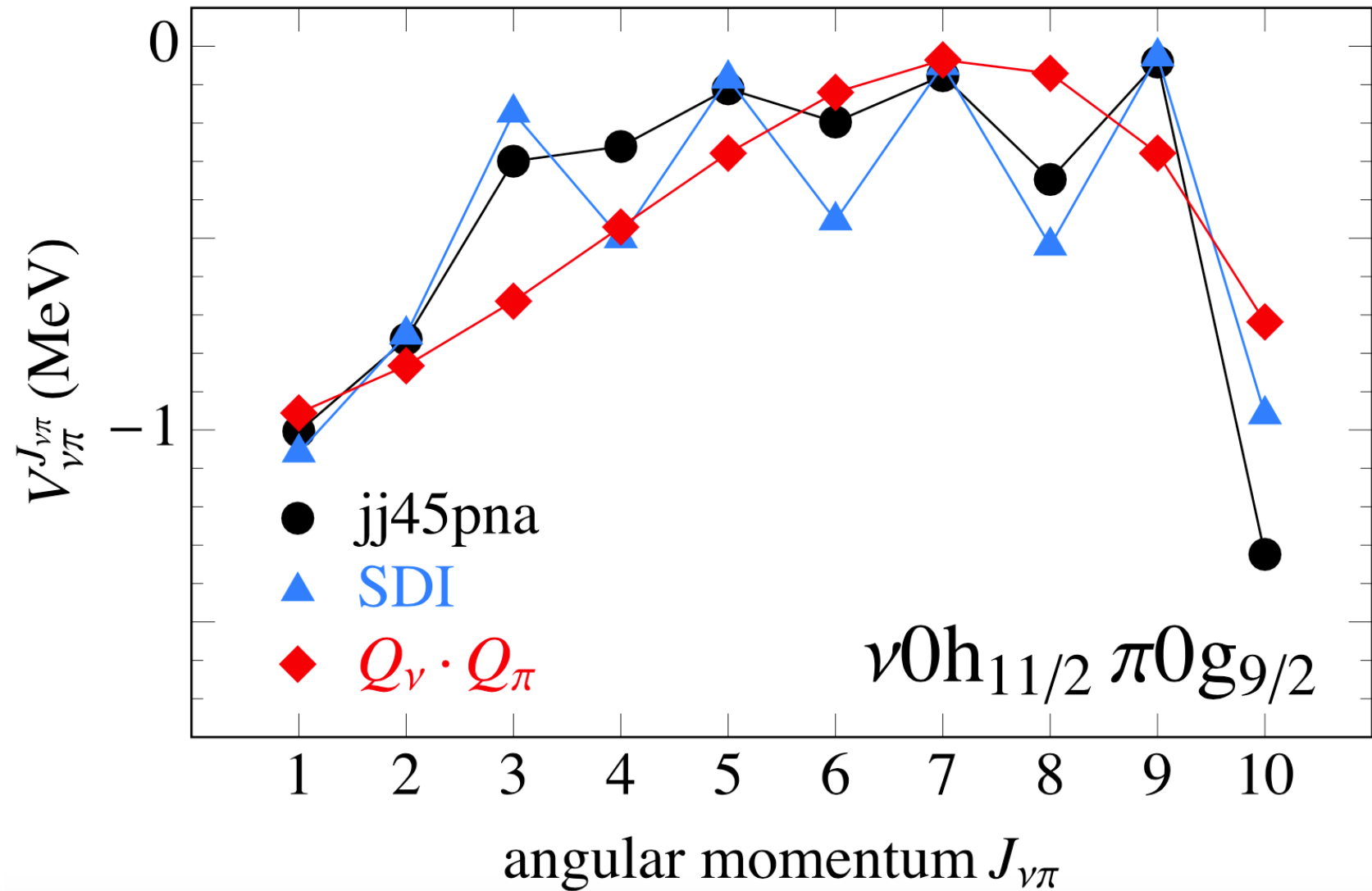


SDI in a $j = 11/2$ orbital

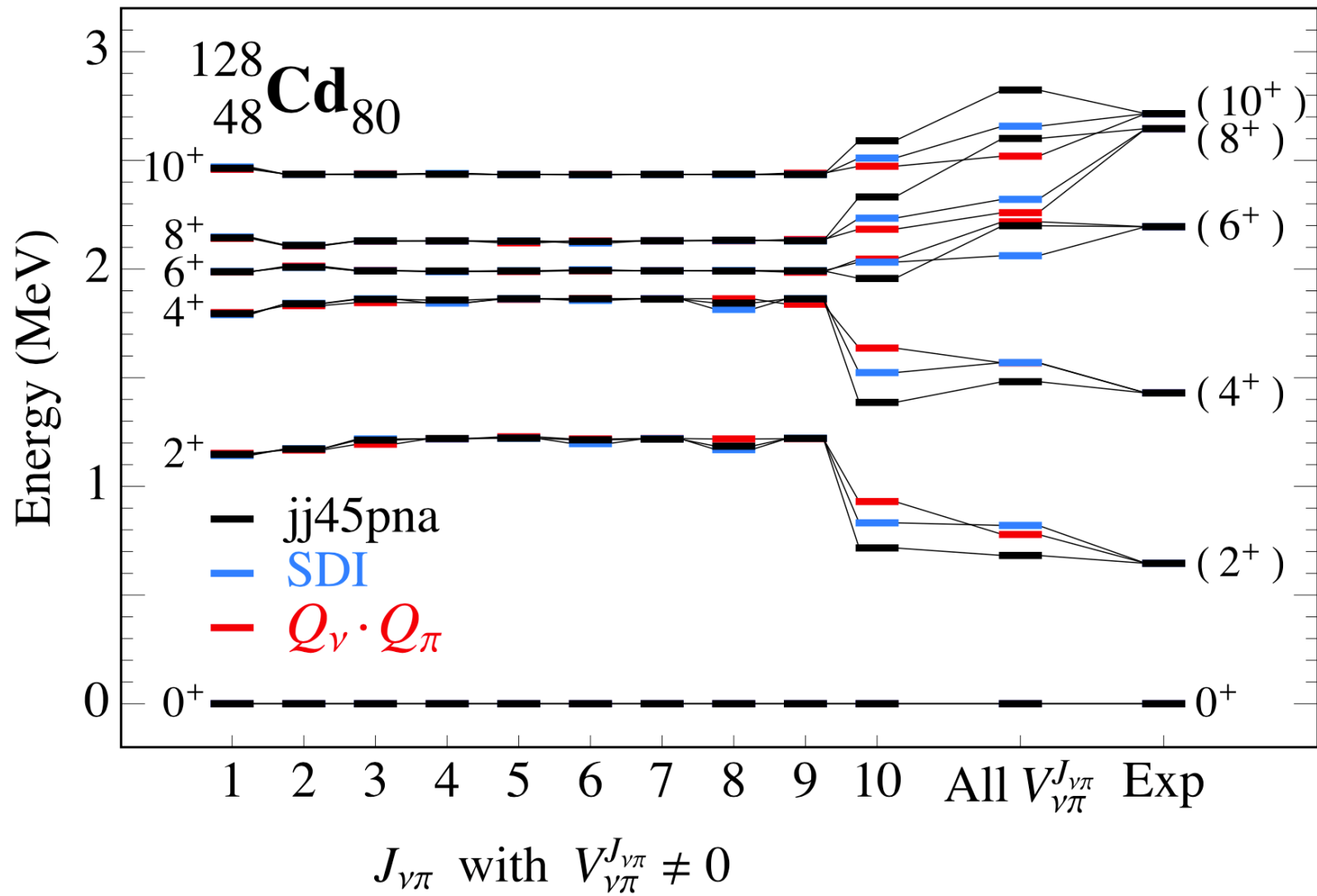
The number of $\nu\pi$ pairs
with $J_2 = 11$ in the
ground state of a SDI
with $a_0 = a_1$.



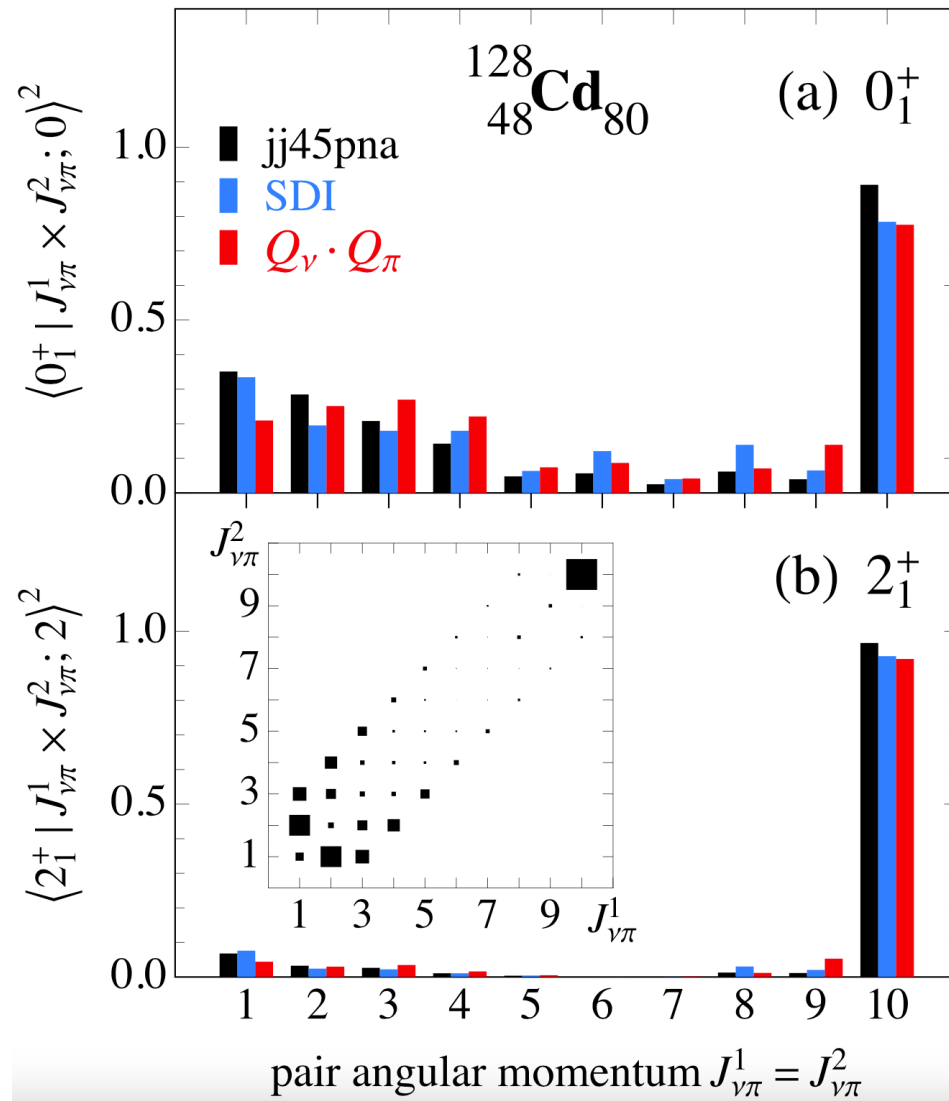
Interaction between high- j orbitals



Spectrum of ^{128}Cd

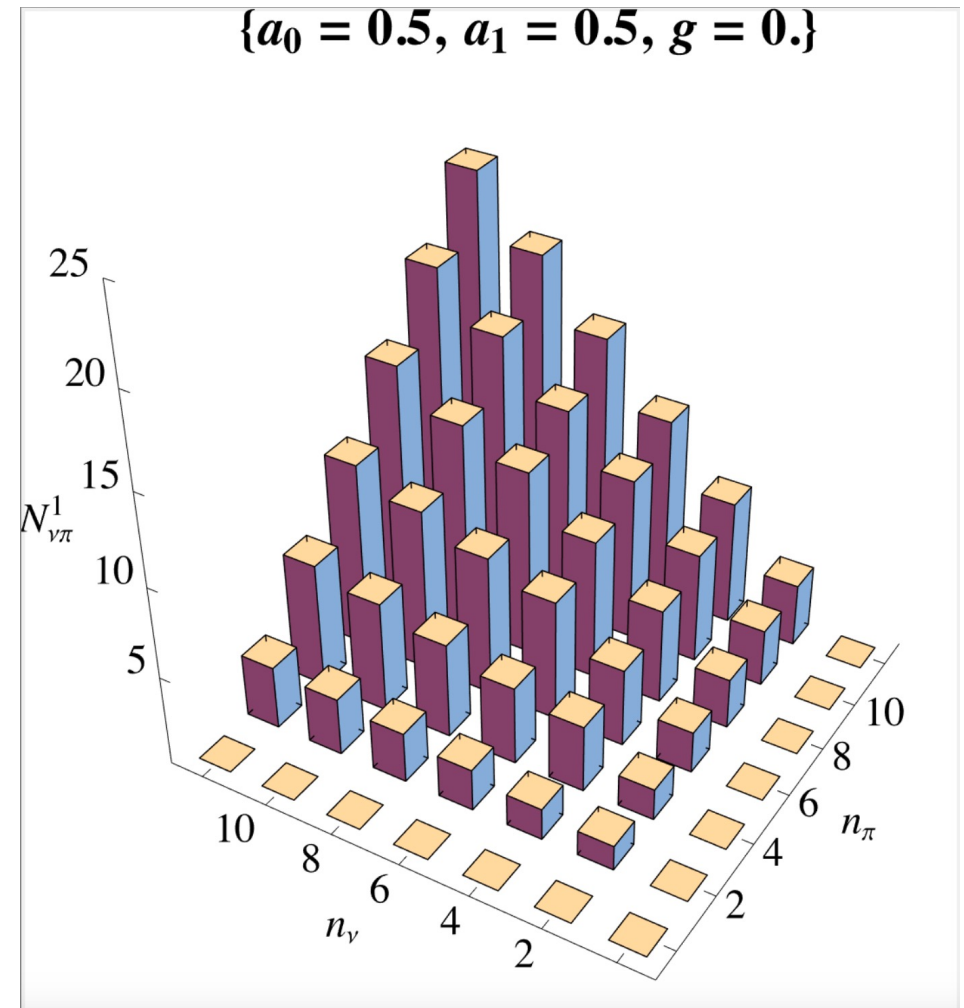


Wave function analysis for ^{128}Cd



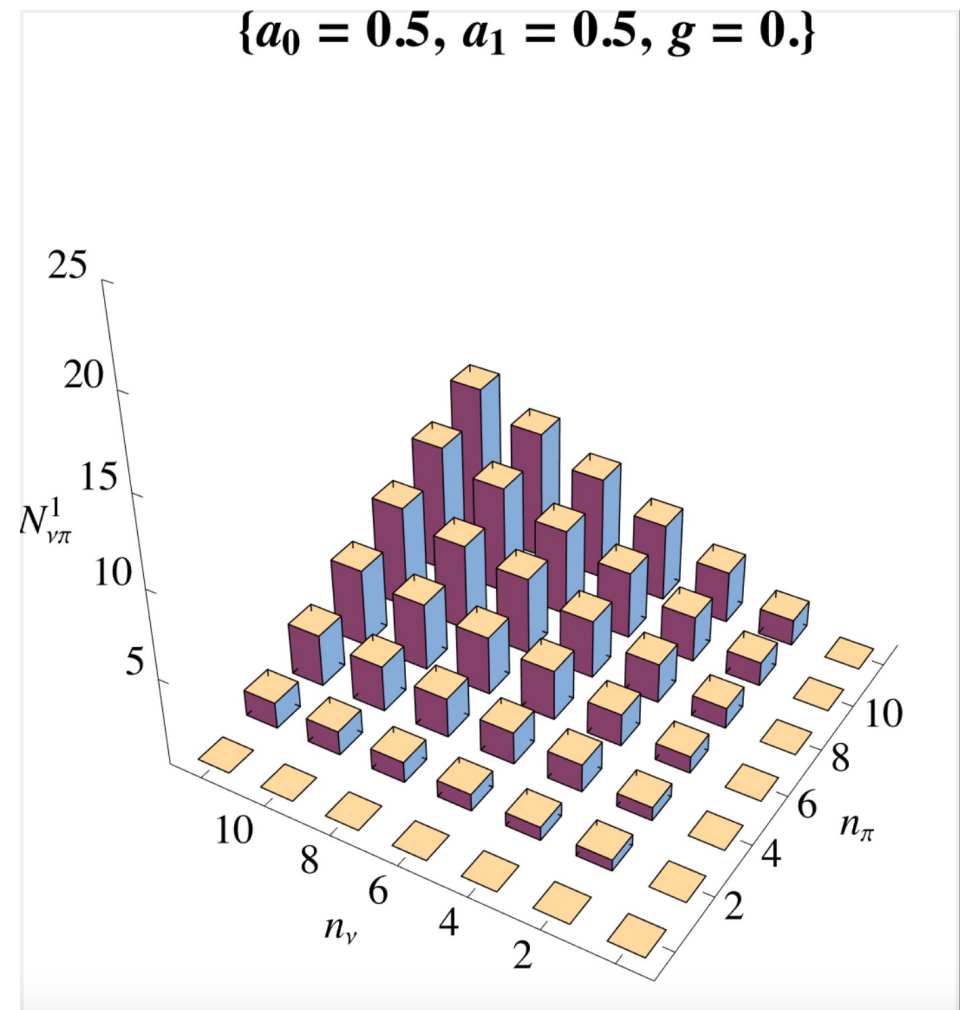
SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



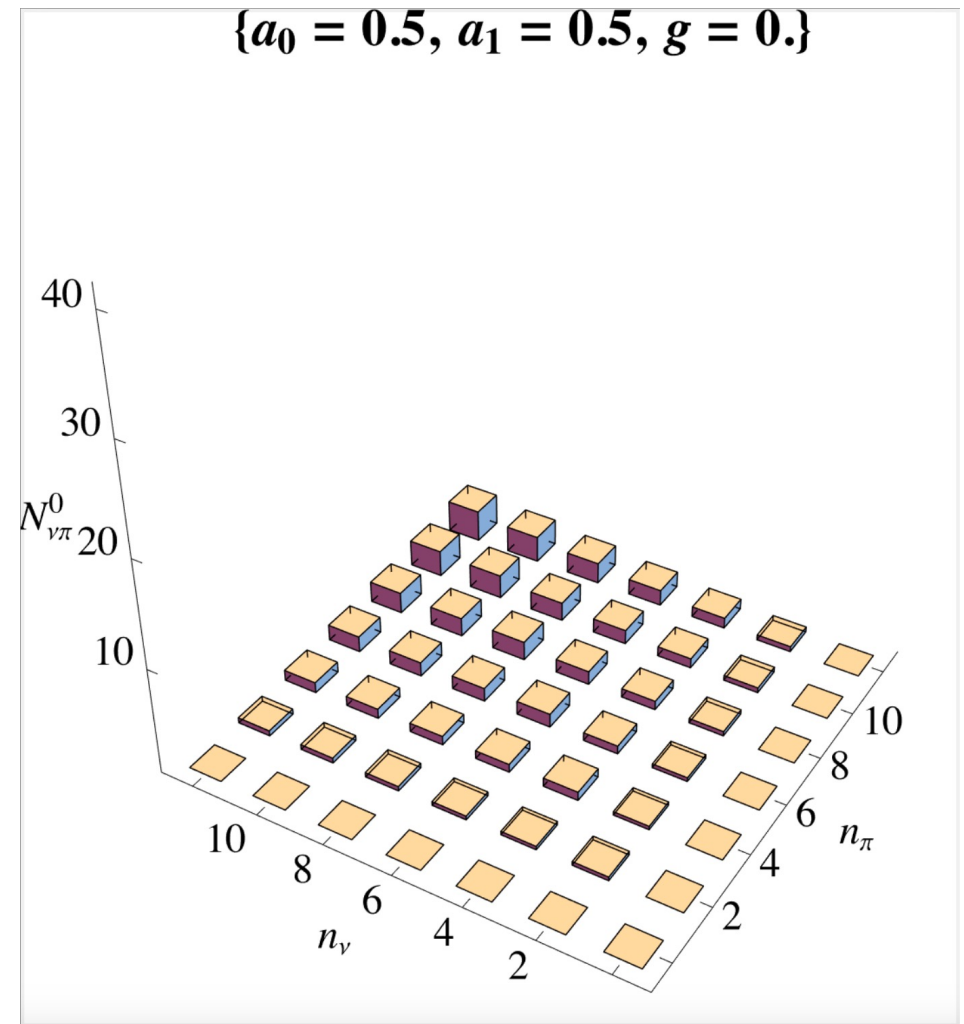
SDI in the sd shell

The number of **isoscalar** $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



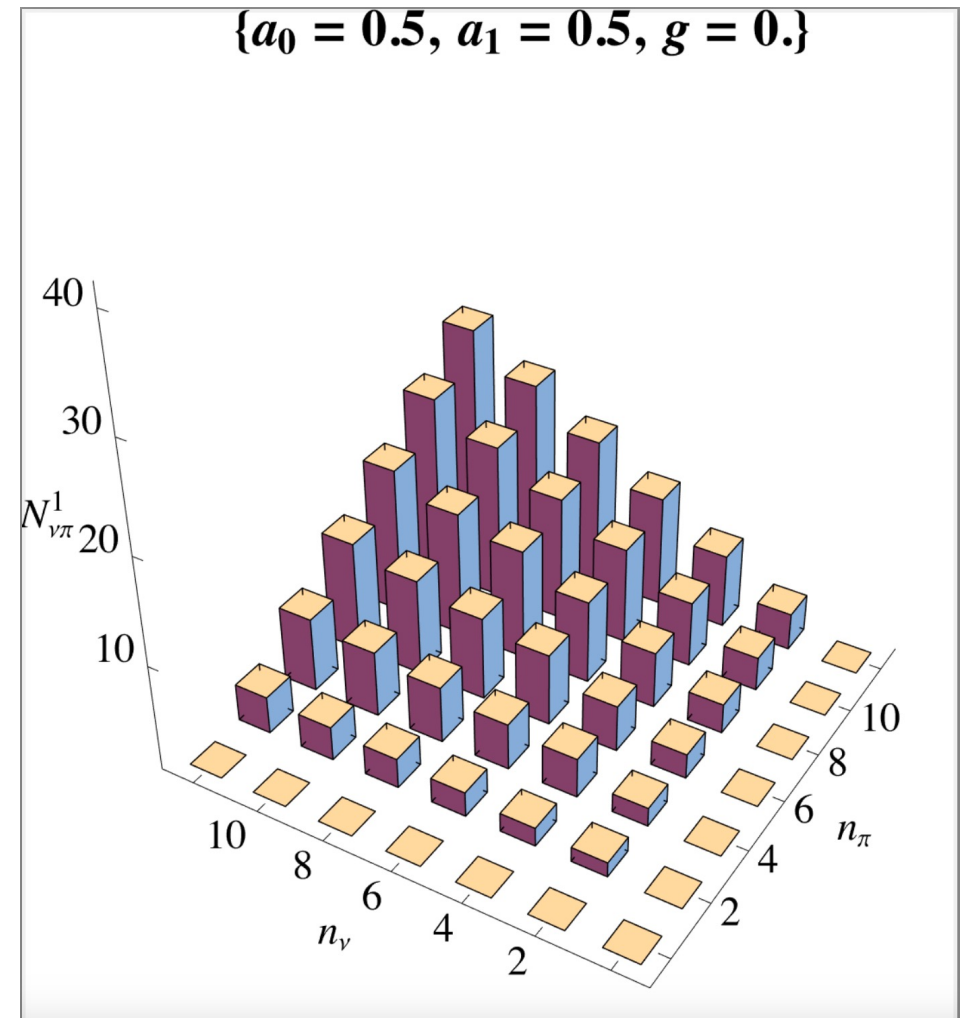
SDI in the *sd* shell

The number of $\nu\pi$ pairs
with $J_2 = 0$ in the
ground state of a SDI
with $a_0 = a_1$ and no
spin-orbit splitting.



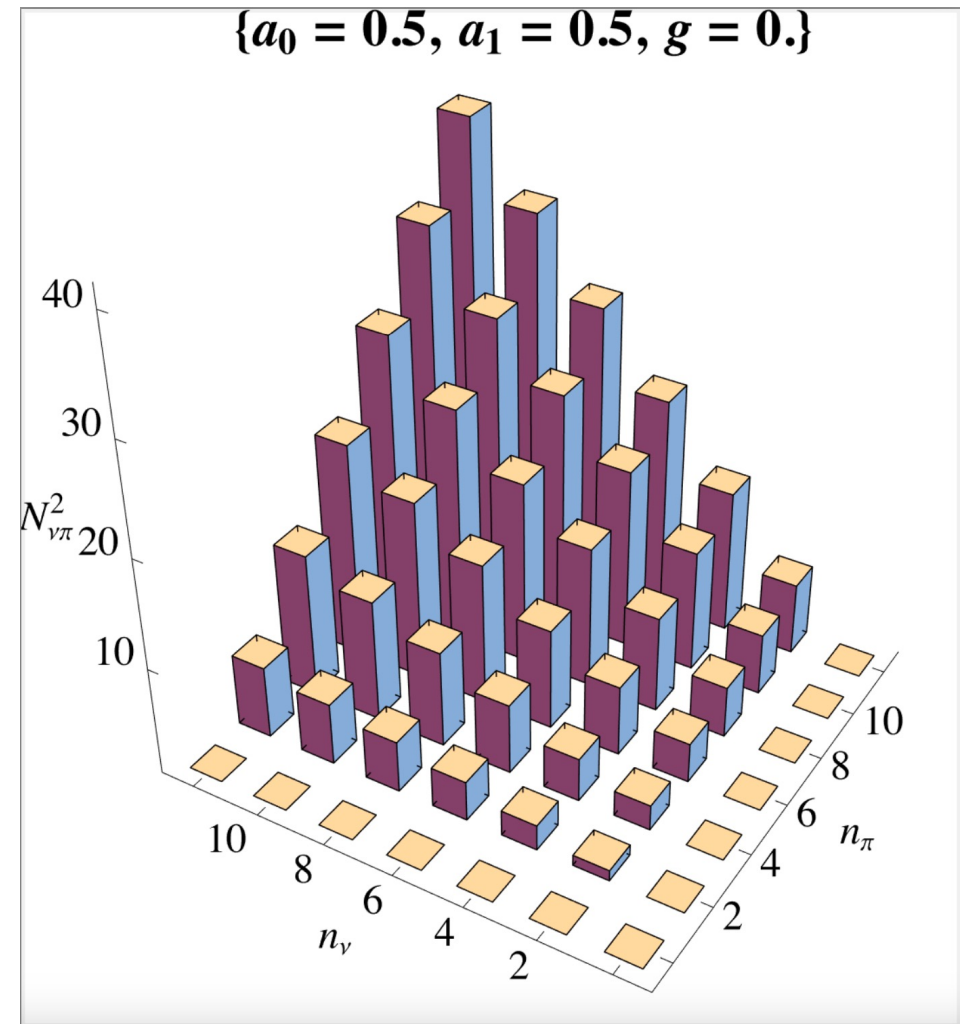
SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 1$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



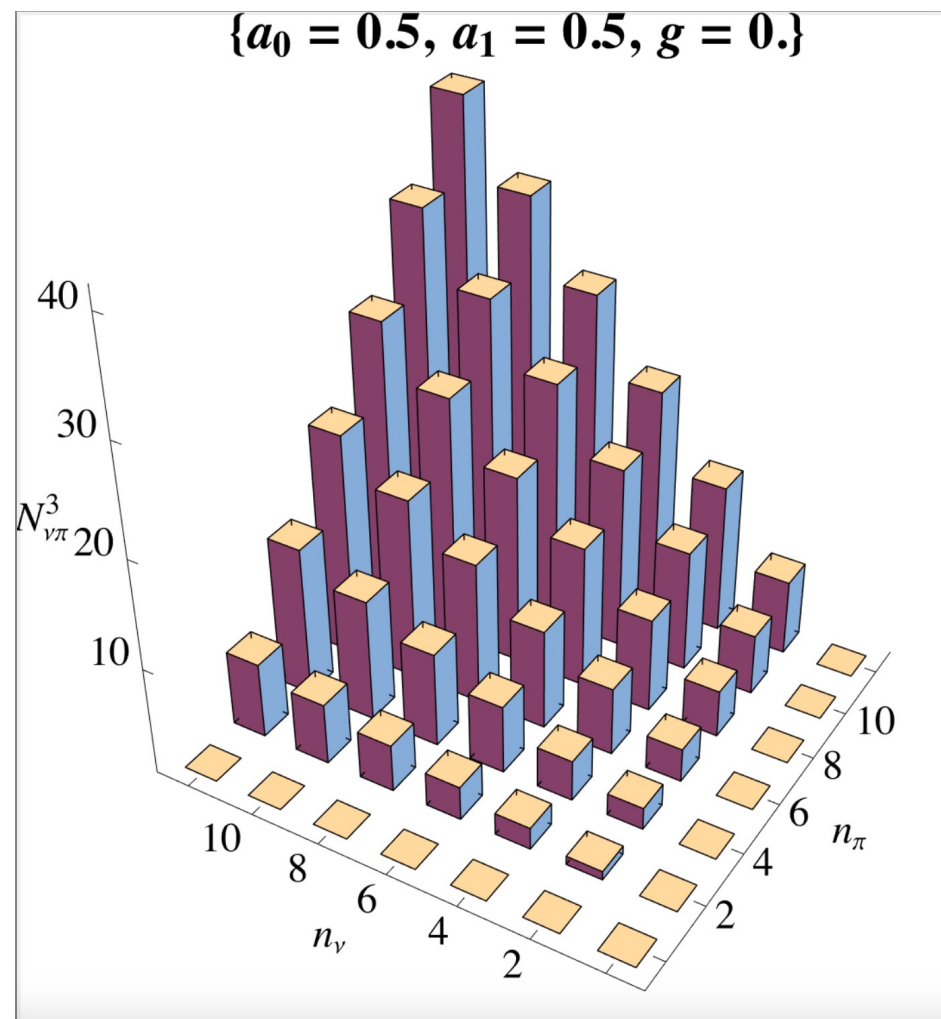
SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 2$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



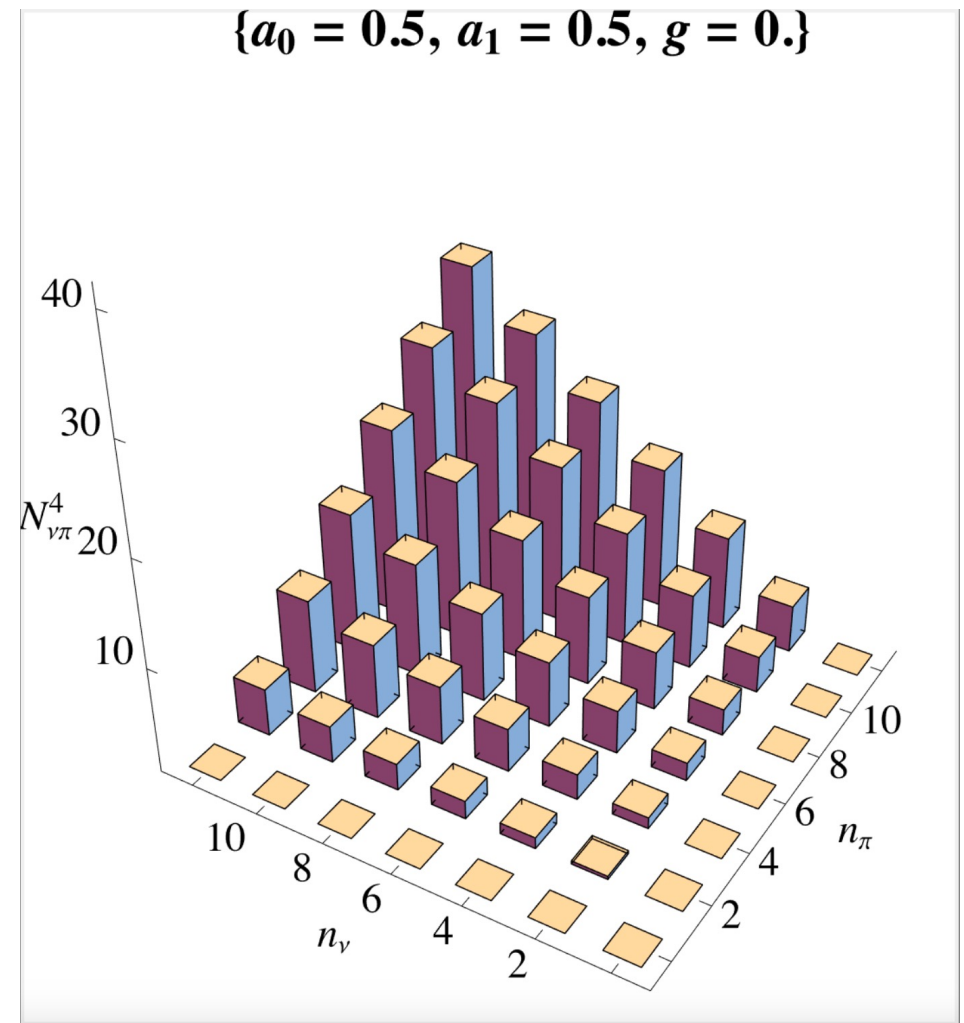
SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 3$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



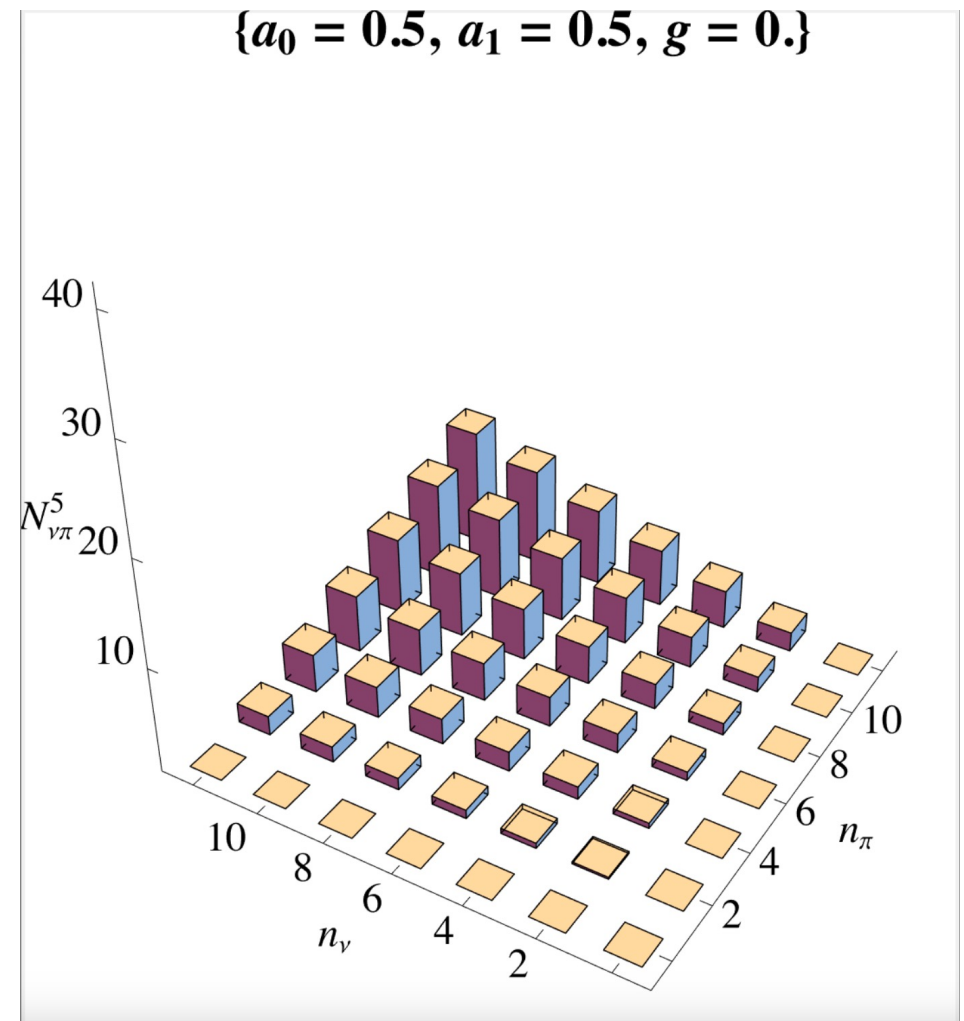
SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 4$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.

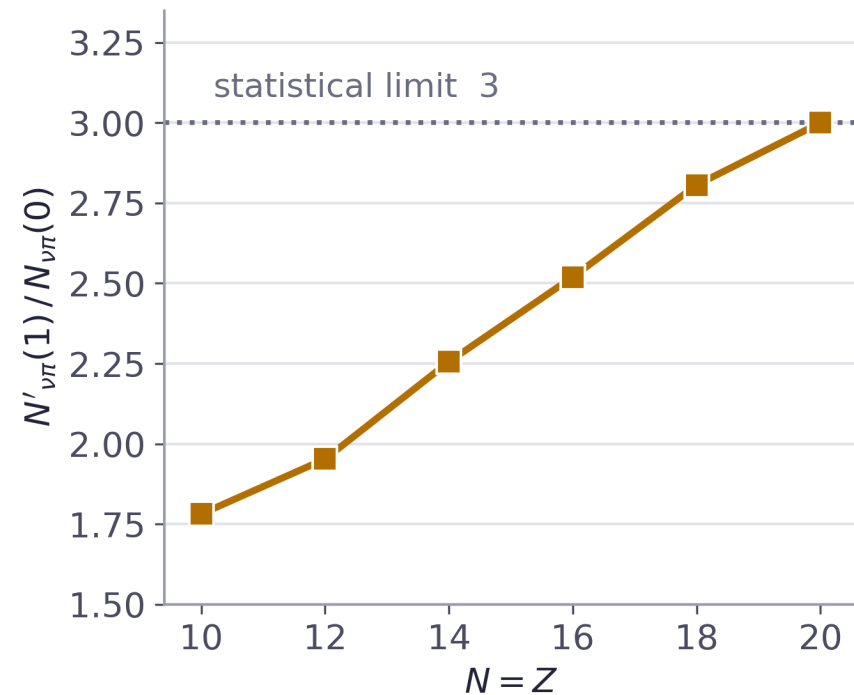
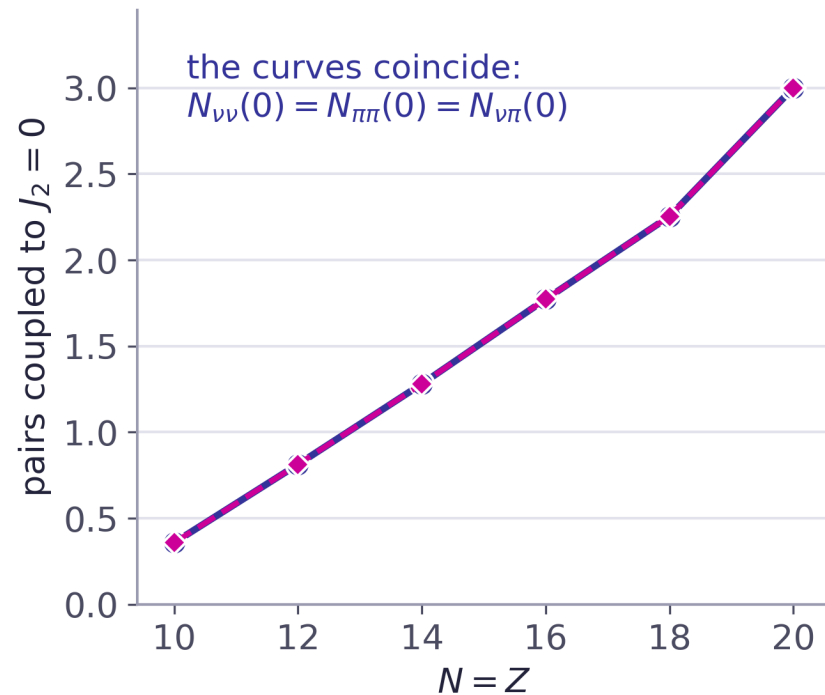


SDI in the sd shell

The number of $\nu\pi$ pairs with $J_2 = 5$ in the ground state of a SDI with $a_0 = a_1$ and no spin-orbit splitting.



What the numbers say (Claude)



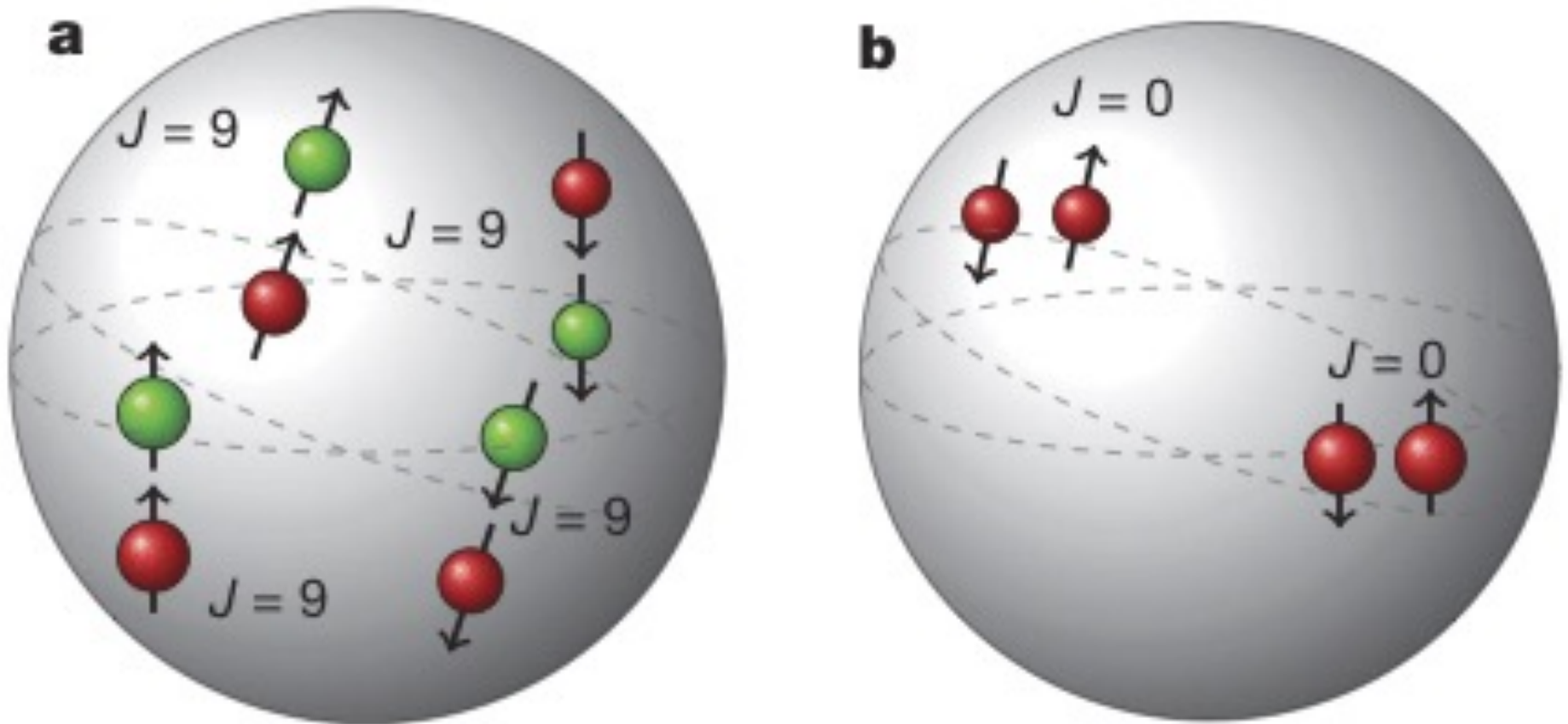
Like-nucleon pairing is weakest at $N = Z$, where the three isovector pair numbers coincide.

The isoscalar-to-isovector ratio grows towards the statistical value 3 as the shell fills.

Take home

Before you start counting pairs you better decide what kind of pairs you would like to count.

Spin-aligned coupling scheme



Approximations

Hypothesis: $N = Z$ nuclei can be described in the (spherical) shell model, in an appropriate model space and with an appropriate interaction.

Approximations:

- (A) Truncate shell model to a single high- j orbital.
- (B) Truncate single- j space to one written in terms of aligned-spin B ($J = 9$ in $0g_{9/2}$) pairs.
- (C) Replace aligned-spin B pairs by b bosons (referred to as “ b -IBM”).

Pair analysis in the shell model

Define different types of nucleon pairs:

$$B_{JT}^+ = \left(a_{j1/2}^+ \times a_{j1/2}^+ \right)^{(JT)}$$

$$S^+ : J = 0, T = 1; \quad D^+ : J = 2, T = 1; \quad B^+ : J = 9, T = 0.$$

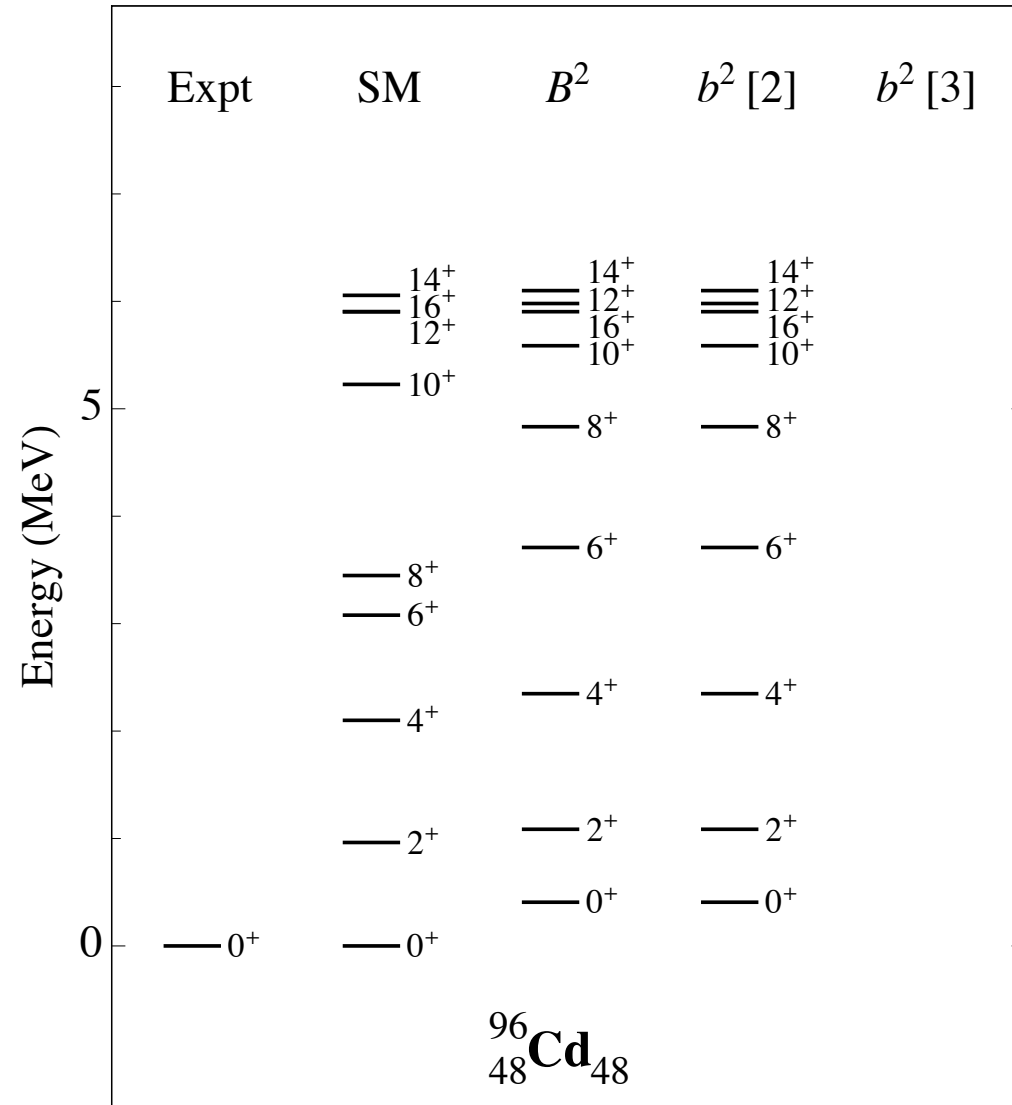
Calculate overlap with shell-model wave functions with the nucleon-pair shell model in an isospin-invariant formulation.

Mapping to bosons

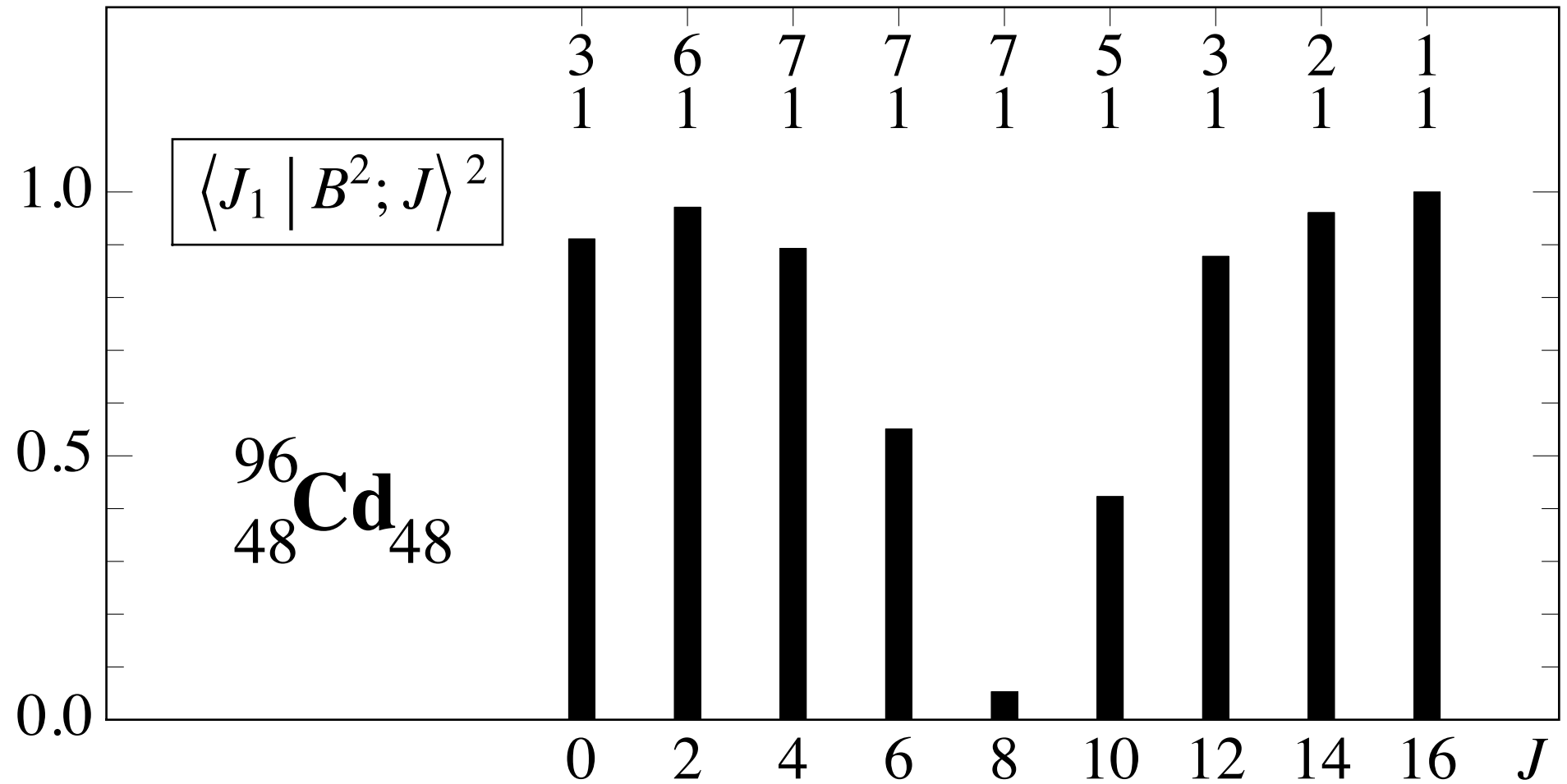
Transform to a much simpler problem in terms of interacting bosons: $B^\dagger \rightarrow b^\dagger$

Boson energies and boson-boson interactions are derived from the shell model (i.e., **not** fitted).

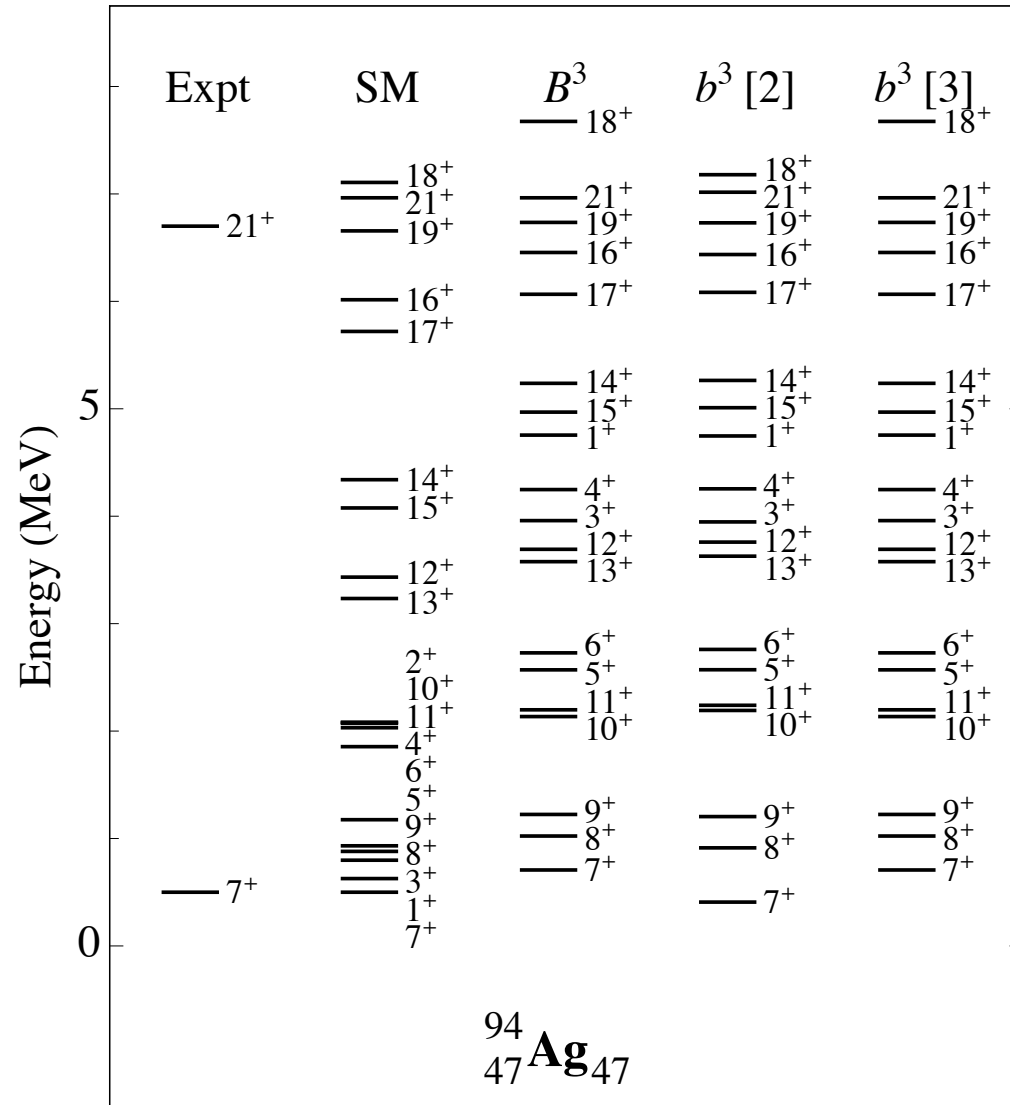
Spectrum of ^{96}Cd



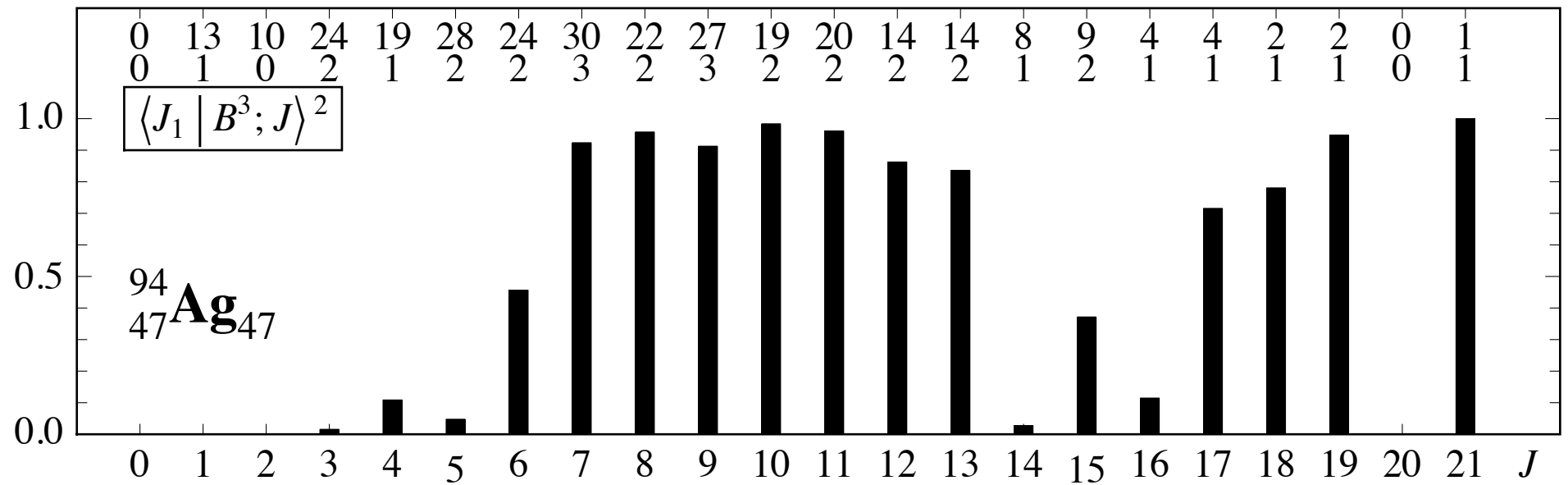
B -pair analysis of ^{96}Cd



Spectrum of ^{94}Ag



B -pair analysis of ^{94}Ag



Conclusion

- (A) The main issue is whether the shell-model space can be reduced to a single- j orbital.
- (B) If so, **selected** states can be written in terms of spin-aligned np pairs.
- (C) The approximation of spin-aligned B pairs by b bosons is generally valid.