

Configuration-Interaction Studies of Nuclear Alpha Clustering

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Random interactions: why, and the simplest model

Why random interactions

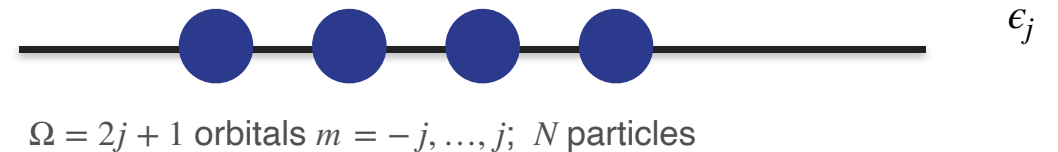
- **Emergence:** how do regular patterns arise from complex interactions of simple components?
- Two-body interactions embedded in the many-body space and constrained by rotational symmetry
- Bosons as effective degrees of freedom: pairs, clusters, IBM, collective shapes
- Same ensemble for **fermions and bosons**: what is due to statistics, what is generic, what is different.

Wigner, Ann. Math. 67, 325 (1958); Dyson, J. Math. Phys. 3, 140 (1962); French and Wong, PLB 33, 449 (1970); Bohigas and Weidenmüller, Annu. Rev. Nucl. Part. Sci. 38, 421 (1988); Zelevinsky, Annu. Rev. Nucl. Part. Sci. 46, 237 (1996); Johnson, Bertsch, Dean, PRL 80, 2749 (1998); Bijker and Frank, PRC 64, 061303 (2001)

The model: one level, two-body forces

- N identical particles on a single level: fermions with half-integer j , bosons with integer ℓ
- Many-body states classified by (J, M) ; $D(J)$ states of spin J

$$H = \sum_{L \text{ even}} V_L \sum_M P_{LM}^\dagger P_{LM}, \quad P_{LM}^\dagger = (a^\dagger a^\dagger)_{LM}$$

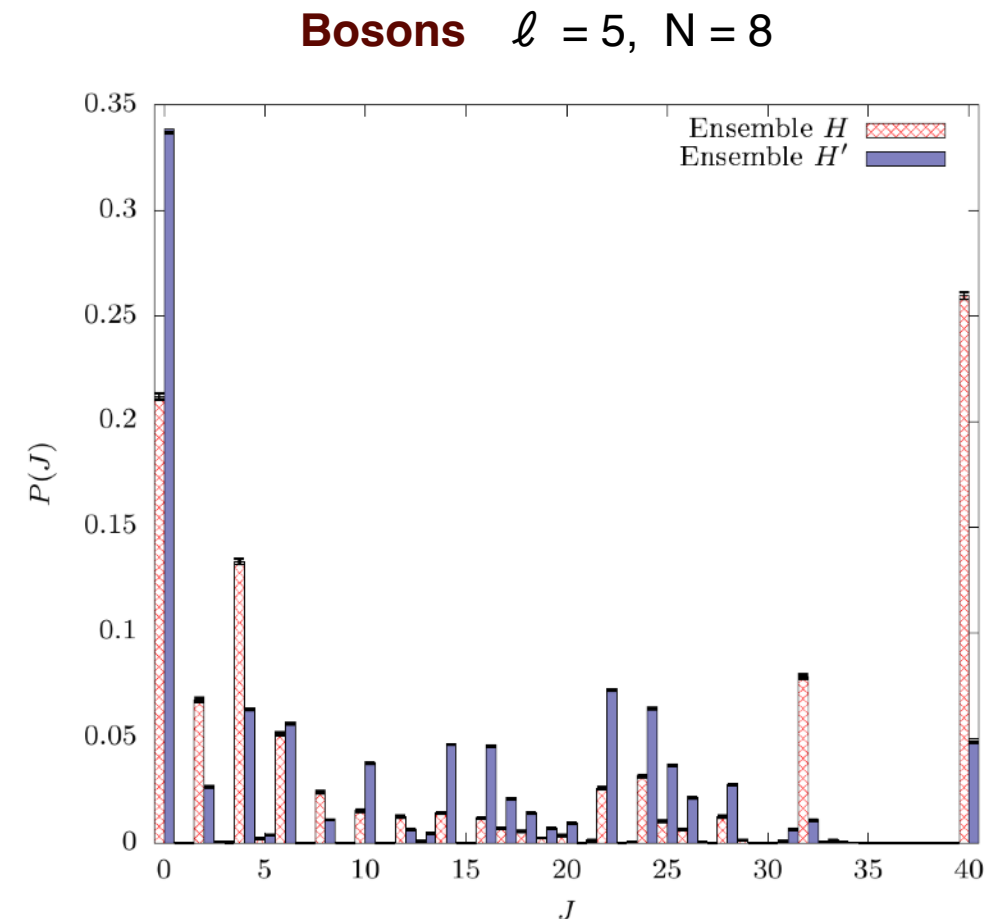
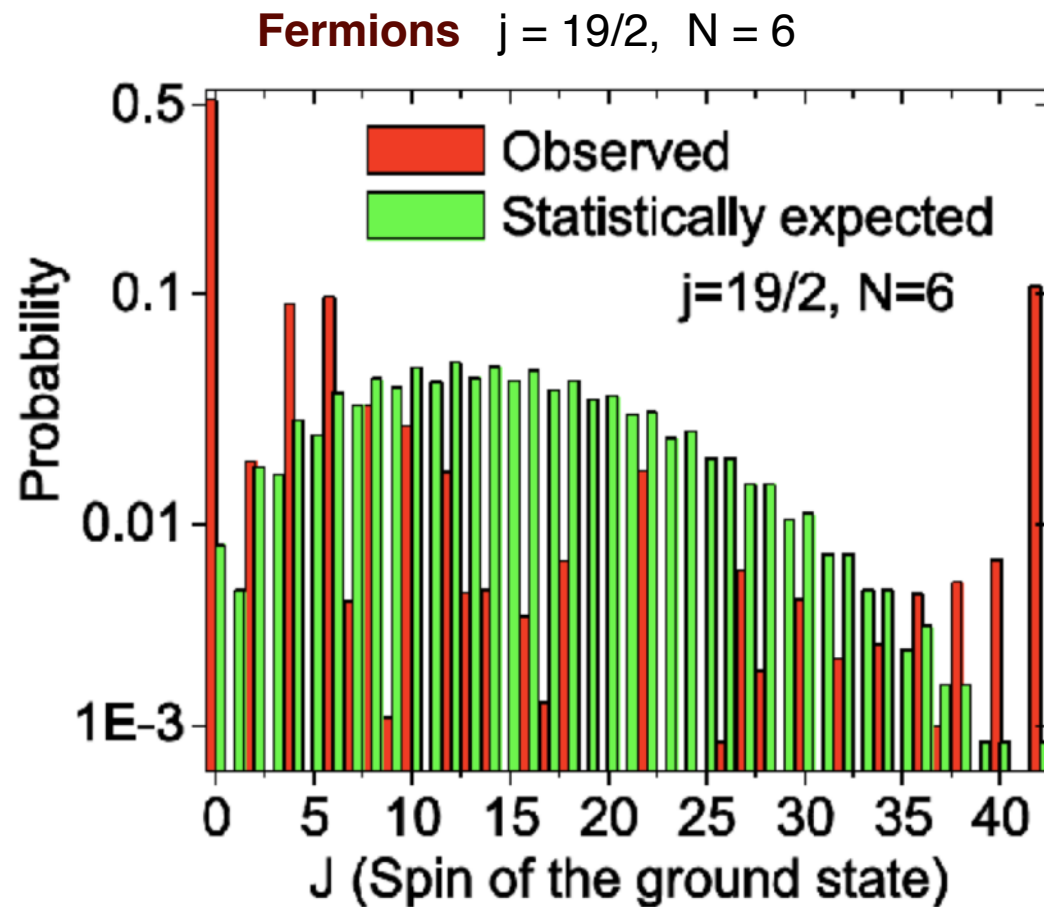


- V_L independent Gaussian numbers, $\overline{V_L} = 0$, $\overline{V_L^2} = 1$; the whole dynamics is fixed by $\ell + 1$ (bosons) or $j + \frac{1}{2}$ (fermions) numbers
- **Primed ensemble H' :** the monopole and J^2 components are removed from each realization; wave functions unchanged, only the $J(J + 1)$ energy shift is gone
- 10^5 realizations per system; ground-state spin, energy and wave function recorded

Question for the rest of this part: what do the ground states of such a structureless ensemble look like, and are they random?

Ground-state spin: $J = 0$ dominates

Probability that the ground state of a random realization has spin J , compared with the fraction of states $D(J)/D$ that spin J would get by chance.



- $J = 0$ is the most probable ground state in both systems, a third to a half of all realizations against a statistical chance (about 1%); for odd N bosons the role of $J = 0$ is taken by $J = \ell$
- The aligned state $J = J_{\max}$ is also enhanced: it is unique, its energy is linear in the V_L and has the broadest distribution
- Removing the J^2 component (primed ensemble H' , blue) suppresses the high-spin ground states and leaves the $J = 0$ preponderance intact

The $J = 0$ preponderance is generic: it needs neither pairing nor a specific force. What are these ground states?

Principal component analysis of ground states

Question: each random interaction $\{V_L\}$ gives a ground state $|\varphi_n\rangle$. Are these states random, or do they repeatedly select the same few many-body structures?

Ensemble of ground states

$$|\varphi_n\rangle = \sum_{i=1}^{D(J)} c_i^{(n)} |J_i\rangle, \quad n = 1, \dots, W$$

W realizations with the same ground-state spin J . The $|J_i\rangle$ are any fixed basis of the $D(J)$ spin- J states.

Ensemble density matrix

$$Q_{ik} = \frac{1}{W} \sum_{n=1}^W \langle J_i | \varphi_n \rangle \langle \varphi_n | J_k \rangle, \quad \text{Tr } Q = 1$$

PCA covariance matrix of the ensemble. Hermitian and positive.

Principal states and weights

$$Q |\phi_\alpha\rangle = q_\alpha |\phi_\alpha\rangle, \quad q_\alpha = \frac{1}{W} \sum_n \left| \langle \phi_\alpha | \varphi_n \rangle \right|^2$$

$q_1 \geq q_2 \geq \dots \geq 0$, $\sum_\alpha q_\alpha = 1$: average weight

of $|\phi_\alpha\rangle$ in a ground state. If every ground state coincided with one $|\phi_\alpha\rangle$, q_α is how often it occurs.

Effective dimension

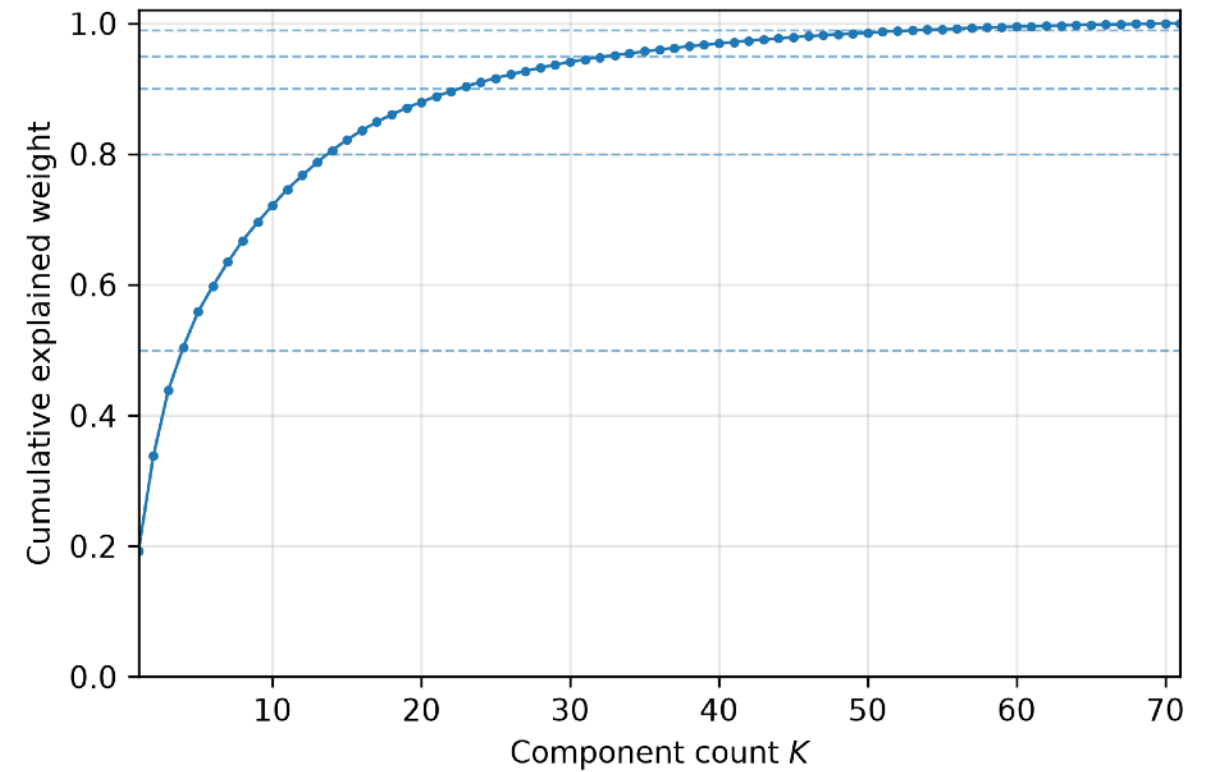
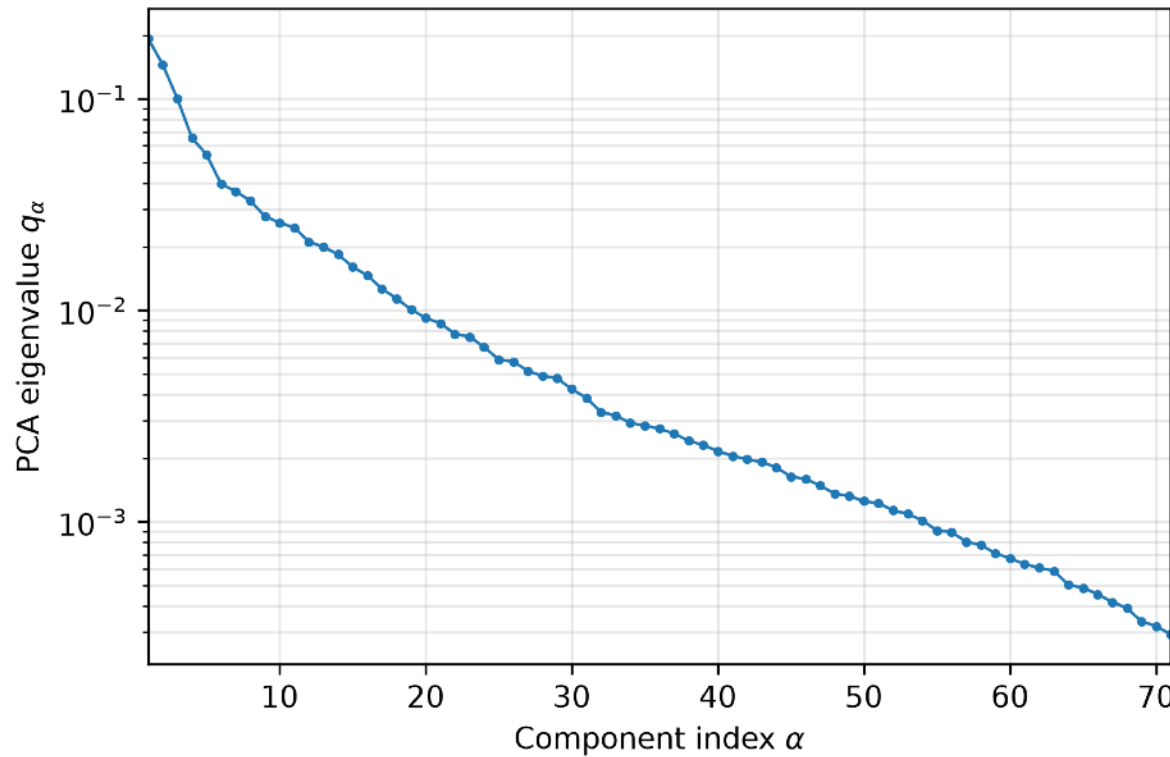
$$D_{\text{ent}} = e^S, \quad S = - \sum_\alpha q_\alpha \ln q_\alpha, \quad 1 \leq D_{\text{ent}} \leq D(J)$$

Random reference: ground states uniform over the J space give $q_\alpha = 1/D(J)$ and $D_{\text{ent}} = D(J)$.

$D_{\text{ent}} \ll D(J)$: ground states are **not** random, they live in a small preferred subspace.

Diagnostics: q_1 , $\eta = D_{\text{ent}}/D(J)$, K_{90} = number of components carrying 90% of the weight

Fermions: principal components of the ground states



Eigenvalues q_α of the ensemble density matrix (left) and cumulative weight $\sum_{\alpha \leq K} q_\alpha$ (right); $j = 25/2$, $N = 8$, primed ensemble, $J_{\text{gs}} = 0$ realizations.

$j = 25/2$, $N = 8$: $D(0) = 71$ states with $J = 0$

- $q_1 = 0.193 \gg 1/71 = 0.014$
- $D_{\text{ent}} = 21.2$, $D_{\text{ent}}/D(0) = 0.30$
- $K_{90} = 23$, $K_{95} = 33$, $K_{99} = 54$: strong concentration, but an extended low-weight tail

Trend with size

- At $j = 25/2$: $D_{\text{ent}}/D(0) \approx 0.90, 0.67, 0.30$ for $N = 4, 6, 8$
- $N = 8$: from 0.71 at $j = 15/2$ to 0.30 at $j = 25/2$
- Matched bosons $\ell = 9$, $N = 8$, $D(0) = 71$: $q_1 = 0.254$, $D_{\text{ent}} = 13.5$, ratio 0.19

Fermion ground states occupy a preferred subspace, about 30% of the $J = 0$ space here.

The concentration grows with system size, but is milder than for bosons.

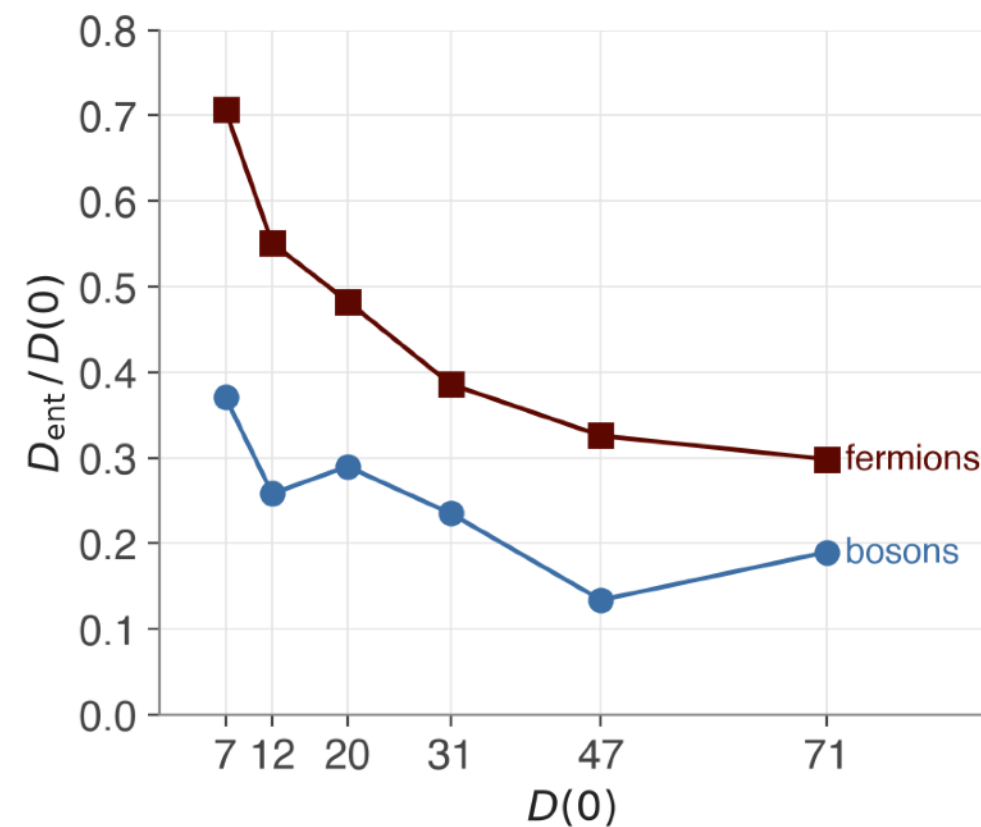
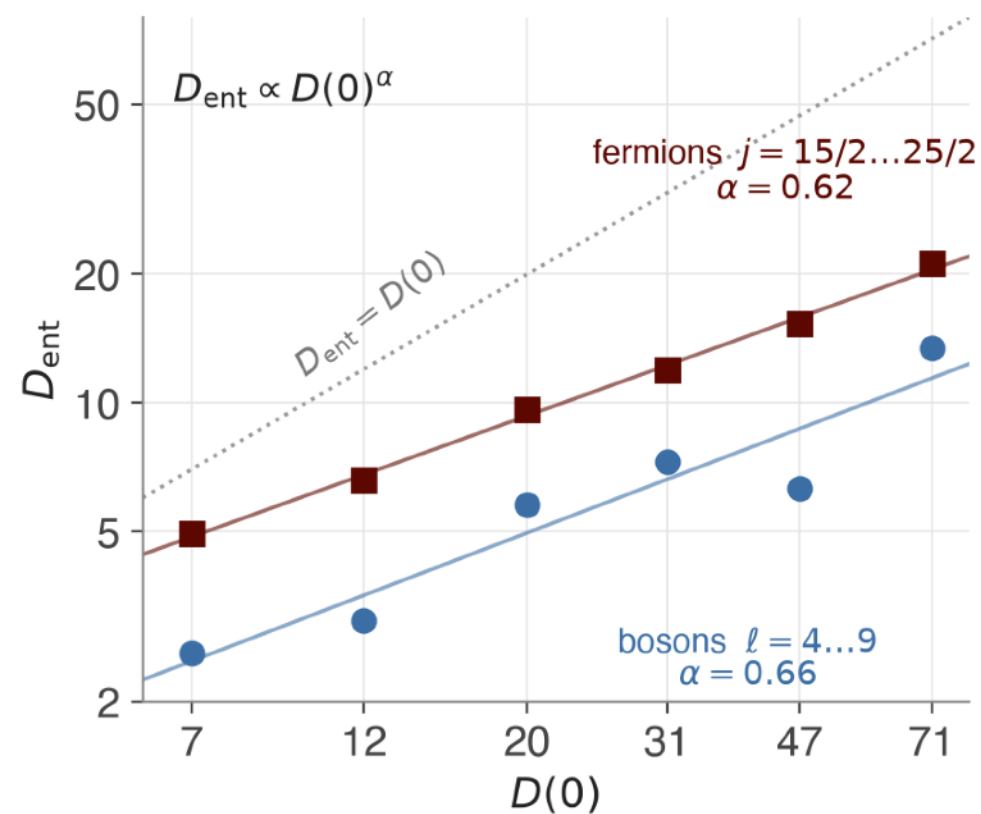
PCA: fermions versus bosons at matched N = 8 and D(0)

Bosons					Fermions			
D(0)	ℓ	q_1	D_{ent}	$D_{\text{ent}}/D(0)$	j	q_1	D_{ent}	$D_{\text{ent}}/D(0)$
7	4	0.562	2.60	0.371	15/2	0.412	4.95	0.707
12	5	0.600	3.10	0.258	17/2	0.317	6.61	0.551
20	6	0.326	5.80	0.290	19/2	0.304	9.66	0.483
31	7	0.340	7.29	0.235	21/2	0.221	11.97	0.386
47	8	0.446	6.30	0.134	23/2	0.219	15.32	0.326
71	9	0.254	13.49	0.190	25/2	0.193	21.20	0.299

Exact duality

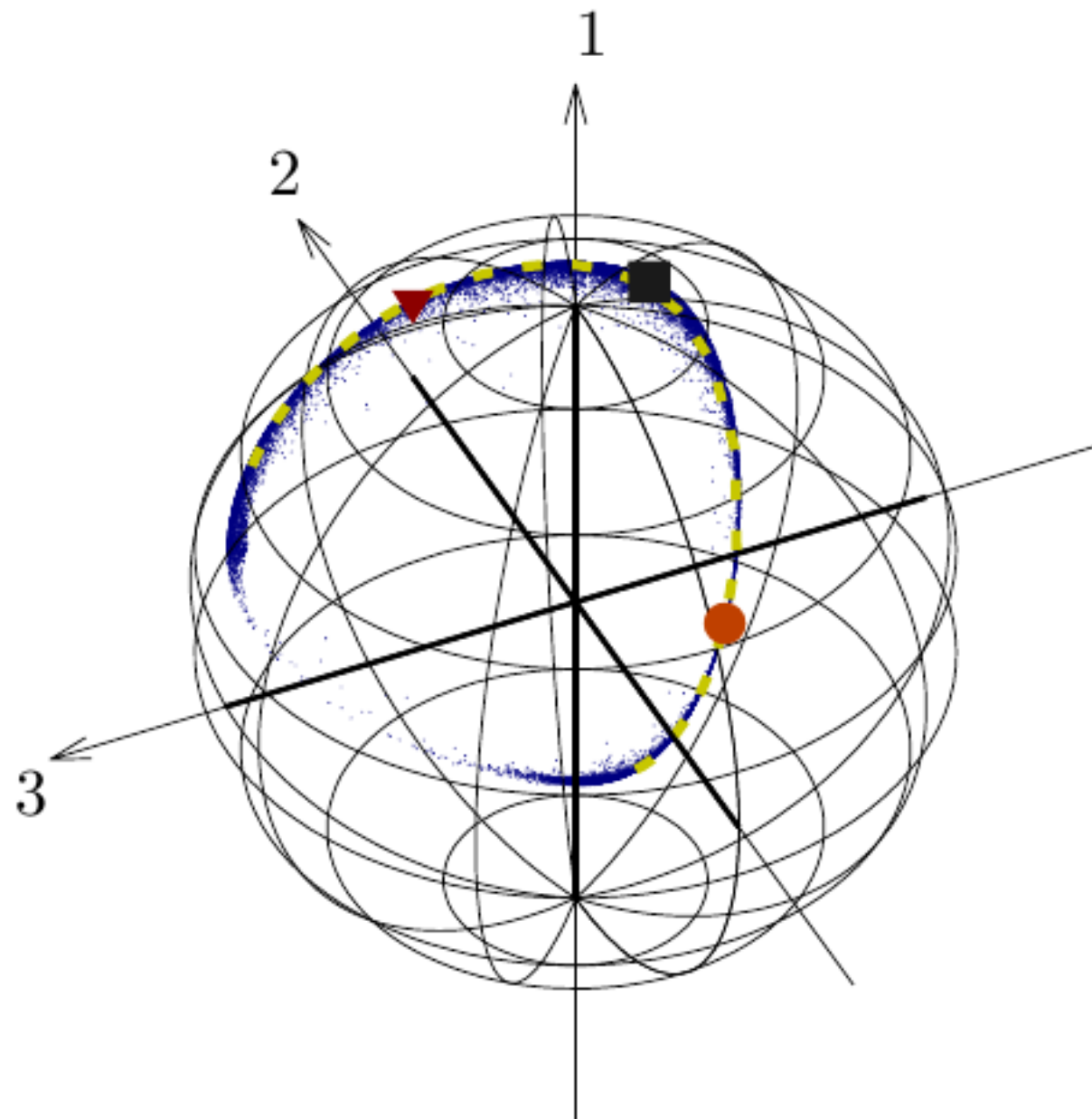
$$D_{\ell,N}^B(J) = D_{j,N}^F(J), \quad j = \ell + \frac{N-1}{2}$$

Identical $D(J)$ for **all** J : the same Hilbert spaces, only the exchange statistics differ.
At every matched dimension fermions have a smaller q_1 and a larger $D_{\text{ent}}/D(0)$.



Pauli statistics weakens, but does not remove, the preferred ground-state structures

One example in full: bosons with $\ell = 5, N = 8$



Every $J = 0$ ground state $|\varphi_n\rangle$ drawn as a point with coordinates $\langle\phi_1|\varphi_n\rangle, \langle\phi_2|\varphi_n\rangle, \langle\phi_3|\varphi_n\rangle$, its overlaps with the three principal states. Dashed line: ground states of Hamiltonians with only V_0 and V_2 . Square: quadrupole-quadrupole, circle: pairing, triangle: $V_2 = -1$ alone.

Two-body random ensemble, 100 000 realizations

- $D(0) = 12$ states with $J = 0$, out of $D_{\ell N} = 1514$ states in total
- $J_{\text{gs}} = 0$ in 33.9 % of realizations
- Principal weights $q_1, q_2, q_3 = 0.60, 0.22, 0.13$
- Effective dimension $D_{\text{ent}} = 3.1$: three of the twelve states are used
- Gumbel fit of E_{gs} : $a = -11.67, b = 0.174$
- Inverted: $\mathcal{D} = 8.7$ competing states, spread $\sqrt{E^2} = 0.35 \sigma_{\text{max}}$

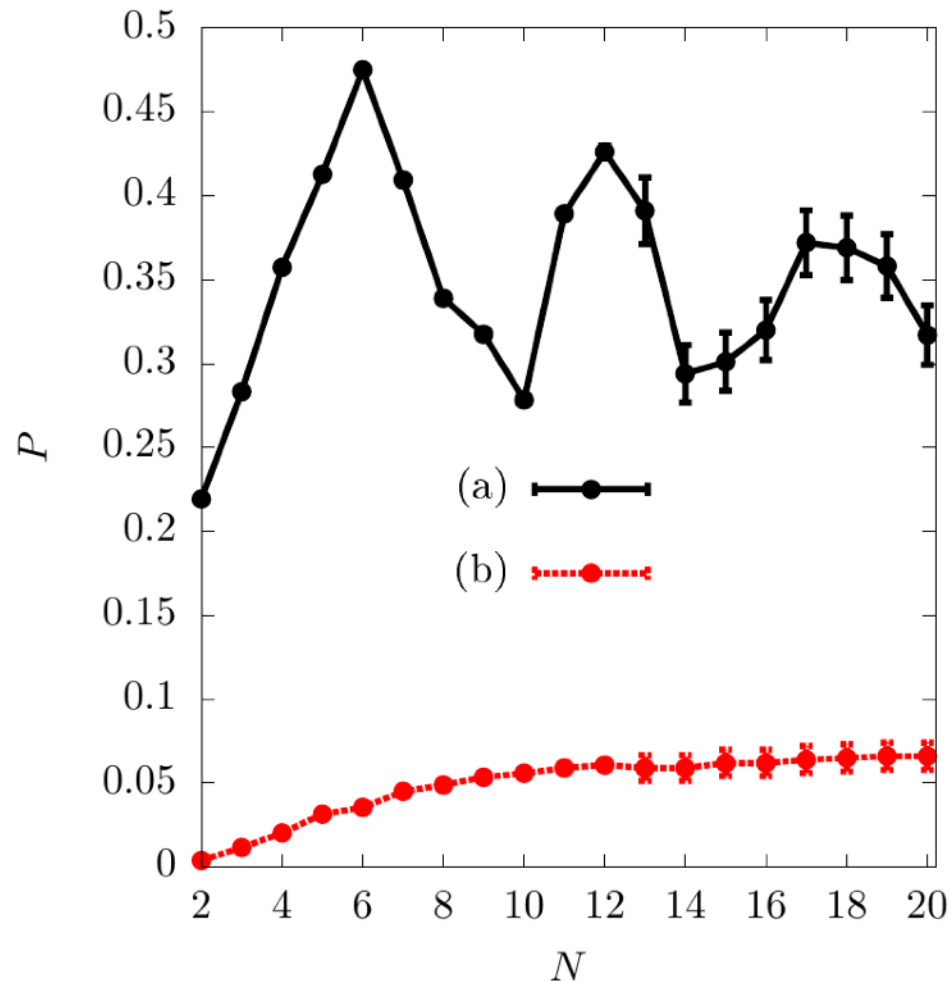
The points do not fill the hemisphere: they trace a **curve**, one parameter effectively. The density is highest near the quadrupole-quadrupole ground state and low near the paired state.

12 states available, 3 effectively effectively happen as ground states, and pairing is not among the favorites.

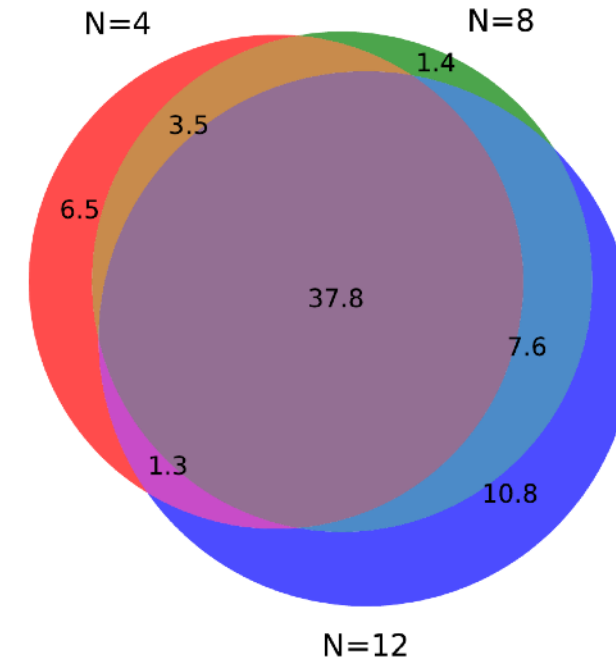
The ground states are far from random.

Few same patterns emerge.

Are the $J = 0$ ground states condensates of clusters?



$\ell = 5$ bosons: probability of a $J = 0$ (top) and $J = J_{\max}$ (bottom) ground state versus N . The $J = 0$ probability oscillates with N .



$\ell = 6$ bosons: sets of interactions $\{V_L\}$ that give a $J = 0$ ground state for $N = 4, 8$ and 12 (percent of the ensemble). Nearly the same interactions do it for all three: 37.8%.

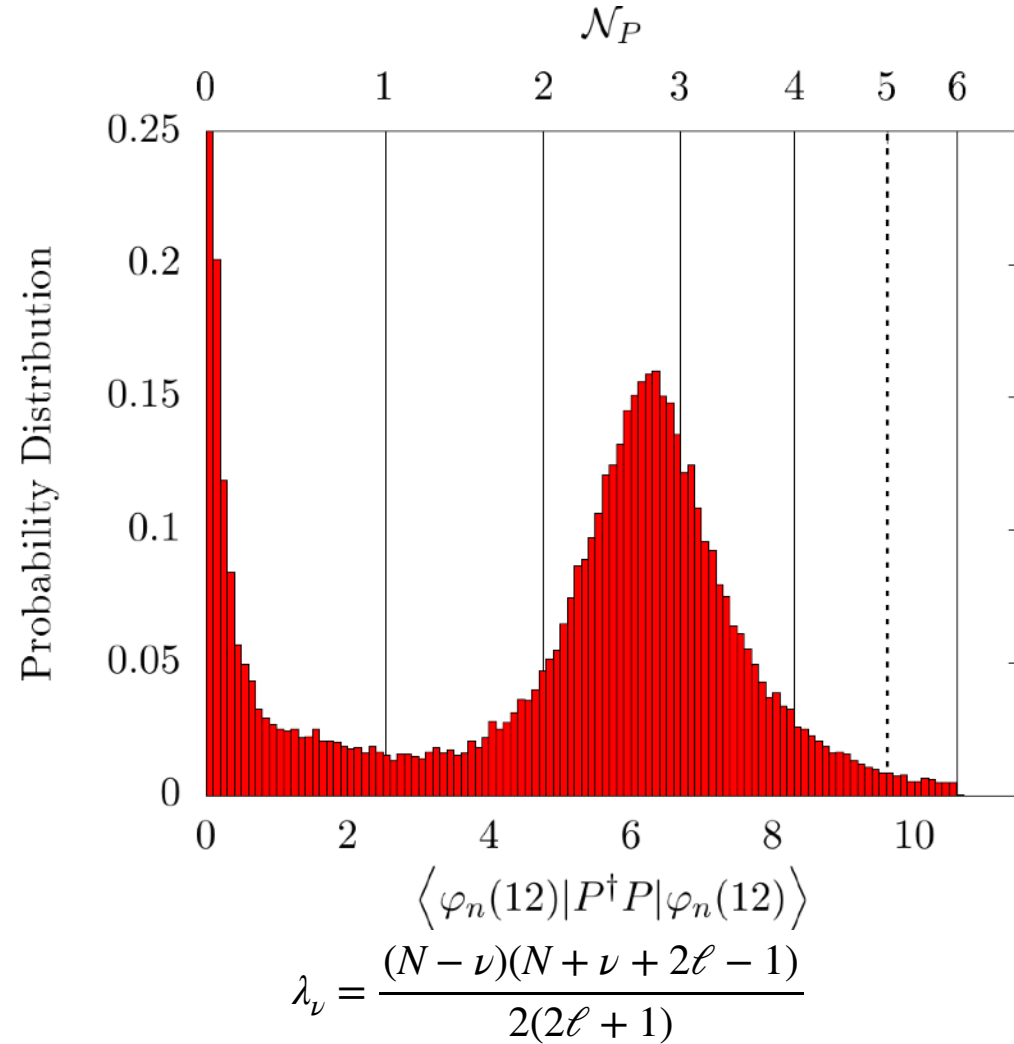
- Recall d bosons (IBM): every $J = 0$ state is a pair or triplet condensate. Oscillation in N and the set overlaps point to triplets and quartets.
- Matched interactions:** the same $\{V_L\}$ in systems of different N ; bosons $N = 4, 8, 12$ and $6, 9, 12$ at $\ell = 6$, fermions $N = 4, 8$ at $2j = 17-23$.
- Remove a cluster $C = P, T, Q_n$ from the larger ground state: do we land on the smaller one?

Diagnostics: cluster number $\langle \varphi_n(N) | C^\dagger C | \varphi_n(N) \rangle$ and removal amplitude $|\langle \varphi_n(N-k) | C | \varphi_n(N) \rangle|^2$

Pairs: pairing does not explain the $J = 0$ ground states

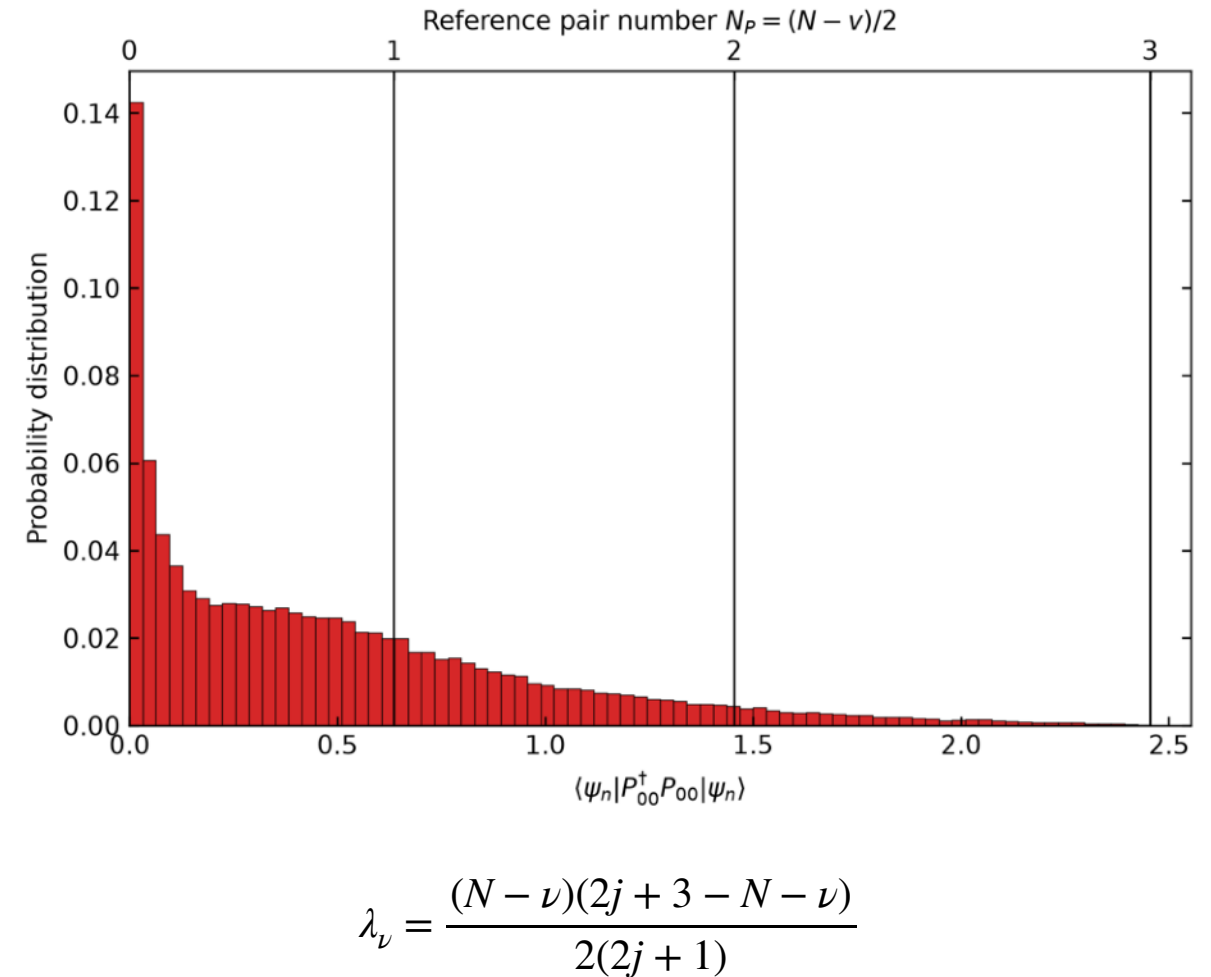
Pair content of every $J_{\text{gs}} = 0$ ground state: $x_n = \langle \varphi_n | P^\dagger P | \varphi_n \rangle$ with $P \equiv P_{00}$, compared with the exact eigenvalues for states of definite seniority ν ($\mathcal{N}_P = (N - \nu)/2$ pairs)

Bosons $\ell = 6, N = 12$



- About a quarter of the $J = 0$ ground states have no pairs at all
- Most of the rest have about 3 pairs of the possible 6

Fermions $j = 21/2, N = 6$



- Strong accumulation near $x_n = 0$, no peaks at the nonzero seniority values
- Most ground states are strongly seniority-mixed

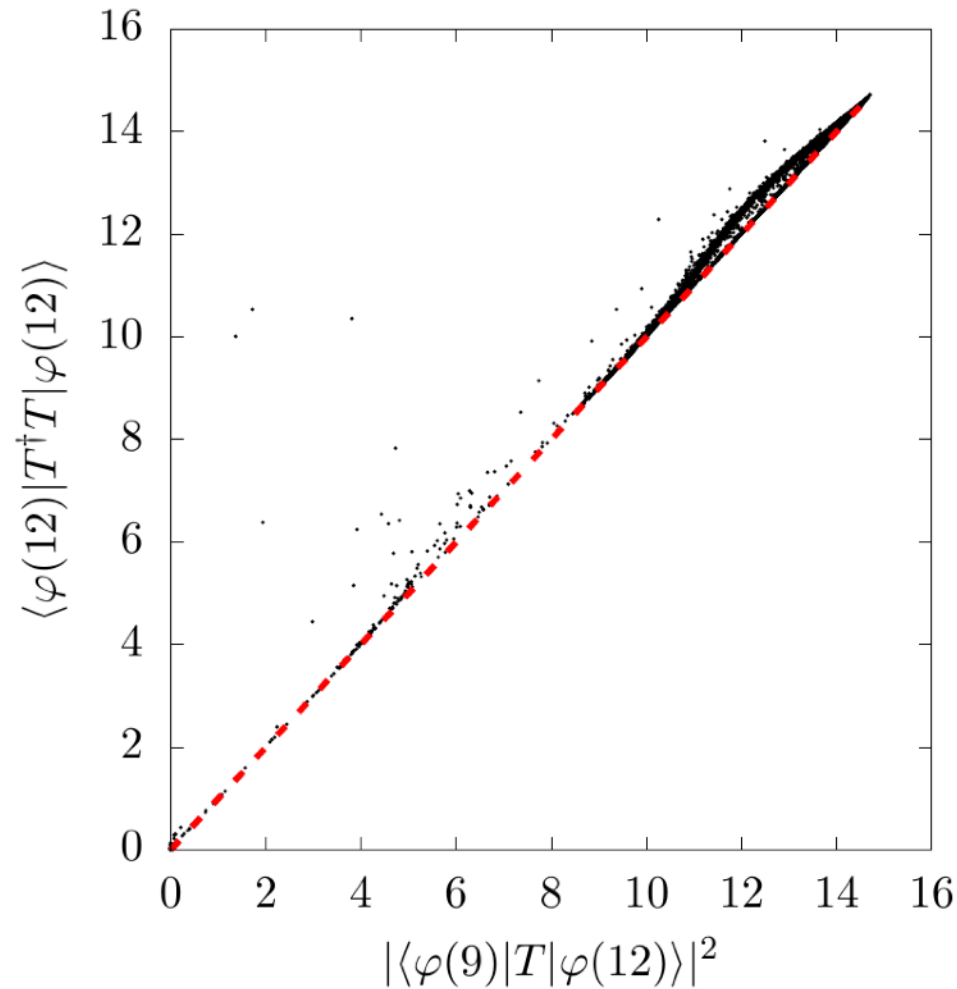
In neither system are the $J = 0$ ground states pair condensates.

Pairing-enhanced realizations exist only in the tail of the distribution.

Triplets: bosons, $\ell = 6, N = 12$

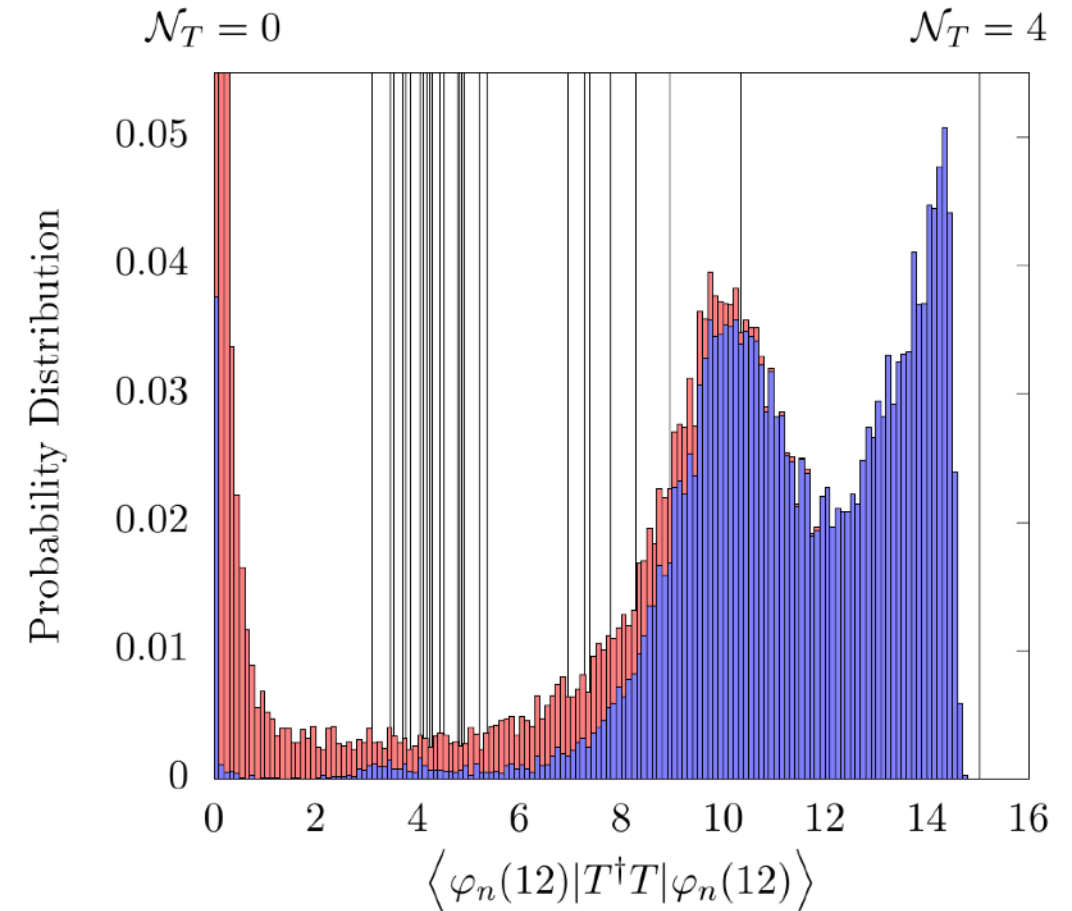
For even ℓ the three-boson $J = 0$ state is unique, so $T^\dagger \equiv T_{00}^\dagger$ is fixed once for all realizations. Possible triplet numbers in a $J = 0$ state of 12 bosons: $\mathcal{N}_T = 0, 1, 2, 4$ (three triplets force a fourth).

Triplet removal $N = 12 \rightarrow 9$



$|\langle \varphi_n(9) | T | \varphi_n(12) \rangle|^2$ against the norm $\langle \varphi_n(12) | T^\dagger T | \varphi_n(12) \rangle$. Points on the diagonal: $T | \varphi_n(12) \rangle \propto | \varphi_n(9) \rangle$, removing a triplet lands on the 9-particle ground state.

Triplet number



Vertical lines: eigenvalues of $T^\dagger T$ among the 127 $J = 0$ states (100 of them are zero). Red: all $J_{\text{gs}} = 0$ cases, 58% of realizations. Blue: the 19% where $N = 6$ and 9 also have $J = 0$. Peak at 15: nearly perfect four-triplet condensate, about 10% of all realizations. Peak at 0: the quartet-type ground states (35%).

Two populations of $J = 0$ ground states: triplet condensates (16%) and quartet-type states (38%), with little overlap.

Fermions: no counterpart on a single level, three identical fermions cannot couple to $J = 0$.

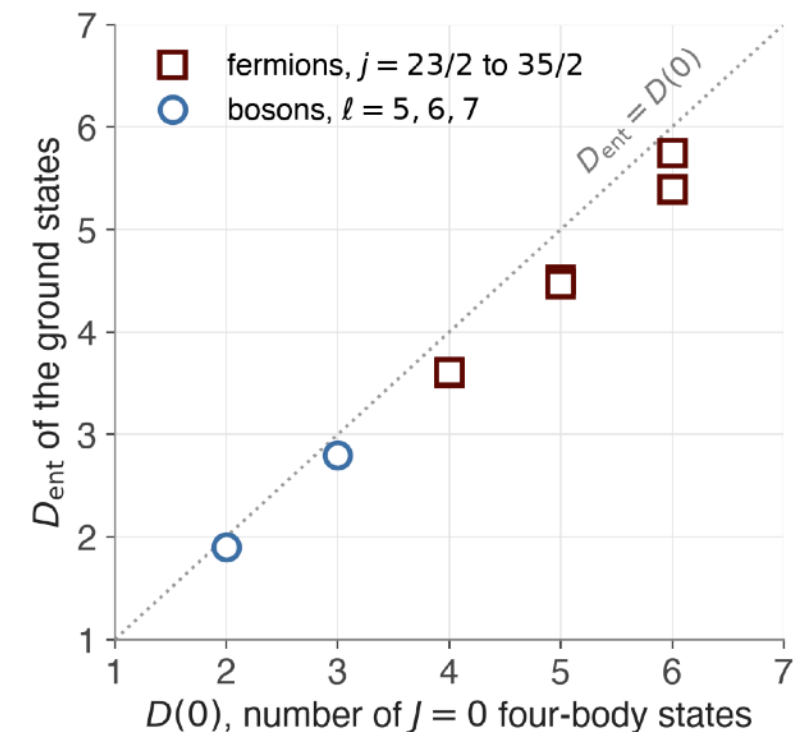
Which quartet? For four particles, any of them

PCA of the $J = 0$ ground states of **four** fermions on a single j level: the effective dimension stays close to the full $J = 0$ dimension, and the leading weight never exceeds 0.44.

j	$D(0)$	D_{ent}	q_1	q_2	q_3	q_4	q_5	q_6
23/2	4	3.60	0.415	0.287	0.185	0.113	—	—
25/2	4	3.61	0.438	0.251	0.182	0.129	—	—
29/2	5	4.52	0.341	0.241	0.193	0.147	0.078	—
31/2	5	4.47	0.288	0.266	0.253	0.133	0.061	—
33/2	6	5.75	0.226	0.223	0.165	0.155	0.144	0.087
35/2	6	5.39	0.285	0.226	0.192	0.138	0.091	0.068

Bosons, $N = 4$: picture is similar.

- No single quartet is preferred: every $J = 0$ four-body state appears as a ground state about equally often
- The quartet is **defined dynamically**, by the interaction of each realization: hence $Q_n^\dagger |0\rangle \equiv |\varphi_n(4)\rangle$
- The collectivity appears only in the larger systems, where the same quartet is repeated



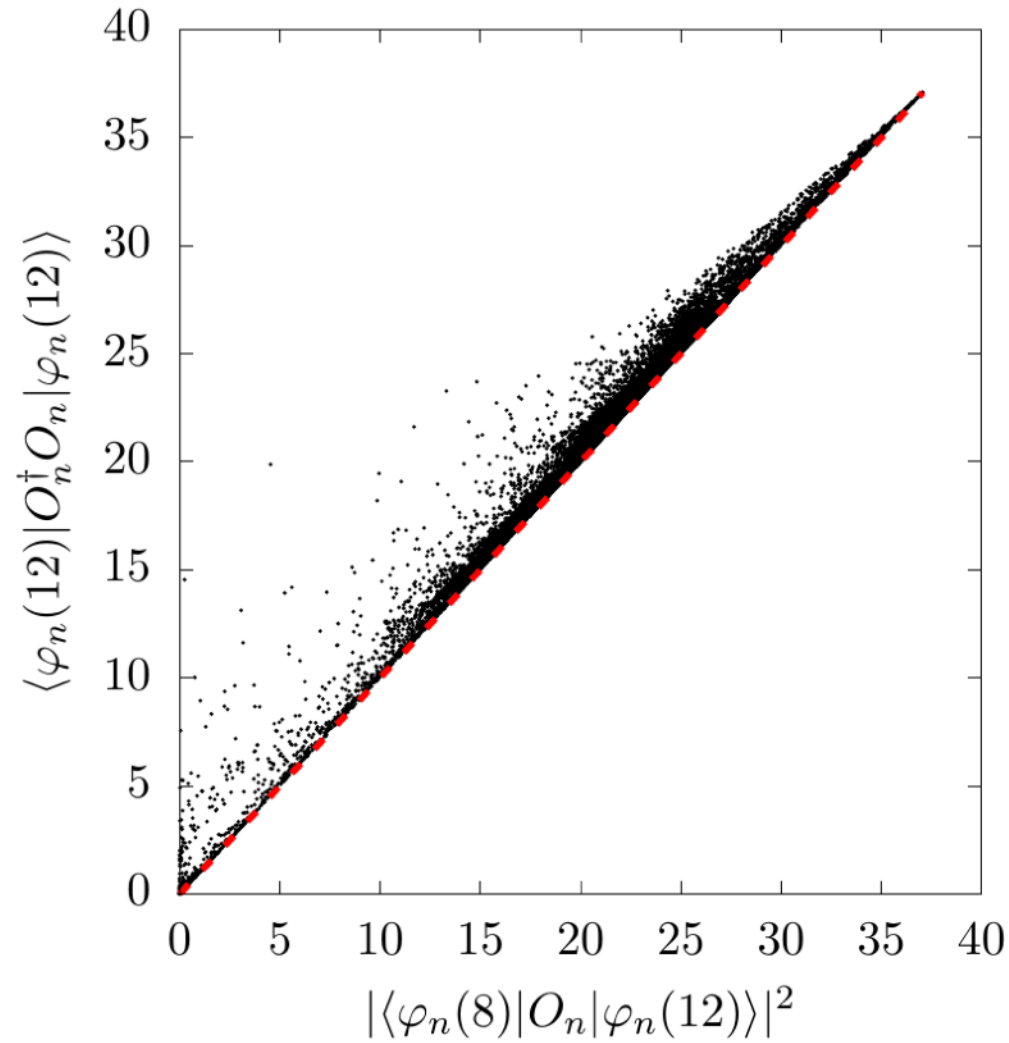
All $N = 4$ systems lie on the diagonal: the ground states fill the whole $J = 0$ space.

Four particles do not know which quartet to form. The preferred structure is a property of the interaction.

Quartets: the same test for bosons and fermions

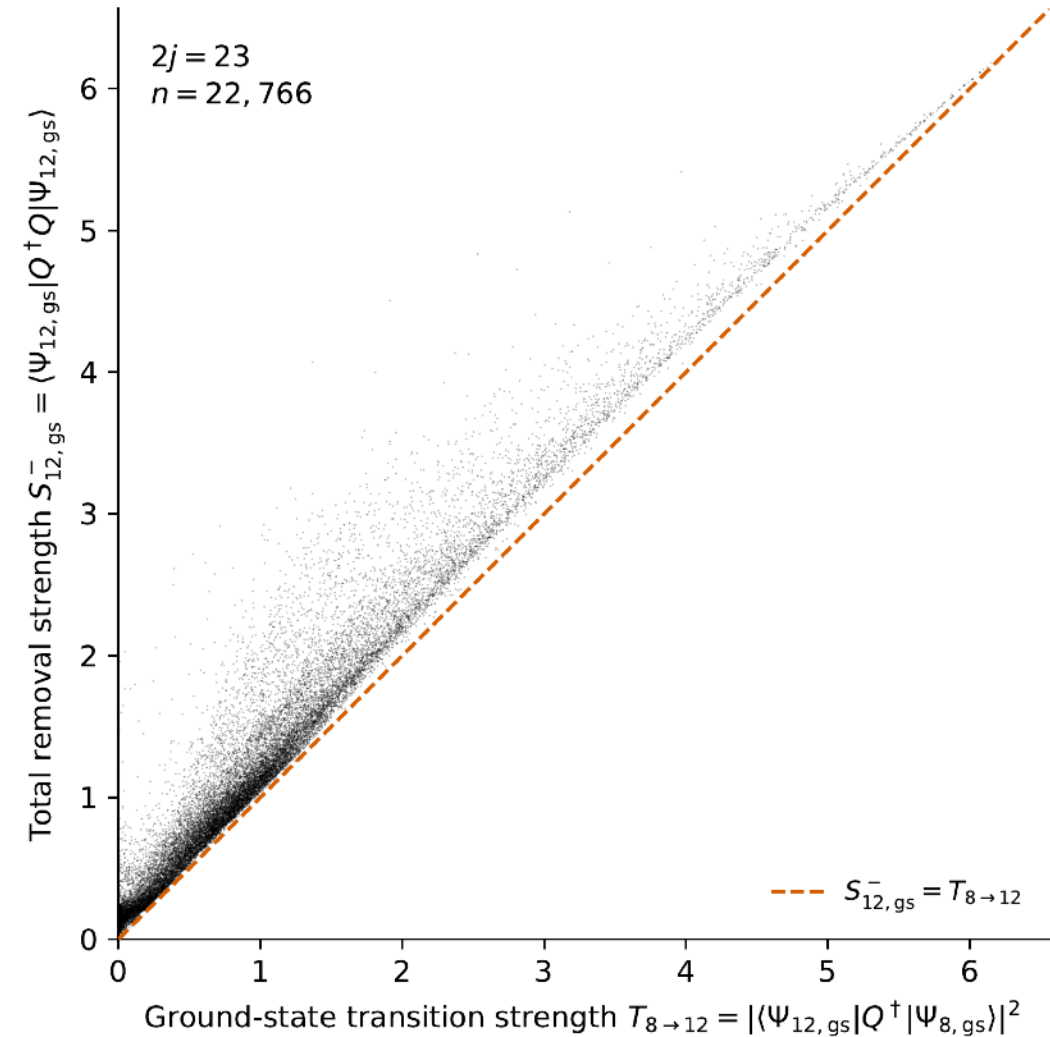
Several $J = 0$ four-particle states exist, so the quartet is defined **per realization** from the 4-particle ground state, $Q_n^\dagger |0\rangle \equiv |\varphi_n(4)\rangle$, using the **same interaction** $\{V_L\}$ for all particle numbers and keeping the realizations with $J_{\text{gs}} = 0$ for every N .

Bosons $\ell = 6$: remove a quartet from $N = 12$



Squared overlap of $Q_n |\varphi_n(12)\rangle$ with $|\varphi_n(8)\rangle$ against its norm; 37.8% of realizations ($O_n \equiv Q_n$ in the figure).

Fermions $j = 23/2$: remove a quartet from $N = 12$



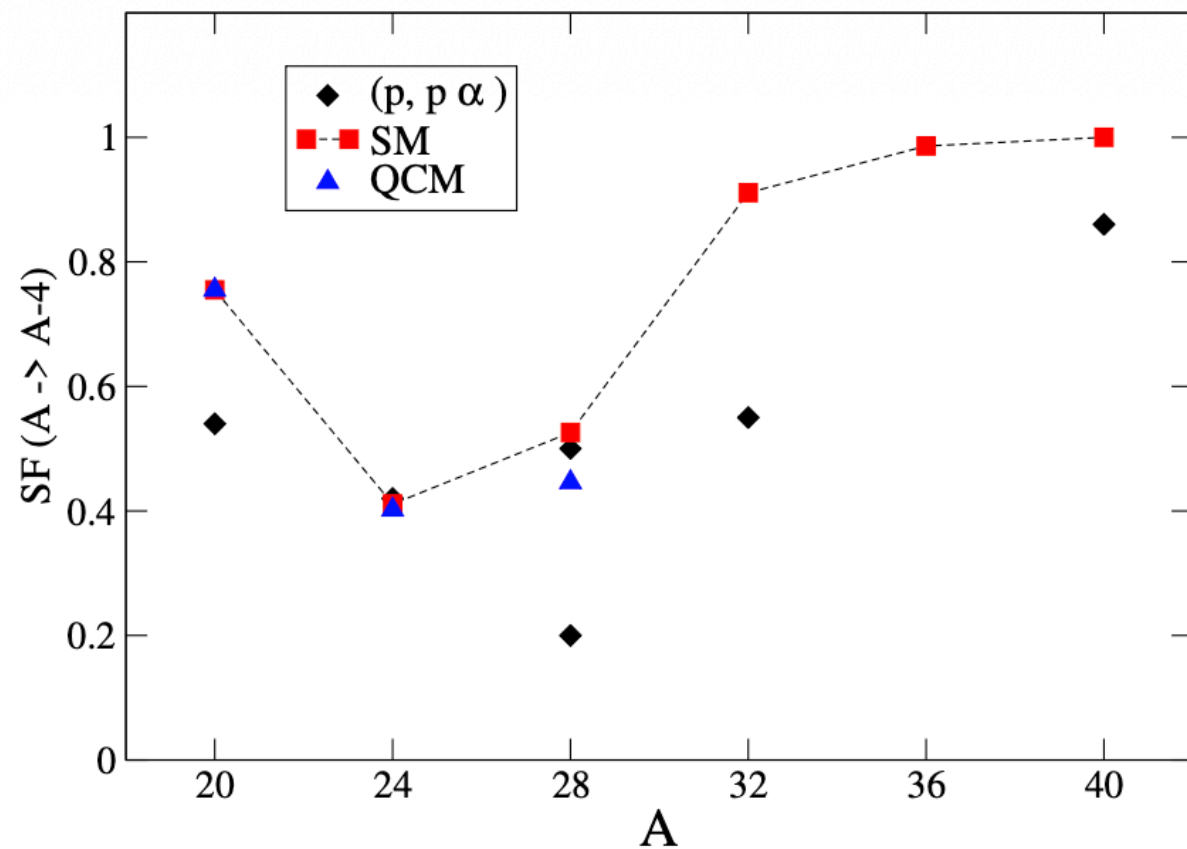
$T_{8 \rightarrow 12} = |\langle \Psi_{12} | Q^\dagger | \Psi_8 \rangle|^2$ against $S_{12}^- = \langle \Psi_{12} | Q^\dagger Q | \Psi_{12} \rangle$; 22.8% of realizations.

Points lie along the diagonal in both cases: removing a quartet from the ground state leads almost entirely to the ground state of the smaller system. $|\varphi_n(8)\rangle \propto (Q_n^\dagger)^2 |0\rangle$, $|\varphi_n(12)\rangle \propto (Q_n^\dagger)^3 |0\rangle$

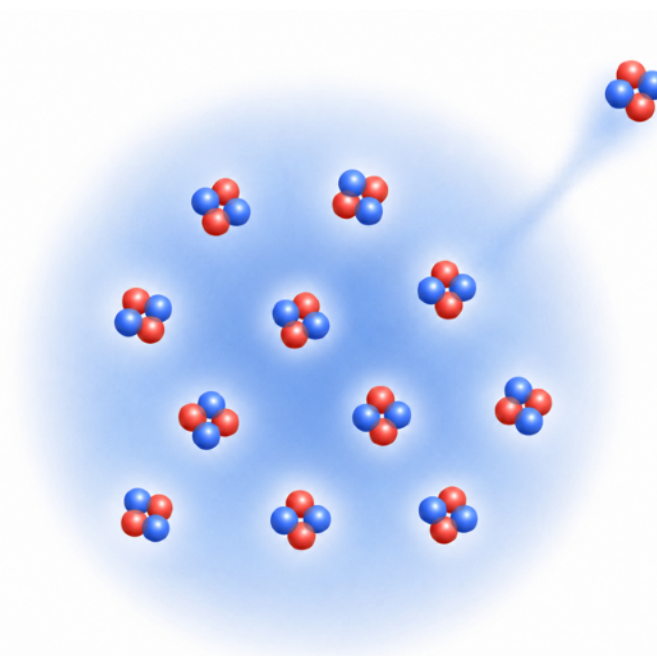
Quartet condensation and alphas

$$Q^\dagger(x, y) = x [P_{T=1}^\dagger P_{T=1}^\dagger]_{T=0} + y [P_{T=0}^\dagger P_{T=0}^\dagger]_{T=0} \quad x, y : \text{variational amplitudes.}$$

$$|\Psi_{\text{QCM}}\rangle = (Q^\dagger(x, y))^{n_q} |0\rangle$$



	$E_{\text{corr}}(SM)$	$E_{\text{corr}}(QCM)$	$\langle SM QCM \rangle$
^{20}Ne	24.77	24.77	1
^{24}Mg	55.70	53.04 (4.77%)	0.85
^{28}Si	88.75	86.52 (2.52%)	0.86
^{32}S	122.51	122.02 (0.40%)	0.98



Spectroscopic factors for the α transfer between the ground states of sd-shell nuclei: experiment [1], SM [2] and quartet condensate state [3].

[1] T. A. Carey, et al. Phys. Rev. C 23, 576(R) (1981), Phys. Rev. C 29, 1273 (1984)

[2] K. Kravvaris and A. Volya, Phys. Rev. C 100, 034321 (2019).

[3] N. Sandulescu, M. Sambataro, and A. Volya, EPJ Web Conf. 292, 1003 (2024).

From random quartets to a clustering

Random two-body ensemble

- $J = 0$ ground states are condensates of one repeated four-body cluster
- no geometry, no realistic force, no continuum
- the Pauli principle only weakens the coherence

α clustering in nuclei

- the quartet becomes the α : discussion of SU(3), center of mass
- quartet correlations vs α spectroscopic factors
- surface, thresholds effects.
- Pauli principle: non-orthogonality, norm kernel
- continuum: decay, reactions, superradiance



Configuration interaction with translationally invariant cluster channels

Our approach: K. Kravvaris and A. Volya, PRL 119, 062501 (2017); J. Phys. Conf. Ser. 863, 012016 (2017); PRC 100, 034321 (2019). A. Volya and Yu. M. Tchuvil'sky, PRC 91, 044319 (2015);

Related approaches: P. Navrátil and S. Quaglioni, PRC 83, 044609 (2011); A. C. Dreyfuss et al., PRC 102, 044608 (2020); T. Otsuka et al., Nat. Commun. 13, 2234 (2022).

Earlier: Smirnov and Chlebowska (1961); Rotter (1968); Hecht (1977); Varga, Lovas, Liotta (1992); Hess, Algora, Cseh, Draayer (2004); Navrátil (2004); Betan and Nazarewicz (2012).

Center of mass in the shell model

Translational invariance

$$H = \frac{\mathbf{P}^2}{2MA} + H_{\text{int}}, \quad H(\omega_0) = H + \frac{AM\omega_0^2}{2}\mathbf{R}^2 = H_{\text{cm}}(\omega_0) + H_{\text{int}}$$

Nothing intrinsic changes, and every eigenstate factorizes:

$$\Psi_{n\ell m} = \phi_{n\ell m}(\mathbf{R}) \Psi', \quad E = \hbar\omega_0\left(N_{\text{cm}} + \frac{3}{2}\right) + E', \quad N_{\text{cm}} = 2n + \ell$$

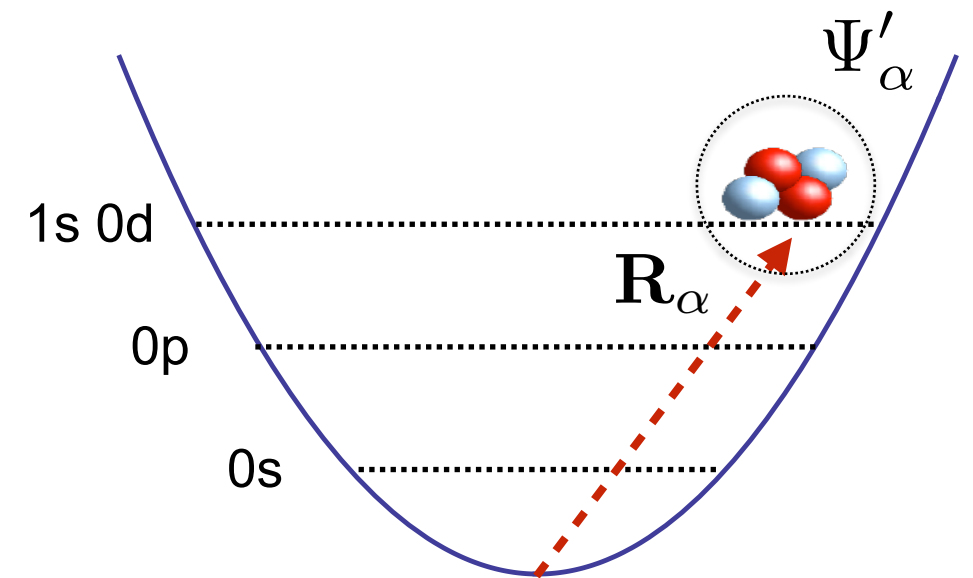
Glöckner–Lawson: physical shell-model states have $N_{\text{cm}} = 0$,

$\Psi_D = \phi_{000}(\mathbf{R}_D) \Psi'_D$; the rest are CM excitations.

The CM is an oscillator we can operate on

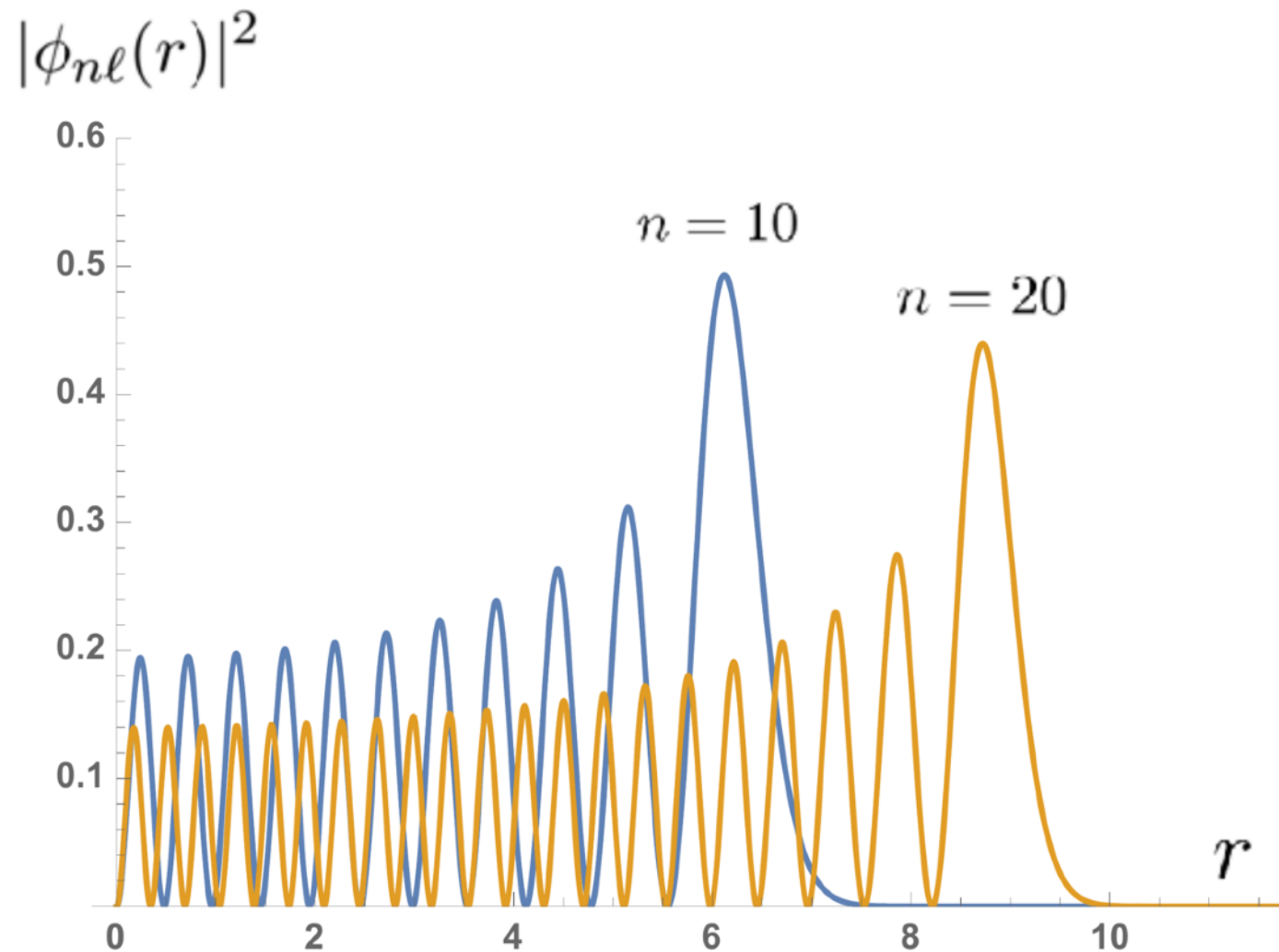
$$\mathbf{R} = \sqrt{\frac{\hbar}{2AM\omega_0}} (\mathcal{B}^\dagger + \mathcal{B}), \quad \Psi_{n+1,\ell m} \propto \mathcal{B}^\dagger \cdot \mathcal{B}^\dagger \Psi_{n\ell m}$$

- $\mathcal{B}^\dagger \cdot \mathcal{B}^\dagger$ adds two CM quanta, $\mathcal{B}^\dagger \times \mathcal{B}$ is the CM angular momentum, $\mathbf{D} = \sqrt{4\pi/3} \mathbf{R}$ the dipole operator
- CM quanta of the whole nucleus, or of a **subsystem**, are controlled exactly inside the shell-model space



CM excitations provide a handle for treating clusters.

More CM quanta, larger distance



CM density $|\phi_{n\ell}(r)|^2$ for $n = 10$ and $n = 20$ quanta: the weight piles up at the classical turning point.

$$\langle \mathbf{R}^2 \rangle = \frac{\hbar}{M\omega_0} \left(N_{\text{cm}} + \frac{3}{2} \right)$$

$$|\phi_{n\ell}(r)|^2 \sim \frac{1}{\sqrt{\langle \mathbf{R}^2 \rangle - r^2}}$$

- The boosted state sits at $r \approx \sqrt{\langle \mathbf{R}^2 \rangle} \propto \sqrt{N_{\text{cm}}}$
- Boosting the CM of an α , $\phi_{n\ell}(\mathbf{R}_\alpha)\Psi'_\alpha$, places it at the surface or in the decay channel
- The four nucleons then spread over many oscillator shells: the shell-model basis must be large enough to hold them

The number of CM quanta is roughly related to distance.

The algebraic limit: $\alpha = (0s)^4$ boosted into the sd shell

Example: α emission with $\ell = 0$ from the sd shell

- 4 nucleons $\times$ 2 quanta = 8 oscillator quanta, conserved in the oscillator basis
- Take the α intrinsically as $(0s)^4$: no intrinsic quanta, all 8 quanta are CM motion
- Then the boosted α is a **single SU(3) (8,0) symmetry state**

$$\begin{aligned} \phi_{n\ell m}(1) \phi_{n\ell m}(2) \phi_{n\ell m}(3) \phi_{n\ell m}(4) &\longleftrightarrow \sum_{\eta} X_{n\ell}^{\eta} \Phi_{(8,0):\ell m}^{\eta} \\ &= \phi_{n'\ell'm'}(\mathbf{R}_{\alpha}) \Psi'_{\alpha} \end{aligned}$$

m-scheme state with $4 \times 2 = 8$ quanta $\leftrightarrow$ SU(3) symmetry state

= CM motion with 8 quanta $\times$ intrinsic state with 0 quanta

Cluster coefficients are analytic

$$X_{n\ell}^{\eta} \equiv \langle \Phi_{(n,0):\ell m}^{\eta} | \phi_{n\ell m}(\mathbf{R}_{\alpha}) \Psi'_{\alpha} \rangle = \sqrt{\frac{1}{4^n} \frac{n!}{\prod_i (n_i!)^{\alpha_i}} \frac{4!}{\prod_i \alpha_i!}}$$

$\eta = \{n_i^{\alpha_i}\}$: α_i nucleons in shell n_i , $n = \sum_i \alpha_i n_i$, $\sum_i \alpha_i = 4$. For $(sd)^4$:

$$X^2 = 1/26 \approx 0.038$$

Configuration content of a realistic ${}^4\text{He}$ (NCSM, $\hbar\omega = 20$ MeV) boosted by 8 quanta, $L = 0$

configuration	$N_{\max} = 0$	$N_{\max} = 4$
$(sd)^4$	0.038	0.035
$(p)(sd)^2(pf)$	0.308	0.282
$(p)^2(pf)^2$	0.103	0.094
$(p)^2(sd)(sdg)$	0.154	0.141
$(s)^2(sd)(sdgi)$	0.000	0.005
$(p)(sd)(pf)(sdg)$	0.000	0.009

$N_{\max} = 0$ is the $(0s)^4$ limit; the realistic α barely moves the numbers.

In the algebraic limit the boosted α is one SU(3) (8,0) operator with analytic cluster coefficients, and only 4% of it lives in the sd shell.

Beyond the limit: the effective α operator in the sd shell

Four $L = S = T = 0$ four-nucleon operators exist in the sd shell, $(\lambda, \mu) = (8,0), (4,2), (0,4), (2,0)$. A realistic α at the frequency suited to sd nuclei ($\hbar\omega = 14$ MeV), projected onto the sd shell, is no longer the pure (8,0) quartet:

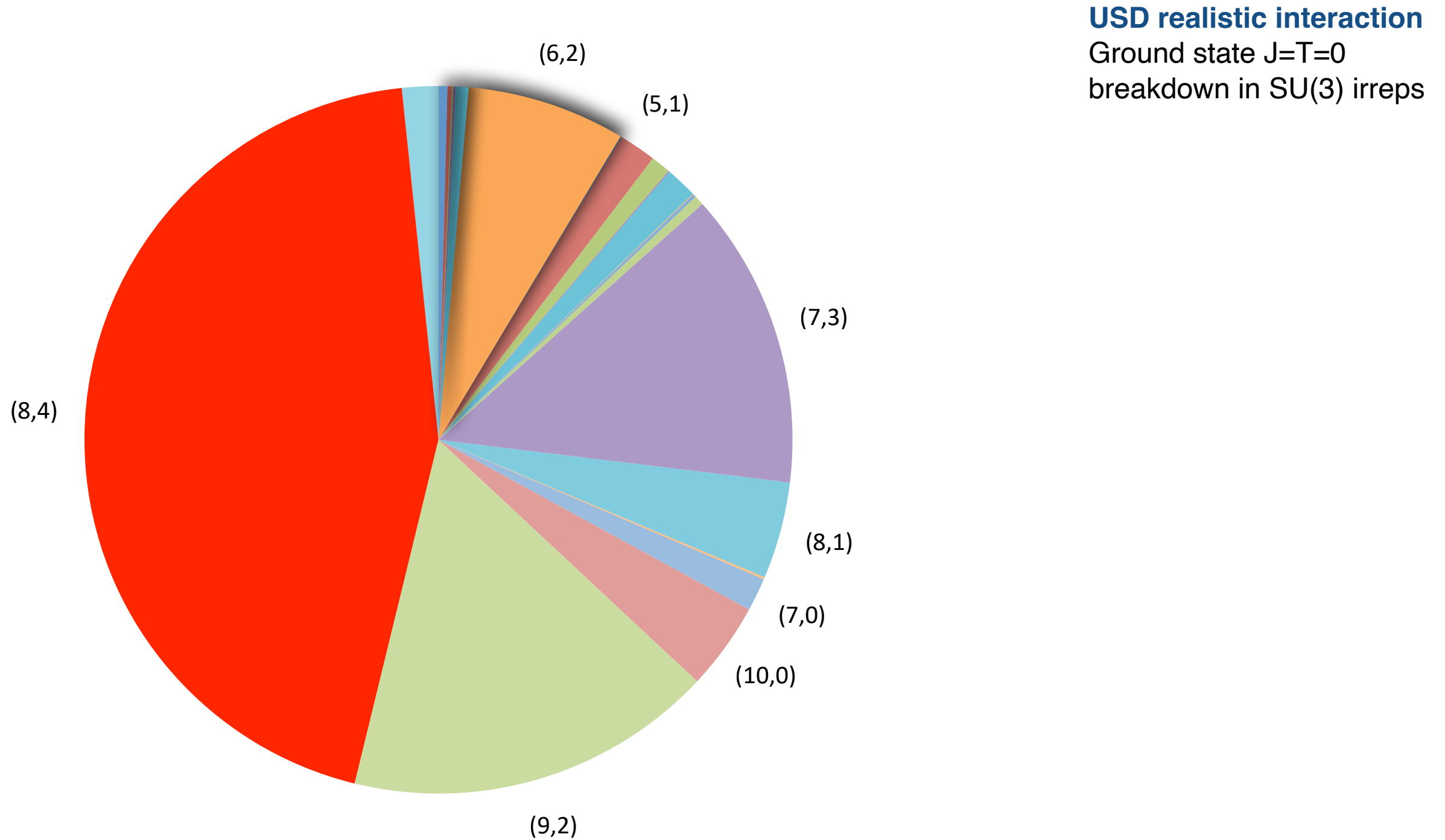
nodes n	X^2	(8,0)	(4,2)	(0,4)	(2,0)
4	0.0285	1.000	0	0	0
3	0.0070	0.562	0.438	0	0
2	0.0017	0.550	0.045	0.336	0.064
1	0.0002	0.069	0.736	0.013	0.147
0	0.0001	0.069	0.261	0.099	0.038

SU(3) decomposition of the α -cluster basis channels with n nodes of relative motion; X^2 is the total norm of the operator in the sd space. Algebraic limit: only $n = 4$, pure (8,0), $X^2 = 1/26$.

- In light nuclei SU(3) symmetry organizes both clustering and deformation
- Norm kernel and Pauli blocking shape the operator; for real alpha the effective α operator departs from the (8,0) limit
- The small components matter for hindered α transitions

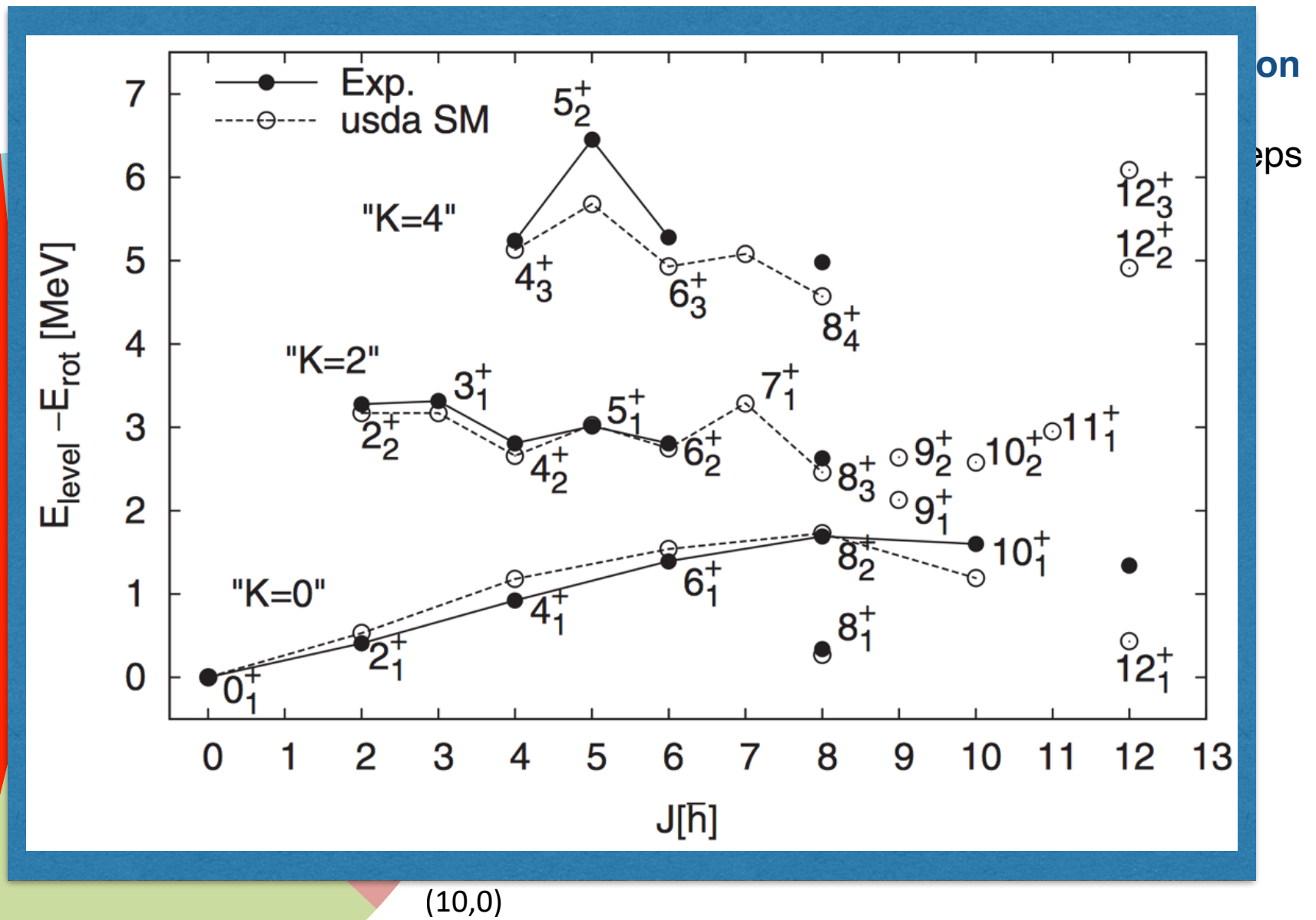
The α in the sd shell is mostly the (8,0) quartet, but not only: the effective operator carries (4,2), (0,4) and (2,0) admixtures.

Classic Example: ^{24}Mg , 8 nucleons in sd-shell



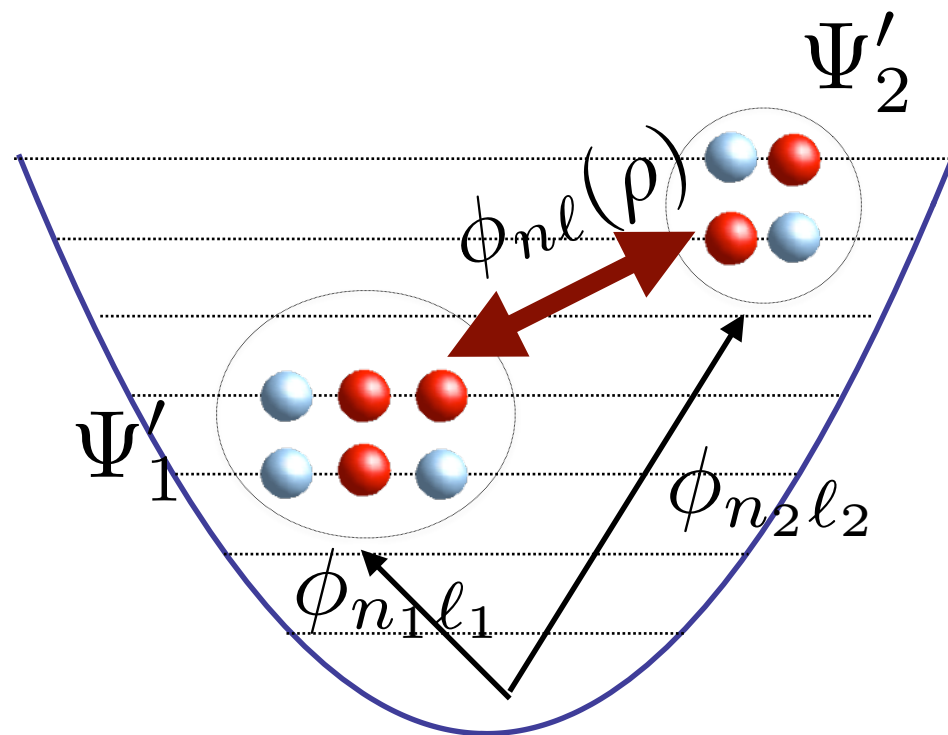
Classic Example: ^{24}Mg , 8 nucleons in sd-shell

(8,4)



Clustering reaction basis channel

(basis states for clustering)



Full Antisymmetrization

$$\Phi_{n\ell m} = \mathcal{A} \left\{ \phi_{000}(\mathbf{R}) \phi_{n\ell m}(\boldsymbol{\rho}) \Psi'^{(1)} \Psi'^{(2)} \right\}$$

Overall CM

Relative
motion

Intrinsic WF
of nucleus 1

Intrinsic WF
of nucleus 1

$$\Psi = \phi_{000}(\mathbf{R}) \Psi'$$

Boost

$$\Psi_{n\ell m} = \phi_{n\ell m}(\mathbf{R}) \Psi'$$

CM-Recouple

Resonating group method

$$\mathcal{F} = +\chi_1 \text{ (diagram)} + \chi_2 \text{ (diagram)} + \chi_3 \text{ (diagram)} + \dots$$

$$\mathcal{F}_\ell(\rho) = \sum_n \chi_n \Phi_{n\ell}$$

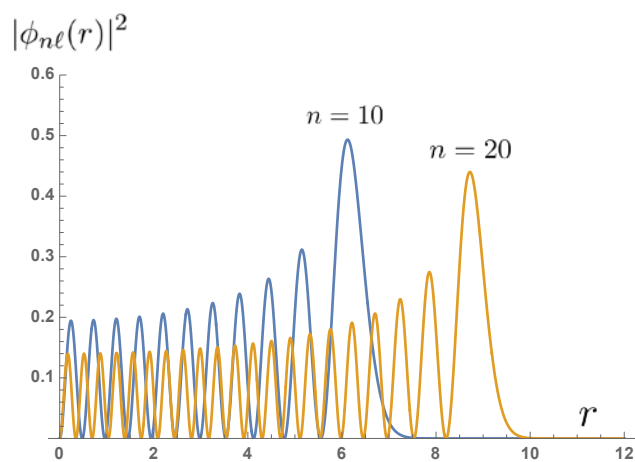
$$\sum_n \mathcal{H}_{nn'}^{(\ell)} \chi_{n'} = E \sum_n \mathcal{N}_{nn'}^{(\ell)} \chi_n$$

$$\mathcal{H}_{nn'}^{(\ell)} = \langle \Phi_{n\ell} | H | \Phi_{n'\ell} \rangle \quad \mathcal{N}_{nn'}^{(\ell)} = \langle \Phi_{n\ell} | \Phi_{n'\ell} \rangle$$

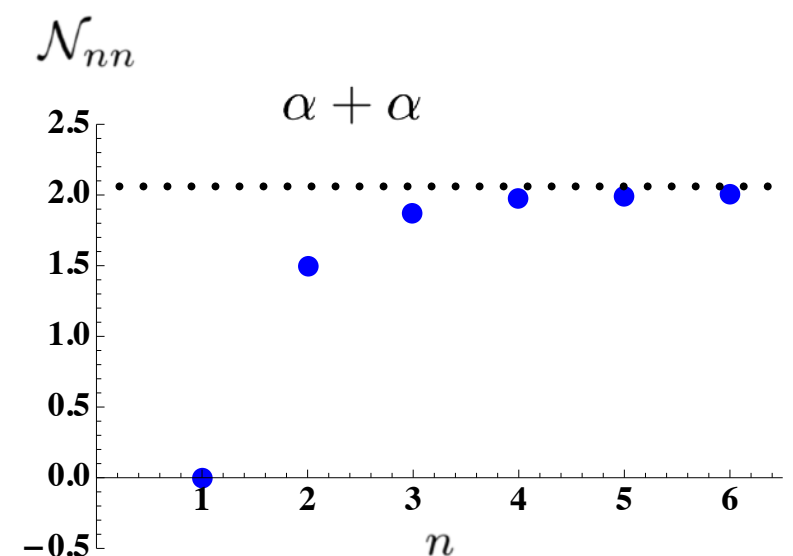
Resonating group method and reactions

$$\sum_n \mathcal{H}_{nn'}^{(\ell)} \chi_{n'} = E \sum_n \mathcal{N}_{nn'}^{(\ell)} \chi_{n'}$$

$$\begin{pmatrix} \mathcal{H}_{00} & \dots & \mathcal{H}_{0n} & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \vdots & \vdots \\ \mathcal{H}_{n0} & \dots & \mathcal{H}_{nn} & T_{nn+1} & 0 & \vdots \\ 0 & 0 & T_{n+1n} & T_{n+1n+1} & T_{n+1n+2} & 0 \\ 0 & \dots & 0 & T_{n+2n+1} & T_{n+2n+2} & \ddots \\ 0 & \dots & \dots & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \end{pmatrix} = E \begin{pmatrix} \mathcal{N}_{00} & \dots & \mathcal{N}_{0n} & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \vdots & \vdots \\ \mathcal{N}_{n0} & \dots & \mathcal{N}_{nn} & 0 & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 & \ddots \end{pmatrix} \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \end{pmatrix}$$

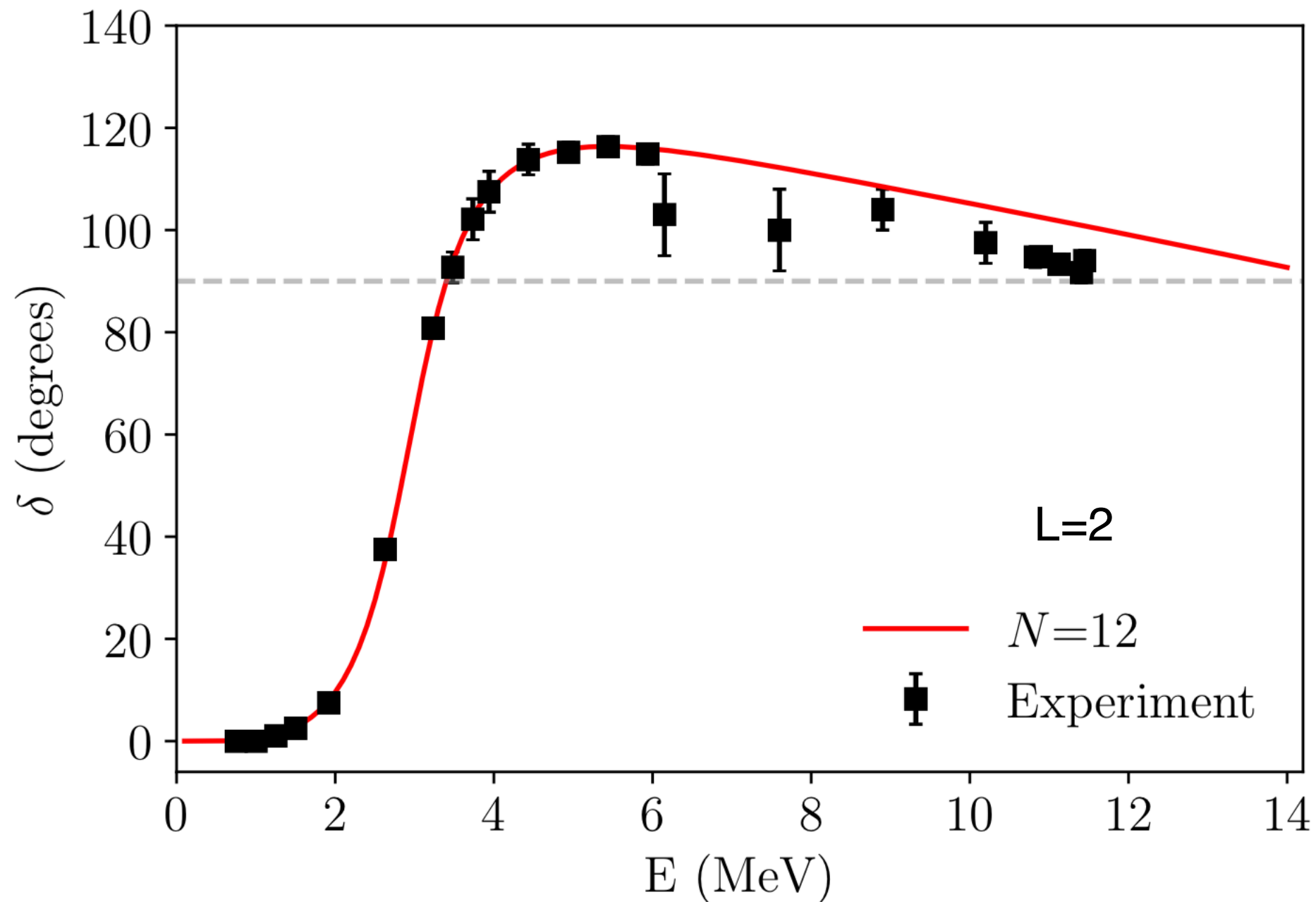


Asymptotic solution with phase shift



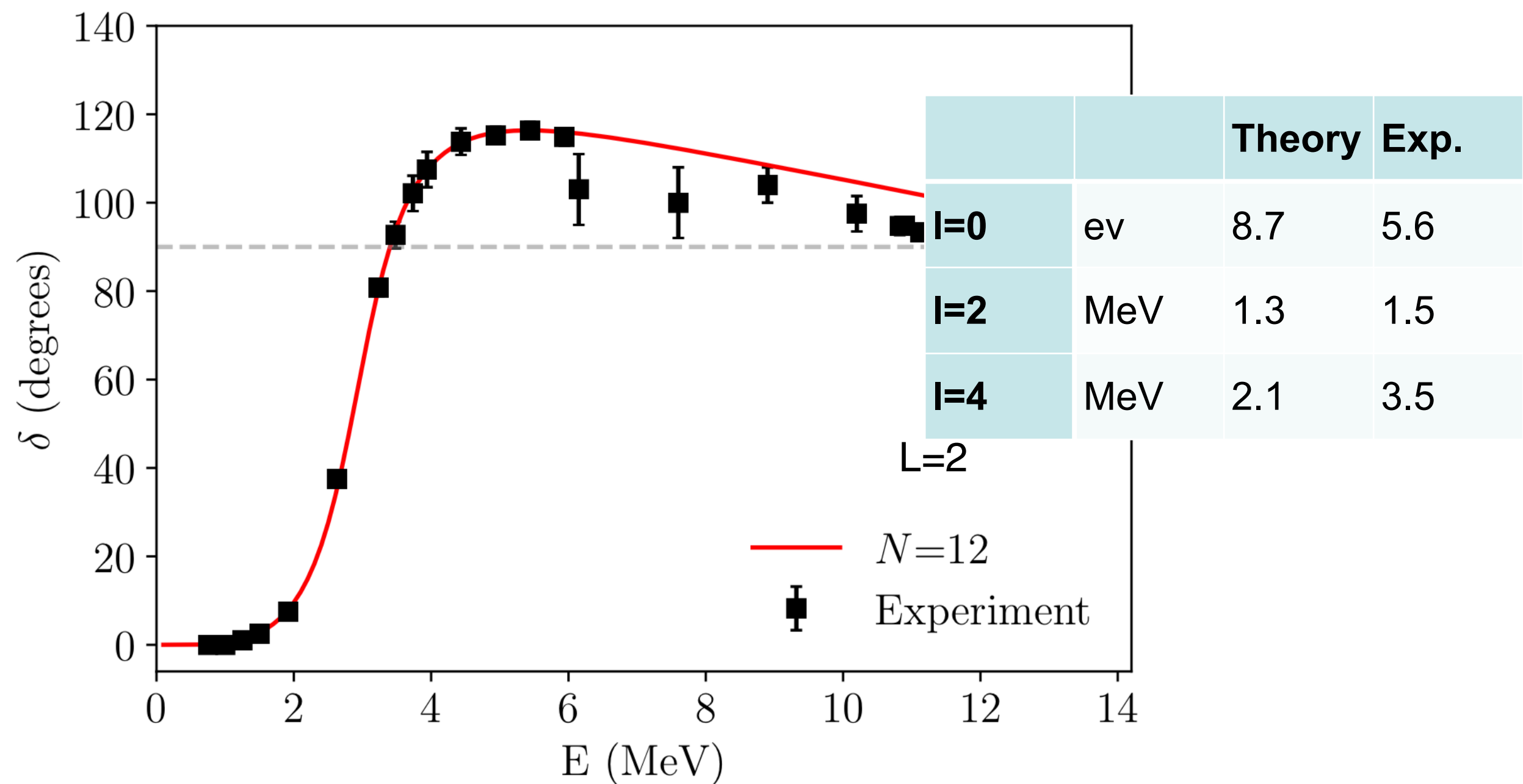
J-matrix (or HORSE) method: J. M. Bang, Annals of Physics **280**, 299 (2000)
Experimental data: Phys. Rev. 168, 1114 (1968); Nucl. Phys. **A287**, 317 (1977)

alpha+alpha scattering phase shifts



Experimental data from S. A. Afzal, A. A. Z. Ahmad, and S. Ali, Rev. Mod. Phys. 41, 247 (1969).

alpha+alpha scattering phase shifts

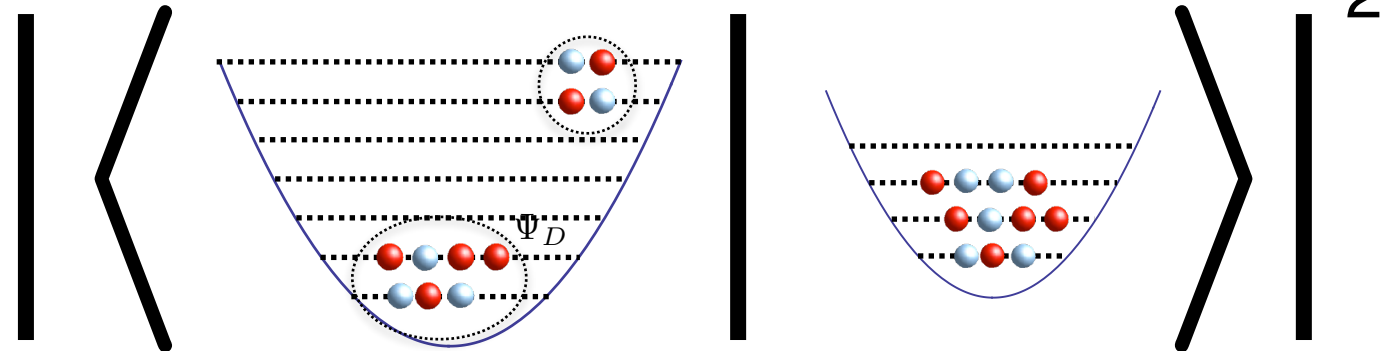


Experimental data from S. A. Afzal, A. A. Z. Ahmad, and S. Ali, Rev. Mod. Phys. 41, 247 (1969).

Cluster Spectroscopic Characteristics

Traditional spectroscopic factor

$$S_{n,\ell} \equiv \left| \langle \Psi^{(A)} | \Phi_{n\ell} \rangle \right|^2 =$$



$$\mathcal{N}_{nn'}^{(\ell)} = \langle \Phi_{n\ell} | \Phi_{n'\ell} \rangle \quad \text{Norm kernel}$$

$$|\Psi_{\nu}^{(\ell, \text{ocm})}\rangle = \sum_n \left(\frac{1}{\sqrt{\mathcal{N}^{(\ell)}}} \right)_{\nu n} |\Phi_{n\ell}\rangle$$

Orthonormalized basis channels

$$S_{\ell}^{(\text{ocm})} \equiv \sum_{\nu} \left| \langle \Psi^{(A)} | \Psi_{\nu}^{(\ell, \text{ocm})} \rangle \right|^2$$

Sum of all new SF from all parent states to a given final state equals to the number of channels

R. Id Betan and W. Nazarewicz Phys. Rev. C 86, 034338 (2012)

S. G. Kadenskaya, S. D. Kurgalina, and Yu. M. Tchuvil'sky Phys. Part. Nucl., 38, 699–742 (2007).

R. Lovas et al. Phys. Rep. 294, No. 5 (1998) 265 – 362.

T. Fliessbach and H. J. Mang, Nucl. Phys. A **263**, 75–85 (1976).

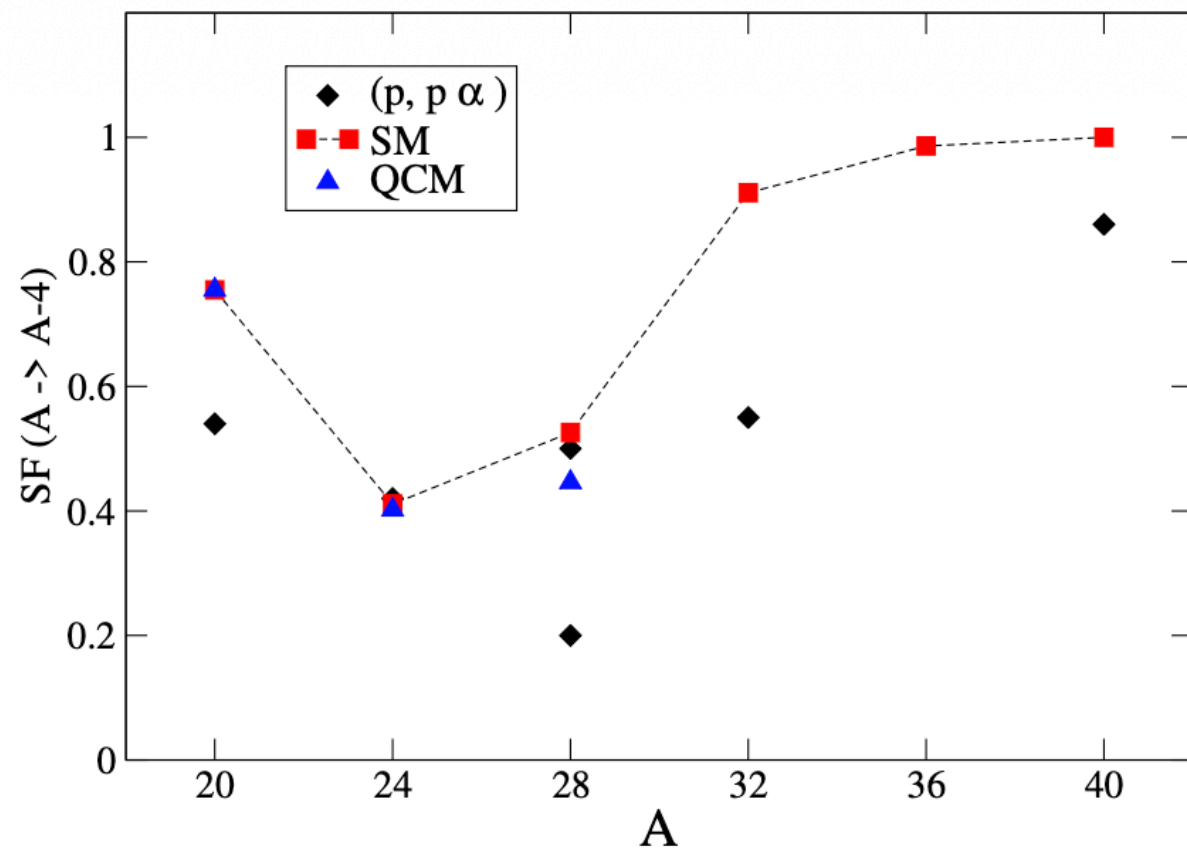
H. Feschbach et al. Ann. Phys. 41 (1967) 230 – 286

A. Volya and Y. M. Tchuvil'sky, Phys.Rev.C 91, 044319 (2015);

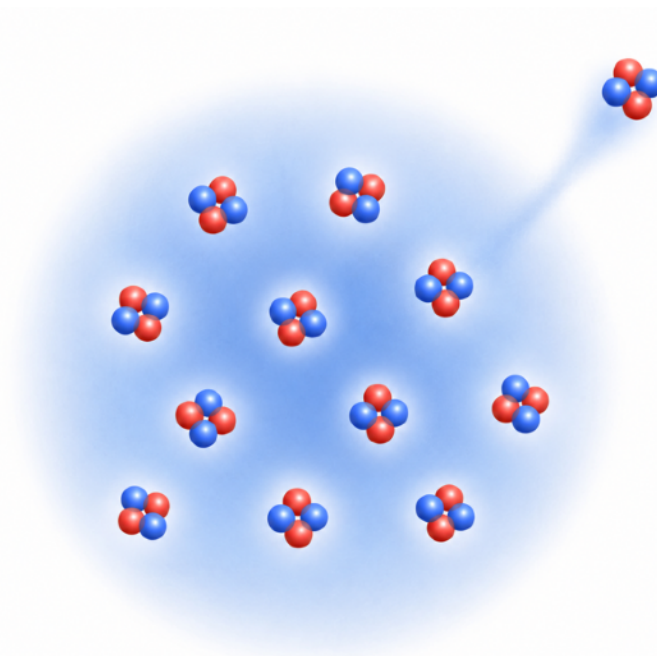
Quartet condensation and alphas

$$Q^\dagger(x, y) = x [P_{T=1}^\dagger P_{T=1}^\dagger]_{T=0} + y [P_{T=0}^\dagger P_{T=0}^\dagger]_{T=0} \quad x, y : \text{variational amplitudes.}$$

$$|\Psi_{\text{QCM}}\rangle = (Q^\dagger(x, y))^{n_q} |0\rangle$$



	$E_{corr}(SM)$	$E_{corr}(QCM)$	$\langle SM QCM \rangle$
^{20}Ne	24.77	24.77	1
^{24}Mg	55.70	53.04 (4.77%)	0.85
^{28}Si	88.75	86.52 (2.52%)	0.86
^{32}S	122.51	122.02 (0.40%)	0.98



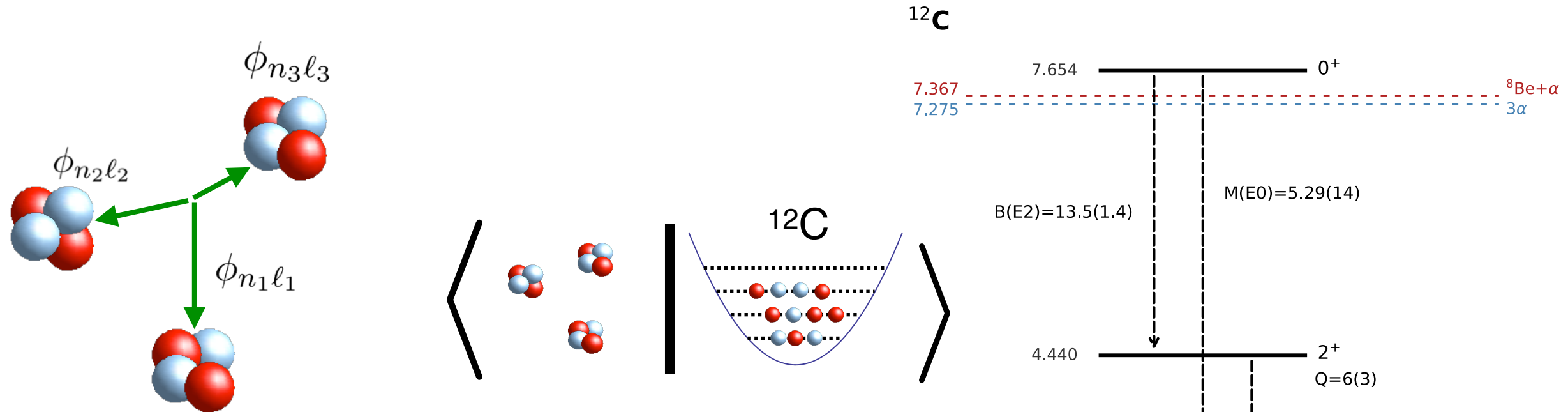
Spectroscopic factors for the α transfer between the ground states of sd-shell nuclei: experiment [1], SM [2] and quartet condensate state [3].

[1] T. A. Carey, et al. Phys. Rev. C 23, 576(R) (1981), Phys. Rev. C 29, 1273 (1984)

[2] K. Kravvaris and A. Volya, Phys. Rev. C 100, 034321 (2019).

[3] N. Sandulescu, M. Sambataro, and A. Volya, EPJ Web Conf. 292, 1003 (2024).

Ttriple-alpha RGM and Hoyle state in ^{12}C

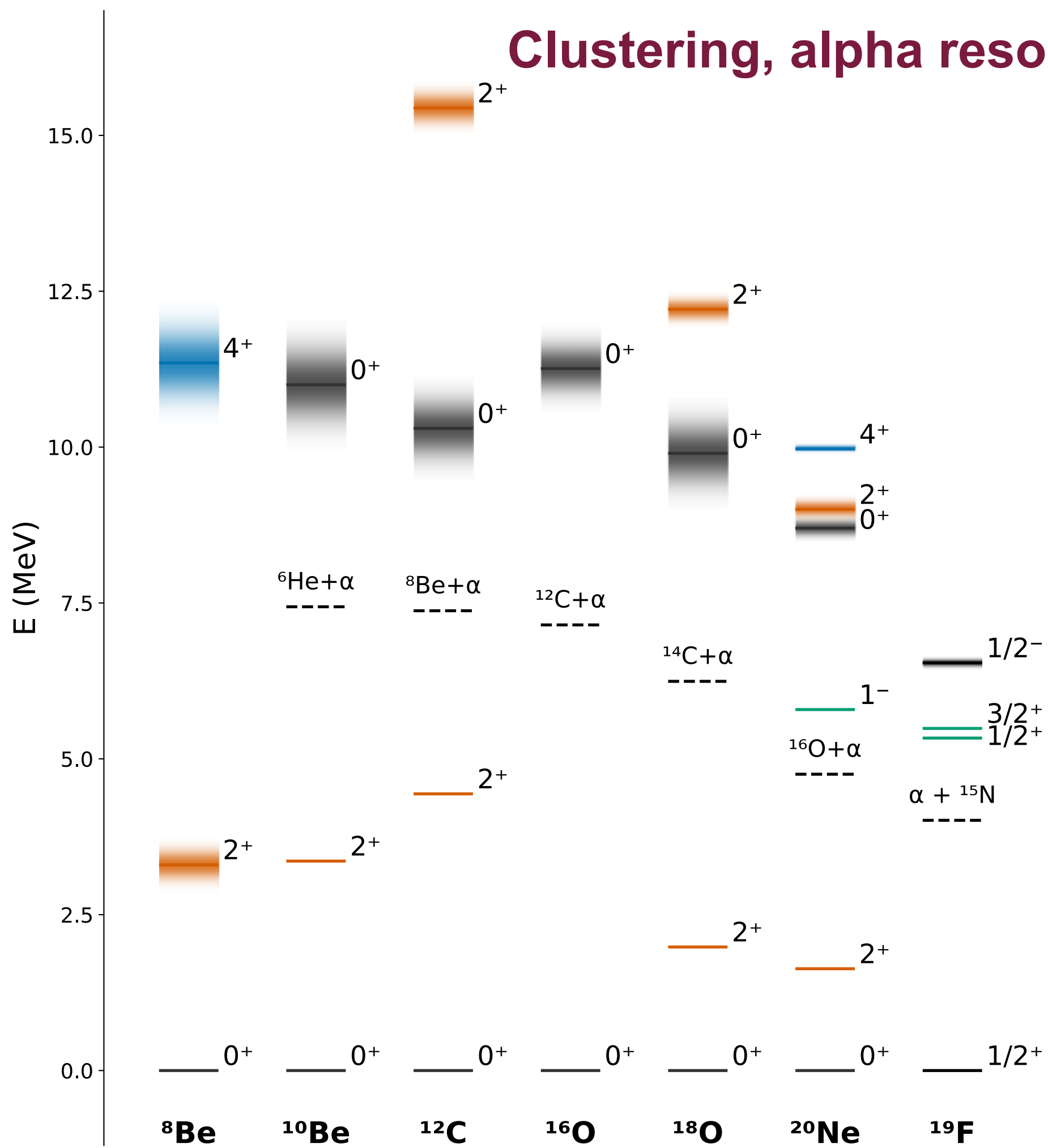


JISP16 interaction ($N_{\text{max}} = 12$, $\hbar\omega = 25$ MeV)

parent	channel	overlap
$^{12}\text{C}[4](0_1^+)$	$\alpha[0] + \alpha[0] + \alpha[0]$	0.841
$^{12}\text{C}[4](0_2^+)$	$\alpha[0] + \alpha[0] + \alpha[0]$	0.229

$$\left\langle \begin{array}{c} ^8\text{Be} + \alpha \\ \alpha + \alpha + \alpha \end{array} \right\rangle^2 = 0.89$$

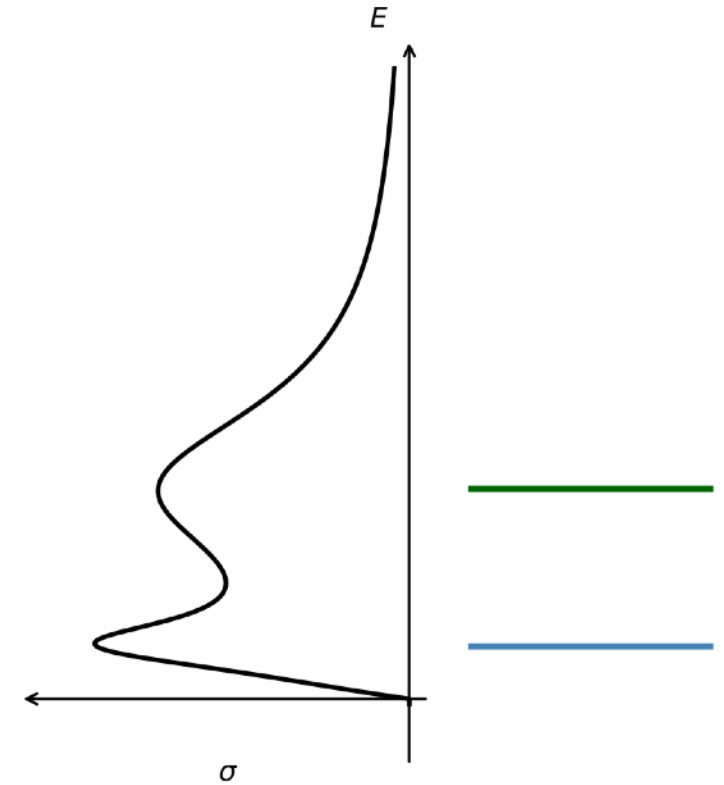
Clustering, alpha resonances



Wave function realignment Superradiance

$$H = \begin{pmatrix} \epsilon - \frac{i}{2}\Gamma & v \\ v & 0 \end{pmatrix} = H_0 - \frac{i\Gamma}{2} A^\dagger A$$

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Stationary system $\Gamma = 0$

Energies $H\Psi_{1,2} = E_{1,2}\Psi_{1,2}$ $E_{1,2} = \frac{1}{2} \left(\epsilon \pm \sqrt{\epsilon^2 + 4v^2} \right)$

Spectroscopic Factors $SF_{1,2} = \frac{1}{2} \left(1 \pm \frac{\epsilon}{\sqrt{\epsilon^2 + 4v^2}} \right) = \left| \langle \Psi_{1,2} | A \rangle \right|^2$

Observing superradiance

$$H = \begin{pmatrix} \epsilon - \frac{i}{2}\Gamma & v \\ v & 0 \end{pmatrix} = H_0 - \frac{i\Gamma}{2} A^\dagger A \quad A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Energies

$$\mathcal{E}_{1,2} = \frac{1}{2} \left(\epsilon - \frac{i}{2}\Gamma \pm \sqrt{\left(\epsilon - \frac{i}{2}\Gamma \right)^2 + 4v^2} \right)$$

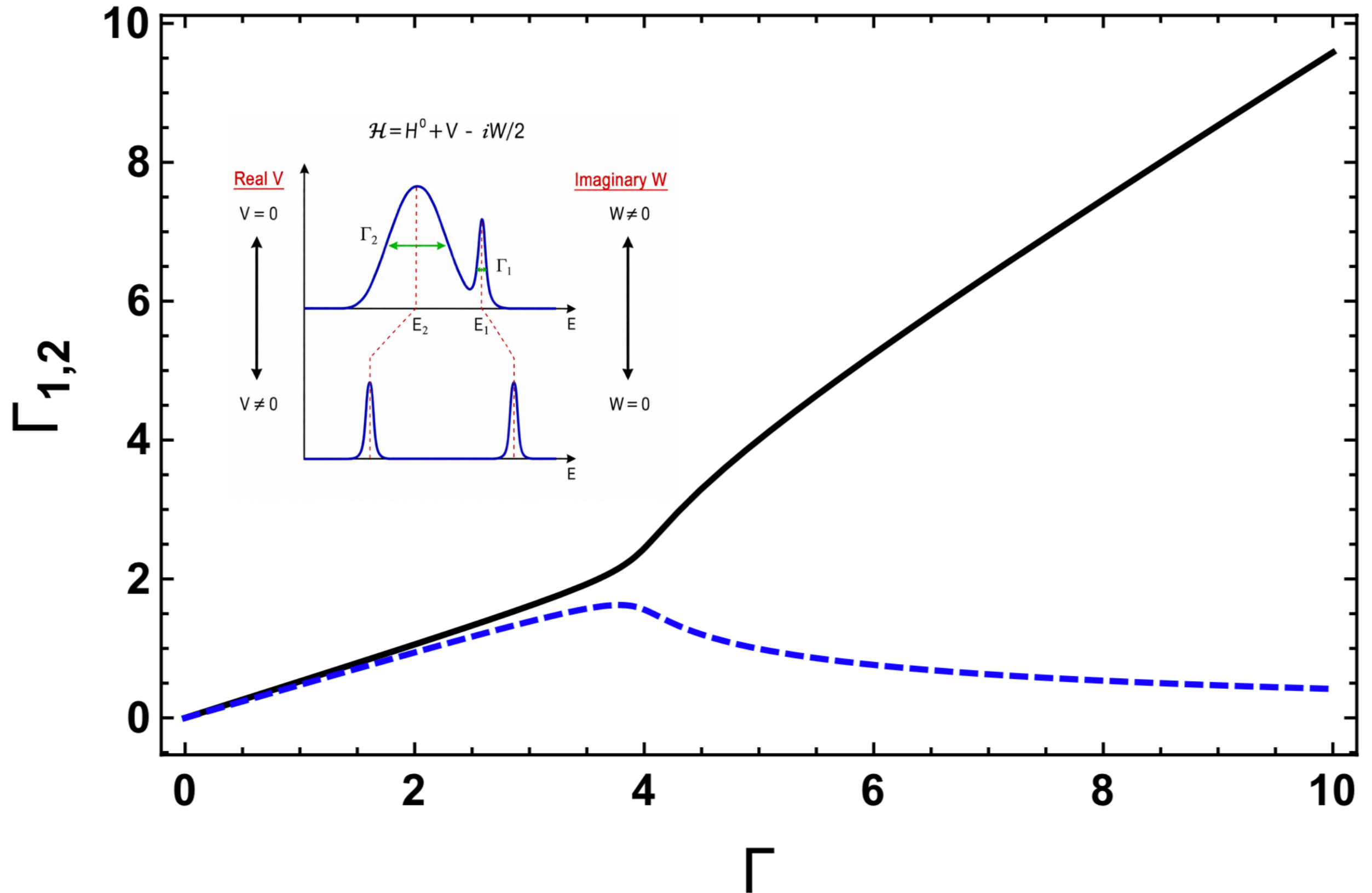
Width

$$\Gamma_{1,2} = -2 \operatorname{Im} (\mathcal{E}_{1,2})$$

Spectroscopic Factors

$$\text{SF}_{1,2} = \Gamma_{1,2} / \Gamma$$

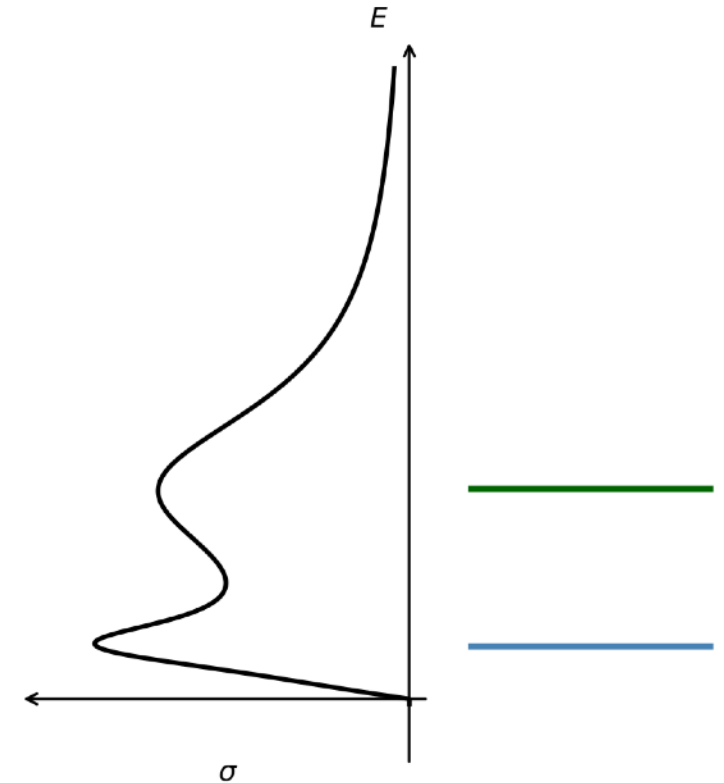
Observing superradiance



Wave function realignment Superradiance

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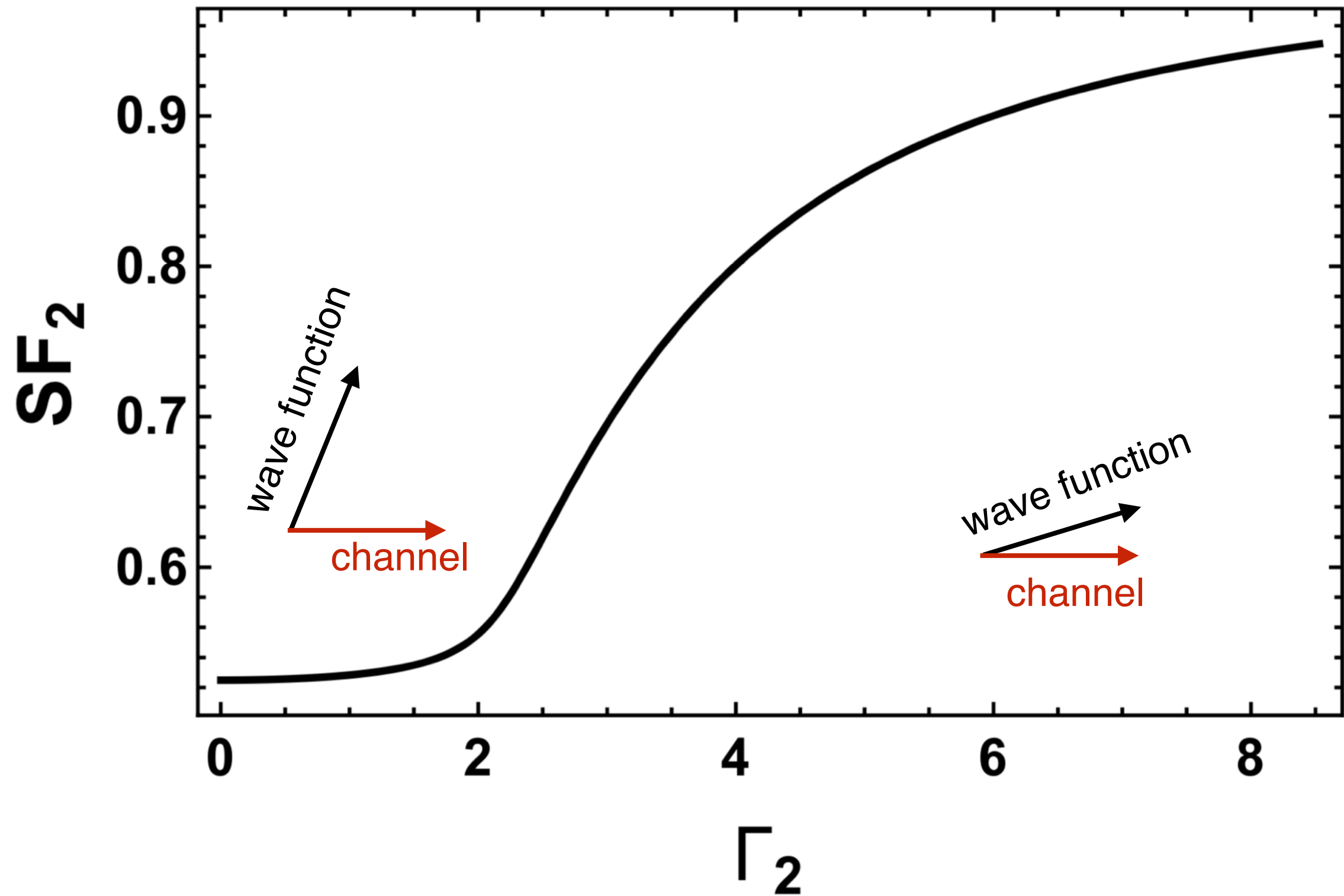
Decaying system $\Gamma \neq 0$

Energies $H\Psi_{1,2} = \mathcal{E}_{1,2}\Psi_{1,2}$

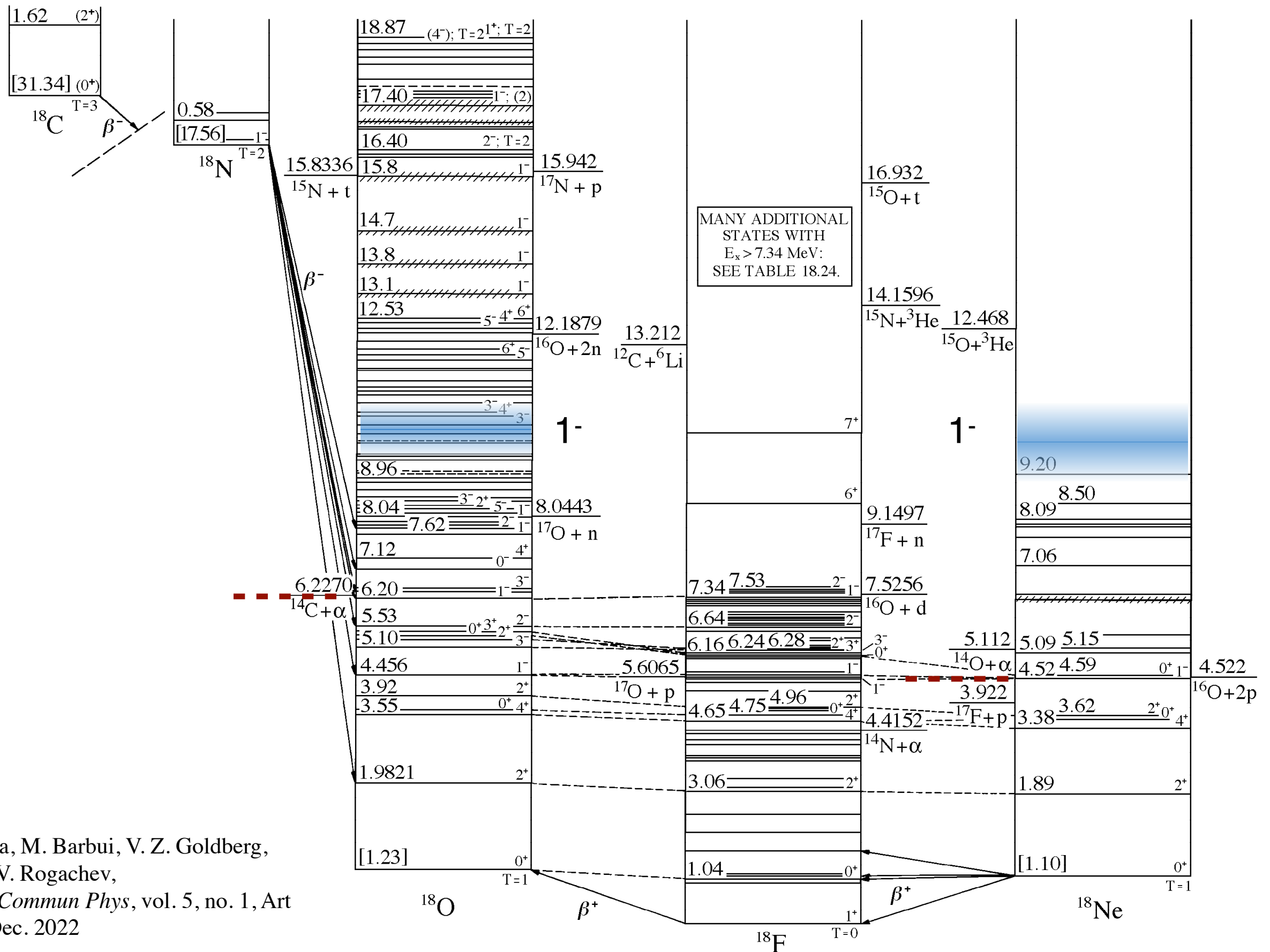
$$\mathcal{E}_{1,2} = \frac{1}{2} \left(\epsilon - \frac{i}{2}\Gamma \pm \sqrt{\left(\epsilon - \frac{i}{2}\Gamma \right)^2 + 4v^2} \right)$$

Spectroscopic Factors $SF_{1,2} = \Gamma_{1,2}/\Gamma = \left| \langle \Psi_{1,2} | A \rangle \right|^2$

Spectroscopic factor for superradiant state



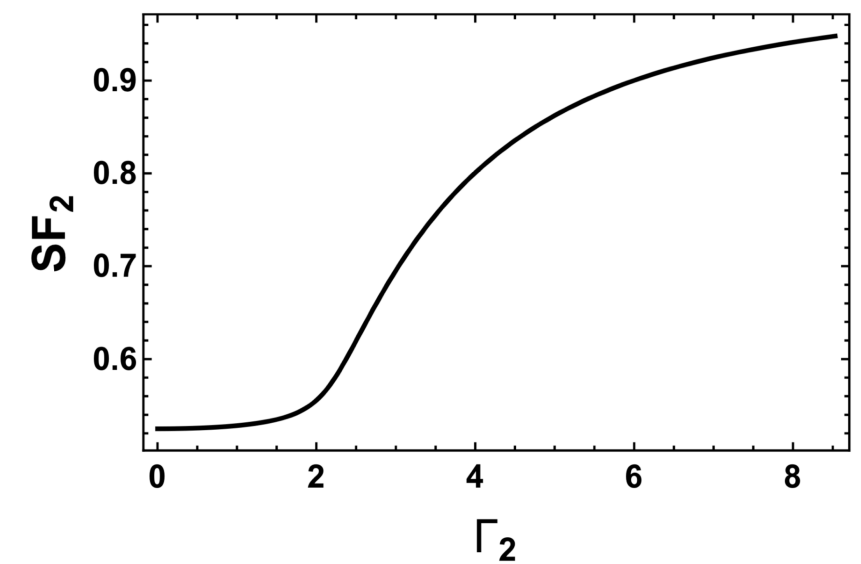
Isospin symmetry



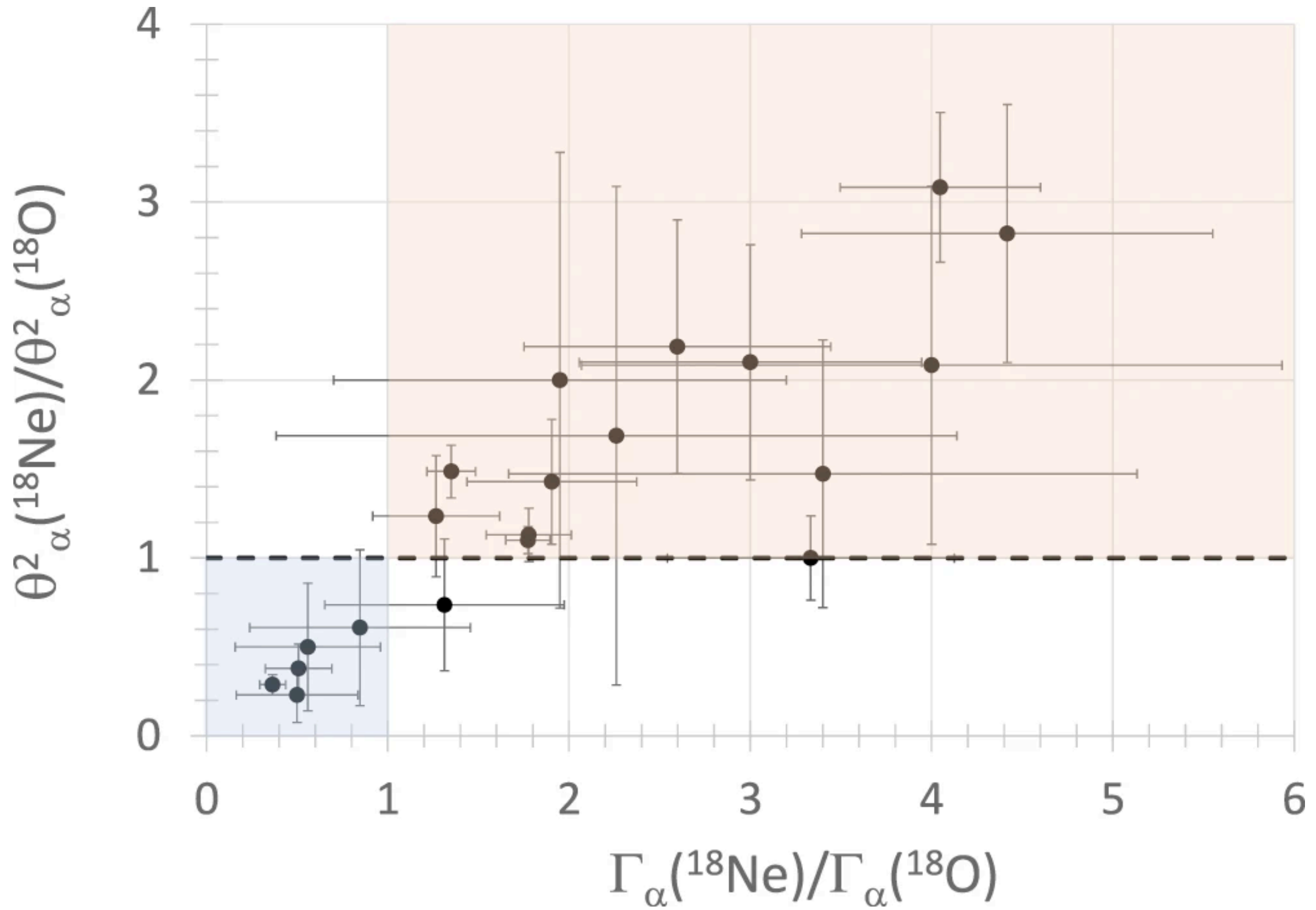
	^{18}Ne			^{18}O		
J	E (MeV)	Γ (keV)	SF	E (MeV)	Γ (keV)	SF
1-	9.08(1)	357	0.21(1)	9.19(2)	200	0.20(1)
1-	9.57(1)	1062	0.51(5)	9.76(2)	630	0.46(4)
1-	10.58(4)	416	0.15(5)	10.8(3)	630	0.29(4)
1-	13.730(2)	780	0.2(1)	14.3(3)	400	0.10(4)
2+	9.19(3)	265	0.21(2)	9.79(6)	90	0.10(3)
2+	10.94(6)	1302	0.52(3)	12.21(8)	1000	0.37(9)
2+	13.4 (2)	1755	0.45(8)	12.8(3)	4800	1.56(13)
2+	16.9(2)	1515	0.3(2)			
3-				8.29(6)	2.9	0.18(1)
3-	8.77(8)	419	1.0(4)	9.35(2)	110	0.48(13)
3-	11.0(1)	497	0.28(7)	11.95(1)	300	0.17(2)
3-	12.7(2)	2025	0.7(2)	12.98(4)	770	0.32(5)
3-	14.8(2)	3967	1.0(2)	14.0(2)	2100	0.7(1)
4+	8.16	31	0.8(3)	7.11*		
4+	13.3(3)	845	0.37(4)	13.46(2)	210	0.12(1)
4+	14.15(21)	375	0.14(10)	14.77(5)	680	0.28(2)
5-	11.31(4)	15	0.03(2)	11.63(1)	30	0.13(1)
5-	12.9(2)	532	0.48(12)	13.08(1)	120	0.17(1)
5-	13.79(8)	219	0.14(10)	14.1(1)	260	0.23(2)
5-	14.6(7)	521	0.27(20)	14.7(1)	220	0.16(6)

M. Barbui *et al.*, “ α -cluster structure of ^{18}Ne ,” *Phys. Rev. C*, vol. 106, no. 5, p. 054310, Nov. 2022, doi: [10.1103/PhysRevC.106.054310](https://doi.org/10.1103/PhysRevC.106.054310).

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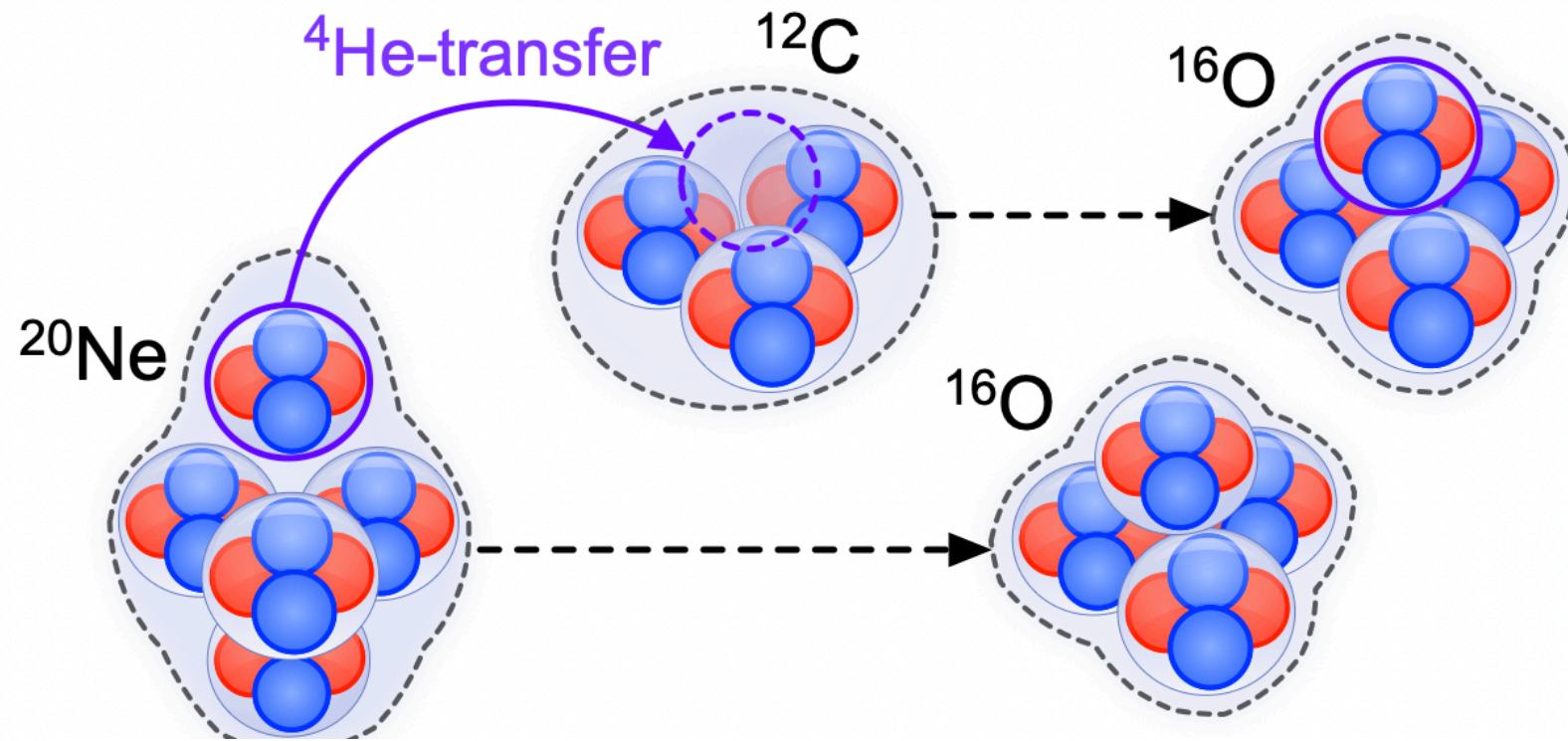
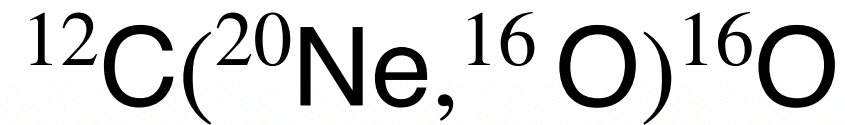


M. Barbui *et al.*, “ α -cluster structure of ^{18}Ne ,” *Phys. Rev. C*, vol. 106, no. 5, p. 054310, Nov. 2022, doi: [10.1103/PhysRevC.106.054310](https://doi.org/10.1103/PhysRevC.106.054310).



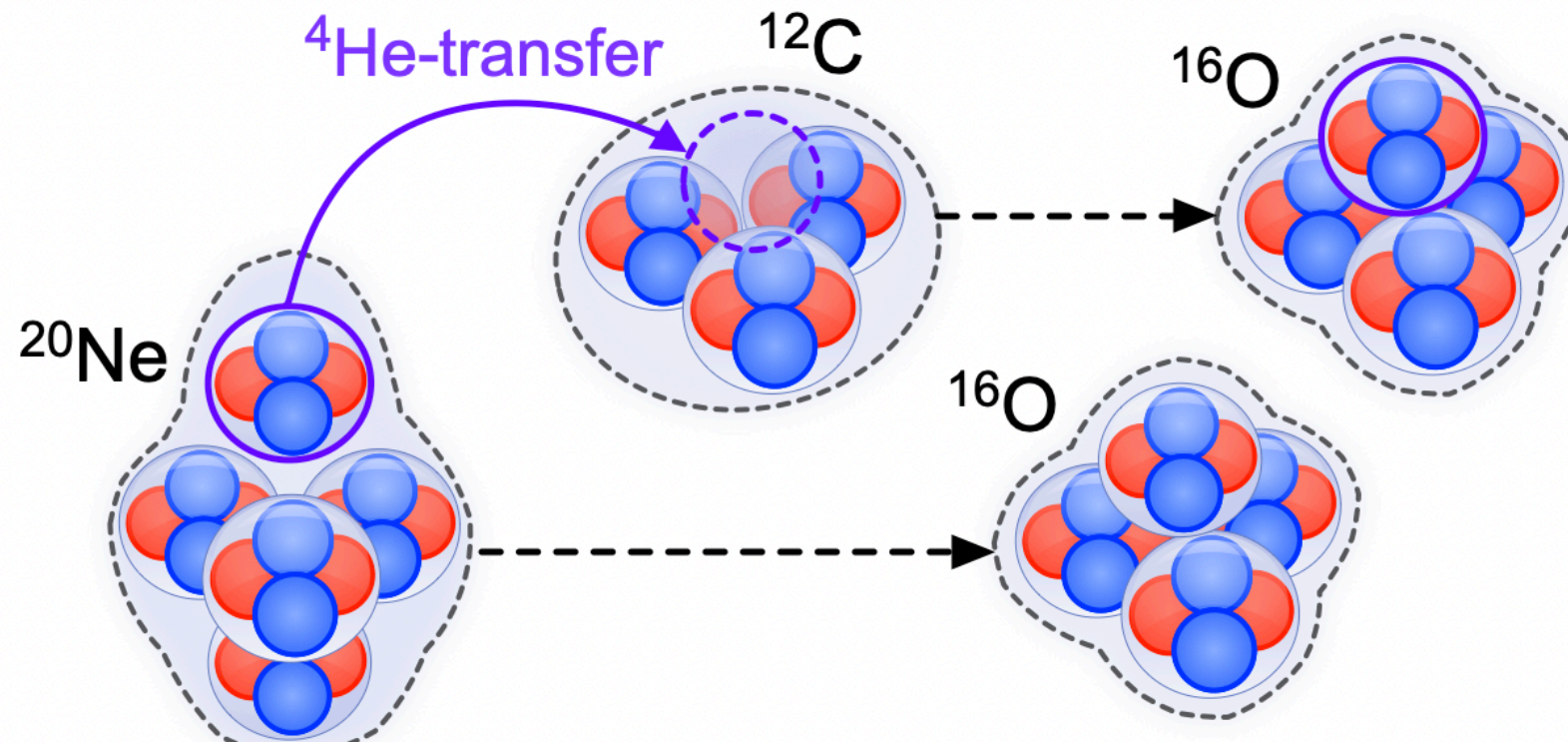
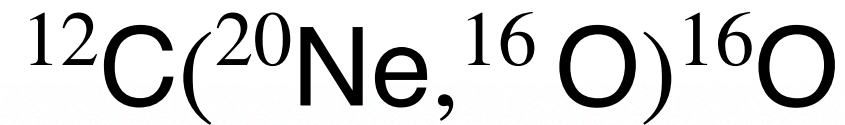
A. Volya, M. Barbui, V. Z. Goldberg, and G. V. Rogachev, “Superradiance in alpha clustered mirror nuclei,” *Nature Commun Phys*, vol. 5, no. 1, Art no. 1, Dec. 2022, doi: [10.1038/s42005-022-01105-9](https://doi.org/10.1038/s42005-022-01105-9).

Clustering in the ground states



	$C_{\alpha, ^{12}\text{C}}^{^{16}\text{O}}$ ($\text{fm}^{-1/2}$)	$C_{\alpha, ^{16}\text{O}}^{^{20}\text{Ne}}$ ($\text{fm}^{-1/2}$)	$[(C_{\alpha, ^{12}\text{C}}^{^{16}\text{O}})^2 (C_{\alpha, ^{16}\text{O}}^{^{20}\text{Ne}})^2]^{1/4}$ ($10^3 \text{ fm}^{-1/2}$)
Experiment			1.75(7)(9)
NLEFT	380(80)	3800(950)	1.20(25)
SA-NCSM	210(30)	3760(470)	0.89(14)
CN-CI	520(30)	4500(500)	1.53(13)
Extreme CM	640	5700	1.9

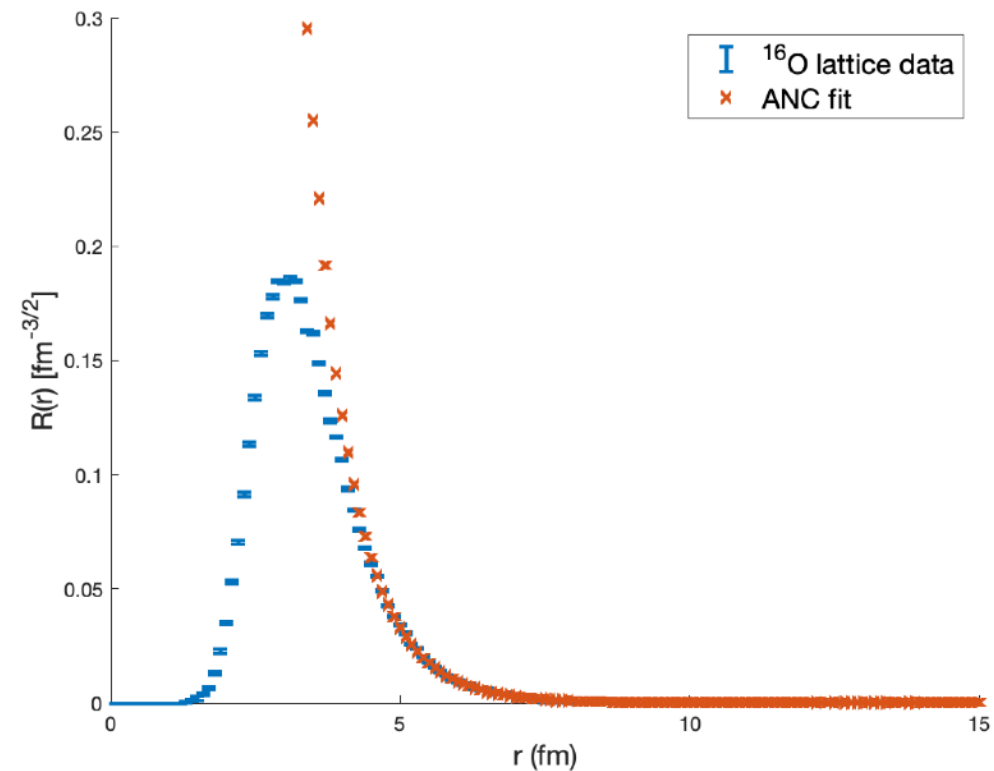
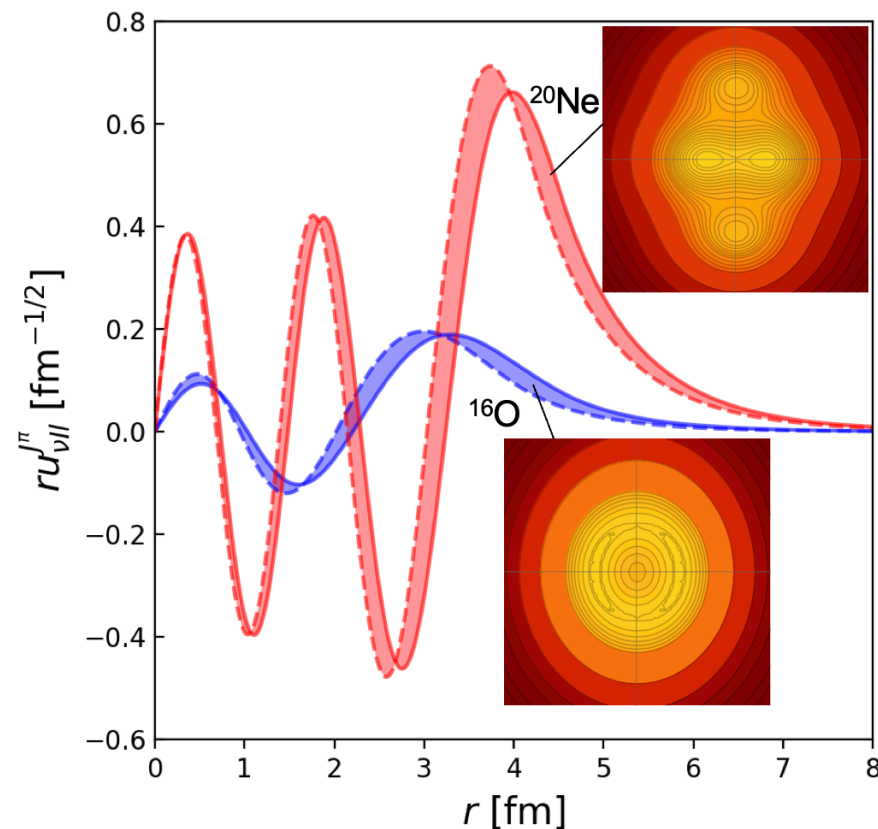
Clustering in the ground states



	$C_{\alpha,^{12}\text{C}}^{^{16}\text{O}}$ (fm ^{-1/2})	$C_{\alpha,^{16}\text{O}}^{^{20}\text{Ne}}$ (fm ^{-1/2})	$[(C_{\alpha,^{12}\text{C}}^{^{16}\text{O}})^2 (C_{\alpha,^{16}\text{O}}^{^{20}\text{Ne}})^2]^{1/4}$ (10 ³ fm ^{-1/2})
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Clustering in the ground states

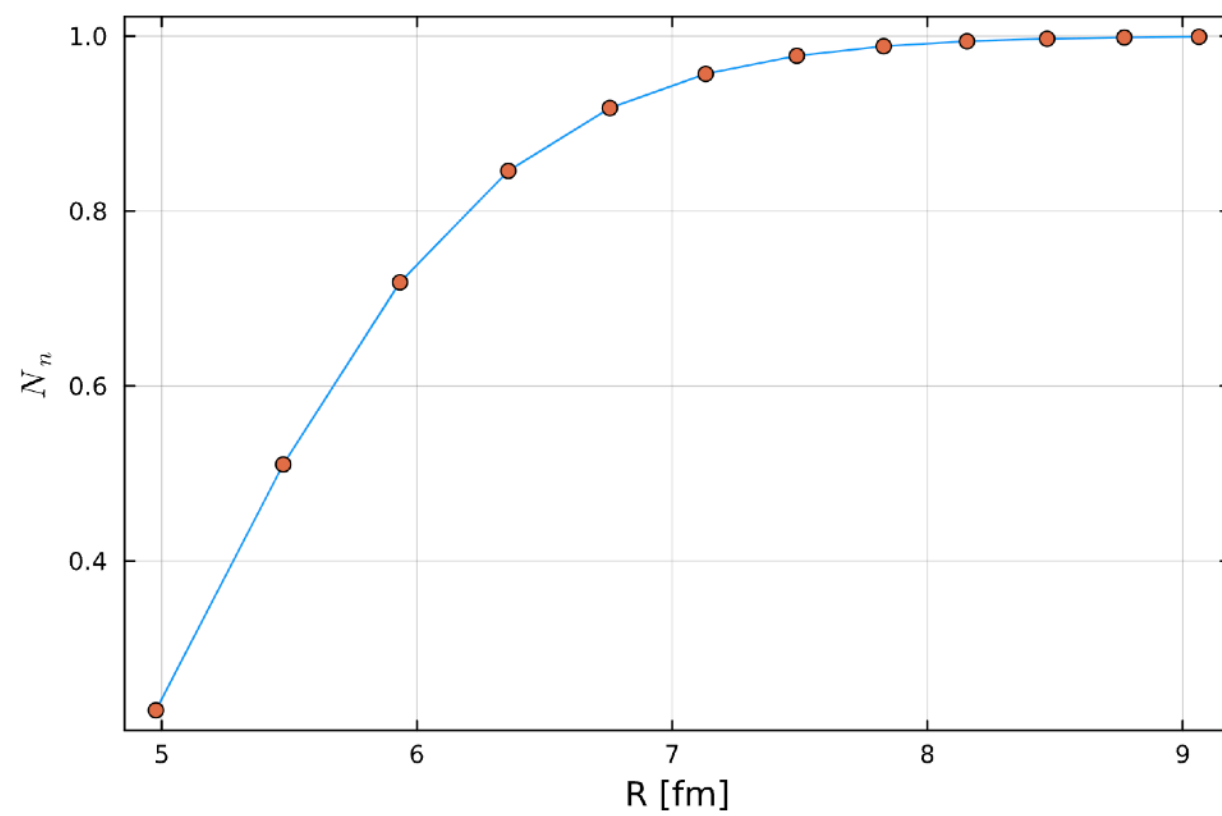
$$^{12}\text{C}(^{20}\text{Ne}, ^{16}\text{O})^{16}\text{O}$$



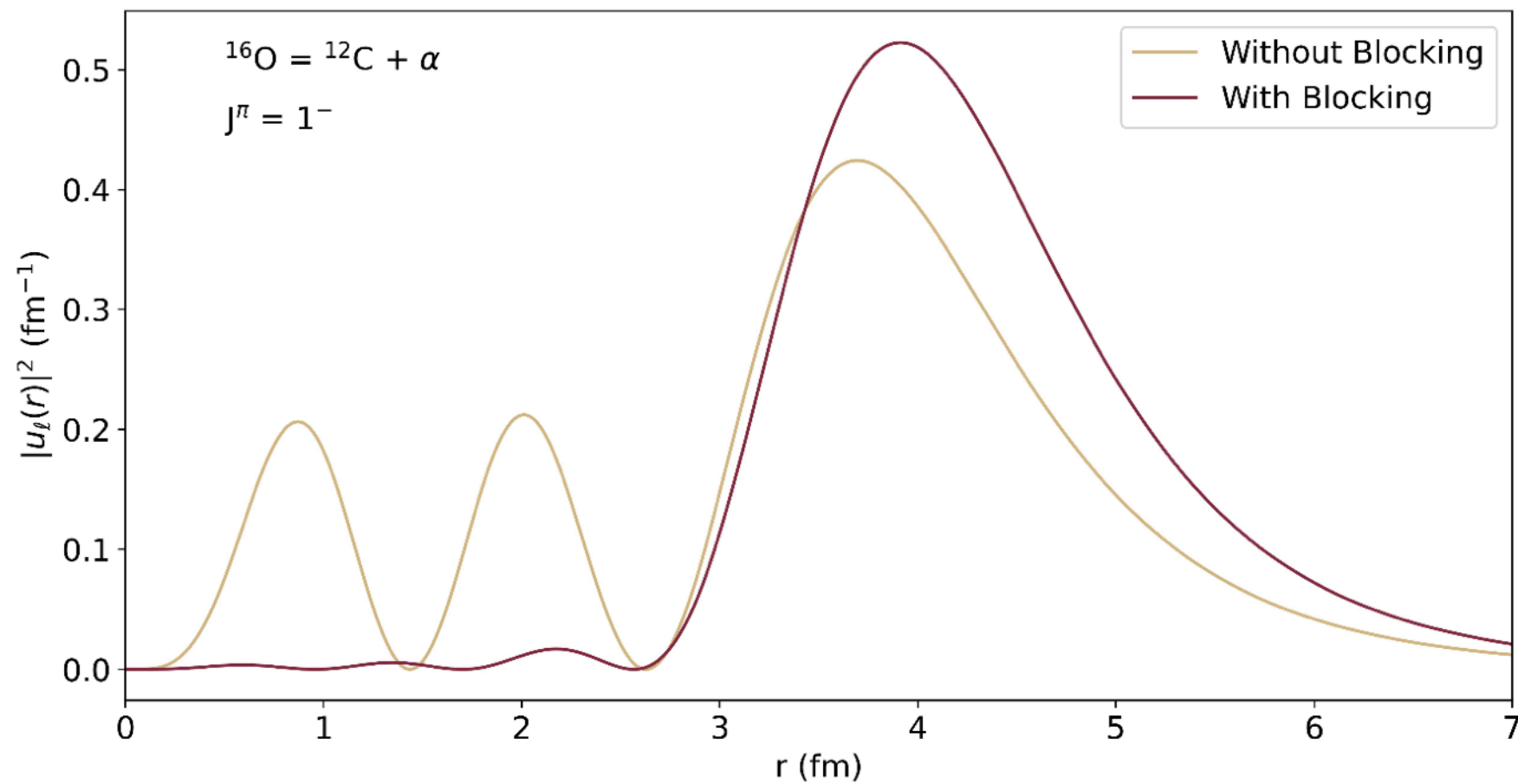
A. C. Dreyfuss et al., Phys. Rev. C 102, 044608 (2020).
K. Kravvaris and A. Volya, Phys. Rev. C 100, 034321 (2019).
E. Epelbaum et al., Phys. Rev. Lett. 109, 252501 (2012).
D. K. Nauruzbayev et al., Phys. Rev. C 96, 014322 (2017).

E. Harris, et al. Quantifying clustering in the ground states of ^{16}O and ^{20}Ne

Pauli blocking and norm kernel



Kevin Liguori, FSU honors thesis.



ANC from the Interior Solution via Green's Function

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2\eta_b\kappa}{r} - \kappa^2 \right] u_b(r) = \frac{2m}{\hbar^2} V(r) u_b(r) .$$

$$M(r) \equiv \frac{\Gamma(\ell+1+\eta_b)}{\Gamma(2\ell+2)} M_{-\eta_b, \ell+1/2}(2\kappa r) = F_\ell(\eta_b, i\kappa r) \quad \text{Regular solution}$$

$$W(r) \equiv W_{-\eta_b, \ell+1/2}(2\kappa r) \sim H_\ell^{(+)}(\eta_b, i\kappa r) \quad \text{Asymptotically decaying solution}$$

$$G_\ell(r, r') = -\frac{1}{2\kappa} M(r_<) W(r_>) \quad \text{Bound state Green's function}$$

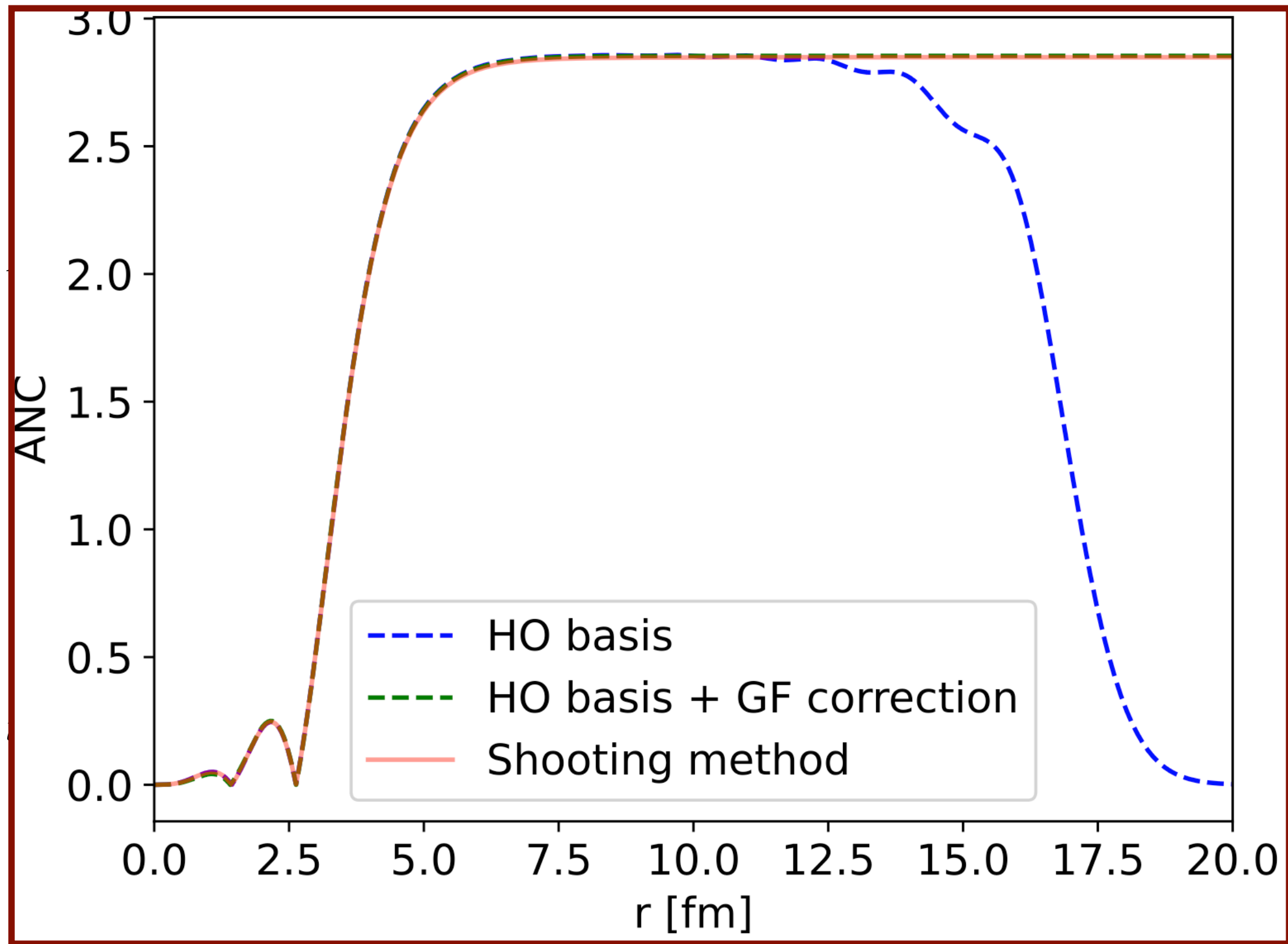
$$u_b(r) = -\frac{m}{\hbar^2\kappa} \left[W(r) \int_0^r M(r') V(r') u_b(r') dr' + M(r) \int_r^\infty W(r') V(r') u_b(r') dr' \right] .$$

$$A_\ell = -\frac{m}{\hbar^2\kappa} \int_0^\infty M(r) V(r) u_b(r) dr .$$

N. K. Timofeyuk, Phys. Rev. Lett. 103, 242501 (2009), Phys. Rev. C 88, 44315 (2013).

A. I. Baz', I. B. Zel'dovich, and A. M. Perelomov, Scattering, reactions and decay in nonrelativistic quantum mechanics., 2nd ed. (Nauka, Moskva, 1971).

ANC from the Interior Solution via Green's Function



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ANC from the Interior Solution via Green's Function

- Green's function method provides a robust extraction of ANCs.
- Additional iterations offer negligible improvement; energy correction is essential.
- The interior wave function is accurately captured with a small HO basis ($N_{\max}$).
- Reliable ANCs are obtained using experimental threshold energies, even when shell-model thresholds are inaccurate.

$N_{\max}$	% diff from exact ANC			
	$^{16}\text{O } 1^-$	$^{20}\text{Ne } 4^+$	$^{16}\text{O g.s.}$	$^{20}\text{Ne g.s.}$
	($E = -0.045$)	($E = -0.48$)	($E = -7.16$)	($E = -4.73$)
5	1.12%	6.13%	2.06%	23.0%
7	0.068%	1.27%	0.188%	5.69%
10	0.042%	0.484%	0.066%	1.32%
25	< 0.01%	< 0.01%	< 0.01%	< 0.01%

Conclusions

Intrinsic correlations

Despite all the many-body complexity, the ground and low-lying states of quantum many-body systems are not random vectors in Hilbert space: coherent structures emerge, and they are not limited to the well-known pairing but include clustering.

Symmetries

Exact symmetries such as angular momentum constrain which clusters can form; dynamical symmetries emerge approximately from the interaction. The oscillator SU(3) makes the α a single operator with analytic cluster coefficients, links clustering to deformation, and, extended to the symplectic algebra, generates the relative motion of cluster and core.

Antisymmetry

The fermionic nature of the constituents plays a major role in shaping the clusters. Channel normalization results in departure from an ideal boson limit, modifies spectroscopic factors and asymptotic normalizations, and pushes clusters out of the interior, so that clustering appears as a surface phenomenon.

Structure and reactions

Cluster preformation in the interior and cluster reactions in the continuum are two sides of one problem: the same channel that defines the cluster component of a bound state determines its decay width, transfer probability and scattering. Near thresholds the coupling reorganizes the states, concentrating the cluster strength in a few that align with the channel: superradiance.

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Principal collaborators

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