Towards an EFT description of alpha-clustering

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JG|U

Motivation

Alpha clustering in light nuclei

- Ab-initio description of clustering in nuclei is an open challenge

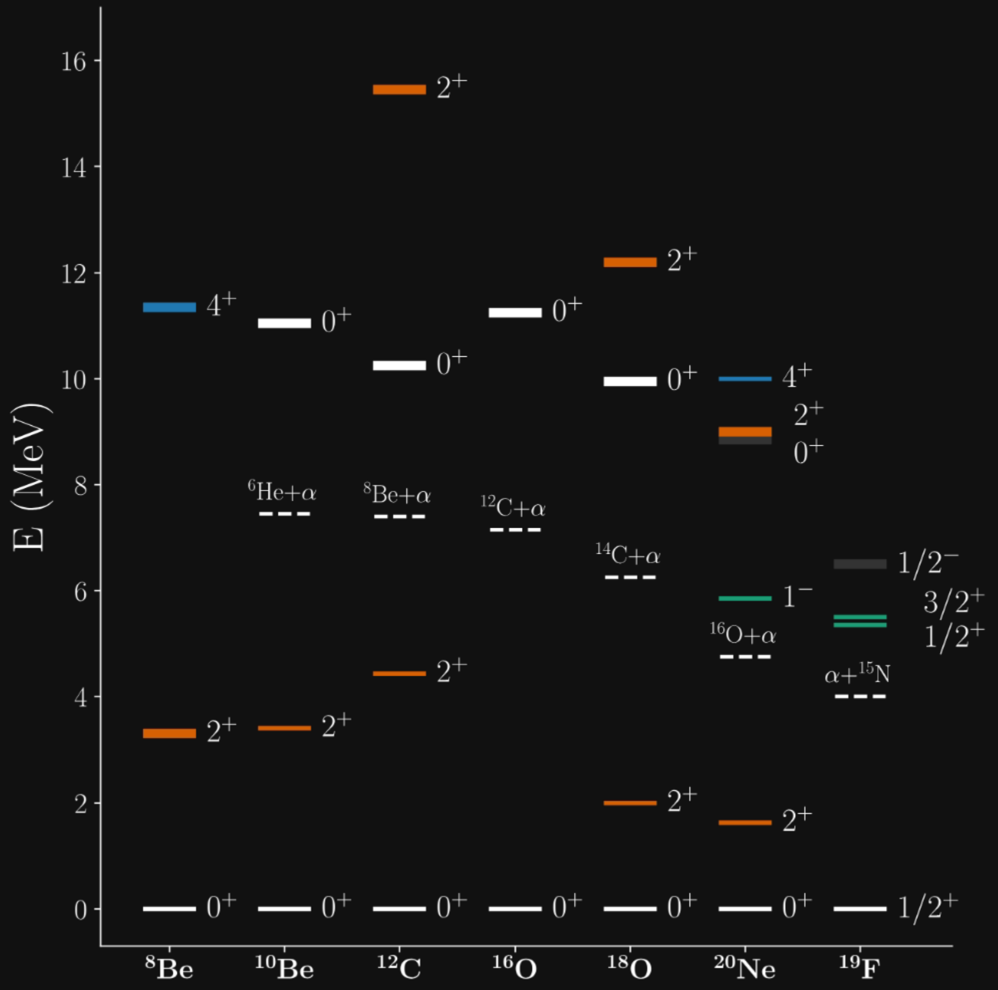


Figure adapted from: Otsuka et. al., The European Physical Journal A 62.6 (2026): 108

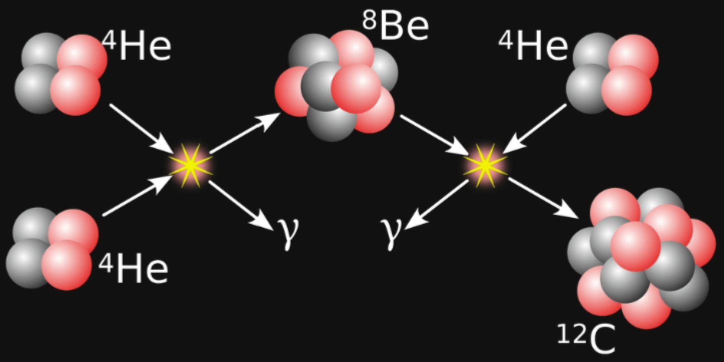
Motivation

Astrophysically relevant reactions

$$^{4}\mathrm{He} + {}^{4}\mathrm{He} \rightleftharpoons {}^{8}\mathrm{Be}$$

$$^{8}\mathrm{Be} + {}^{4}\mathrm{He} \rightarrow {}^{12}\mathrm{C}^{*} \rightarrow {}^{12}\mathrm{C} + \gamma$$

$$^{12}\mathrm{C} + \alpha \rightarrow {}^{16}\mathrm{O}^{*} \rightarrow {}^{16}\mathrm{O} + \gamma$$



Effective Field Theories

1. Select relevant degrees of freedom
2. Construct the Lagrangian
3. Derive the interaction / calculate observables

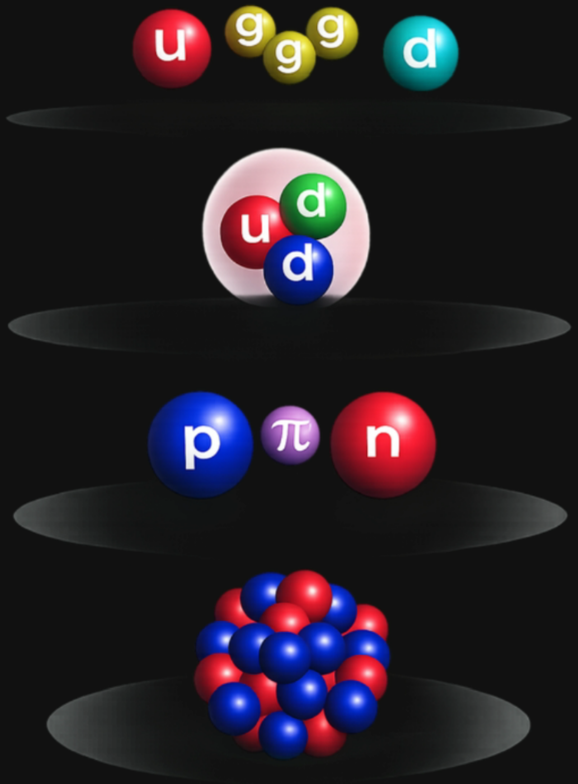


Figure adapted from: G. F. Bertsch, et. al., SciDAC Rev. 6, 42 (2007)

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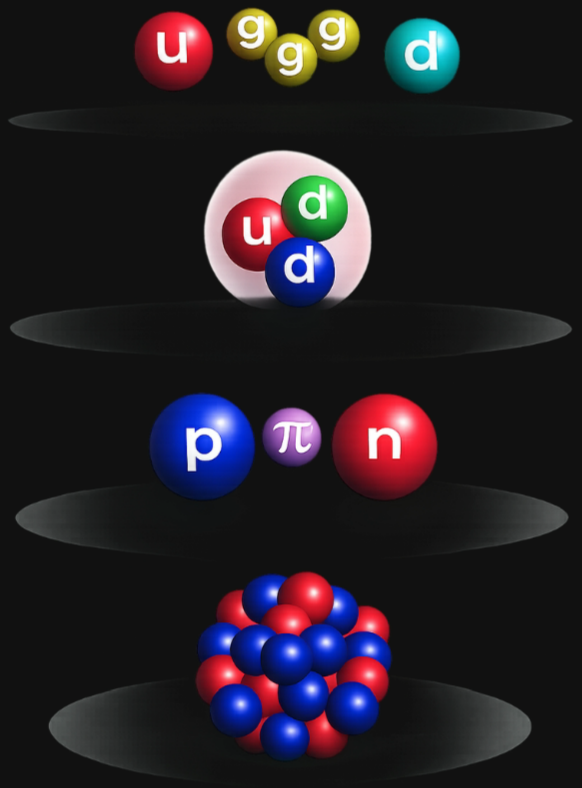

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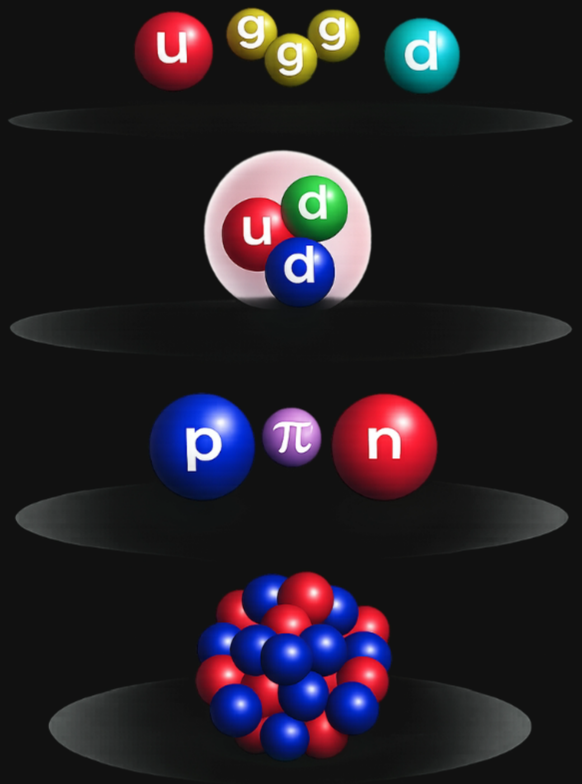

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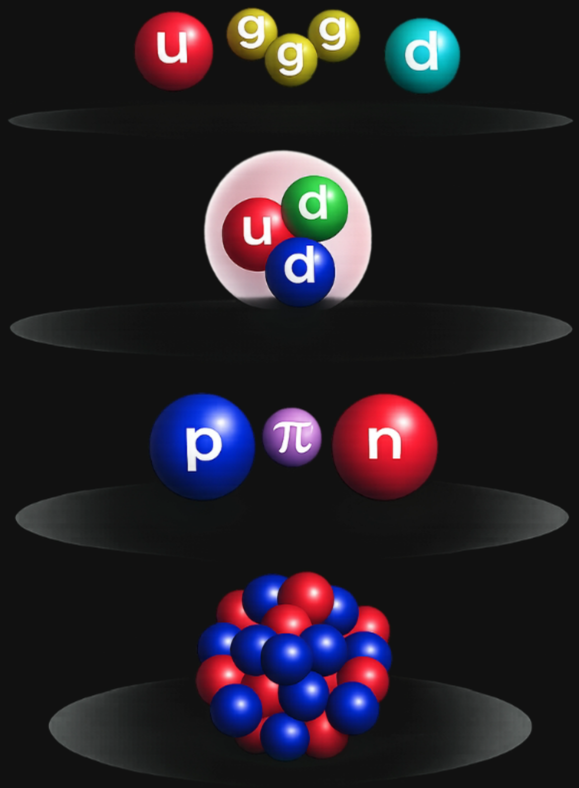

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Contact Effective Field Theories

Rev. Mod. Phys. 92, 025004

- Lagrangian contains only zero-range interactions

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi - \frac{1}{2} C_0 (\psi^\dagger \psi)^2 - \frac{1}{6} D_0 (\psi^\dagger \psi)^3 - \frac{1}{16} C_2 \Big[(\psi\psi)^\dagger (\psi \nabla^2 \psi) + h.c. \Big] + \cdots$$

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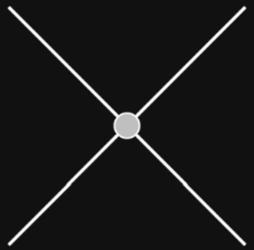

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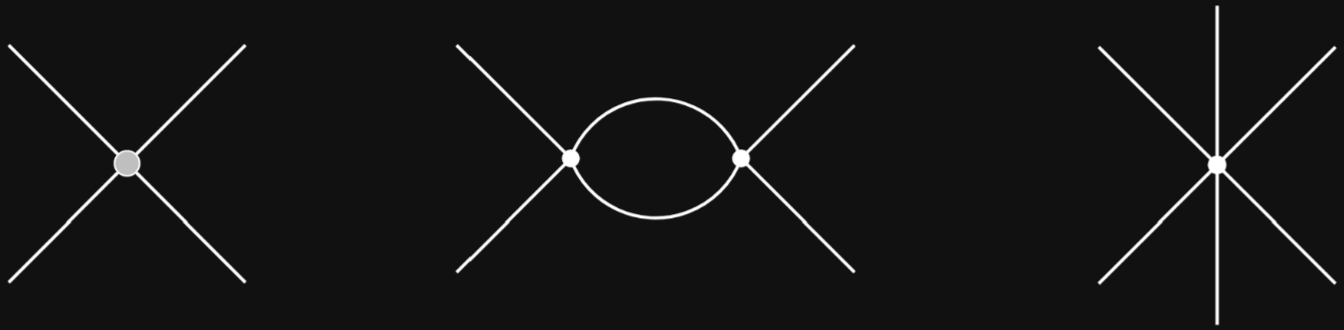

Effective Range Expansion

- Most general parametrization of low-energy scattering

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r k^2 + \frac{1}{4} p_1 k^4 + \ldots$$

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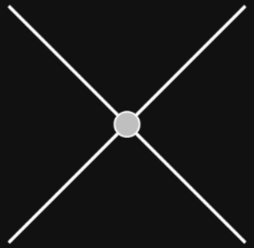
Shape parameter

Effective range

- Usually relatively small radius of convergence

Contact Interactions



$$\longrightarrow \quad \langle \boldsymbol{p}' | V | \boldsymbol{p} \rangle = C_0 + C_2 \left(p^2 + p'^2 \right) + \cdots$$

$$\longrightarrow \quad \langle \boldsymbol{p}' \boldsymbol{q}' | V | \boldsymbol{p} \, \boldsymbol{q} \rangle = D_0 + \cdots$$

The Wigner Bound and RG-Invariance

- EFT Interactions need to be regularized

$$\langle \boldsymbol{p}' | V | \boldsymbol{p} \rangle \longrightarrow e^{-p'^2/\Lambda^2} \langle \boldsymbol{p}' | V | \boldsymbol{p} \rangle \, e^{-p^2/\Lambda^2}$$

$$\langle \boldsymbol{p}' \boldsymbol{q}' | V | \boldsymbol{p}\, \boldsymbol{q} \rangle \longrightarrow e^{-p'^2/\Lambda^2 - \frac{3}{4} q^2/\Lambda^2} \langle \boldsymbol{p}' | V | \boldsymbol{p} \rangle \, e^{-p^2/\Lambda^2 - \frac{3}{4} q^2/\Lambda^2}$$

- Value of r is then restricted

$$r_{\max}(R, a) \leq 2 \left(R - \frac{R^2}{a} + \frac{R^3}{3a^2} \right)$$

The Wigner Bound

Coulomb case

$$b_\ell^C(r) - r_\ell^C \geq 0 \qquad \forall\, r \geq R$$

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$$b_0^C(r) = \frac{2r^3}{3}\,(a_0^C)^{-2}\,{}_1F_2\left(\frac{3}{2};\,2,\,4;\,4\gamma r\right) - \frac{4r^2}{\sqrt{\pi}}\,(a_0^C)^{-1}\,G_{1,3}^{2,1}\left(4\gamma r \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0,\,1,\,-2 \end{matrix}\right) + \\ + 4\sqrt{\pi}\,\gamma r^2\,G_{2,4}^{3,1}\left(4\gamma r \,\middle|\, \begin{matrix} -1,\,\frac{1}{2} \\ -1,0,1,-2 \end{matrix}\right)$$

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System	Channel	Causal range / fm
p–p	${}^1S_0$	1.38 ± 0.01
p–p	${}^3P_0$	2.33 ± 0.05
p–p	${}^3P_1$	≈ 0.03
α–α	S	$2.58^{+0.14}_{-0.13}$
α–α	D	$2.03^{+0.09}_{-0.07}$

Table adapted from: Sebastian König et al 2013 J. Phys. G: Nucl. Part. Phys. 40 045106

Poles of the S-Matrix

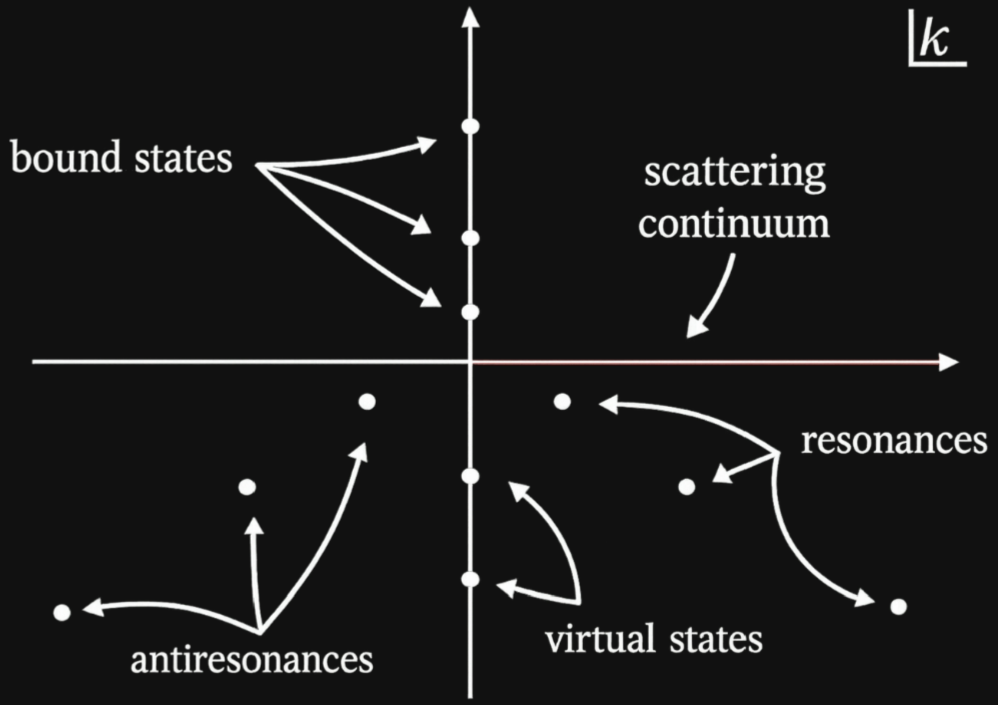

Figure adapted from: arXiv:2605.03977

Effective range expansion

From the EFT point of view

- We assume $a \gg r$ (holds for nucleons, alpha particles, ^{4}He atoms . . .)

$$-\frac{\mu}{2\pi}T(k) = \frac{1}{k\cot\delta - ik} = \left[\frac{1}{-1/a - ik} + \ldots\right]$$

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$$T^{(0)} = V^{(0)} + V^{(0)} G_0 T^{(0)}$$

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NLO: $$T^{(1)} = (1 + T^{(0)}G_0)\,V^{(1)}\,(G_0\,T^{(0)} + 1)$$

Effective range expansion with Coulomb

From the EFT point of view

- Long range forces modify the form of the ERE

$$k \cot \delta_c = \frac{1}{C_\eta^2}\left[-\frac{1}{a_c} + \frac{1}{2} r_c k^2 + \cdots\right] - \frac{2k\eta h(\eta)}{C_\eta^2}$$

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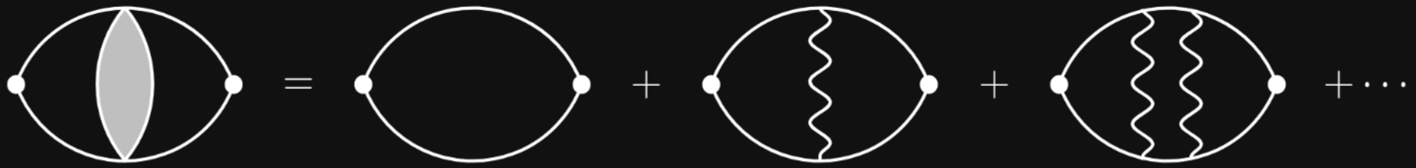

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$$T_{cs}^{(0)} = V^{(0)} + V^{(0)} G_C T_{cs}^{(0)}$$

Alpha-alpha scattering

- $^{8}\mathrm{Be}$ at $E_r = 92.08(4)$ keV
  $\Gamma = 5.6(3)$ eV
- $M_{lo} \approx k_r \approx 20$ MeV
- $M_{hi} \approx \sqrt{m_n E^*} \approx 140$ MeV
- $a = 1920(90)$ fm
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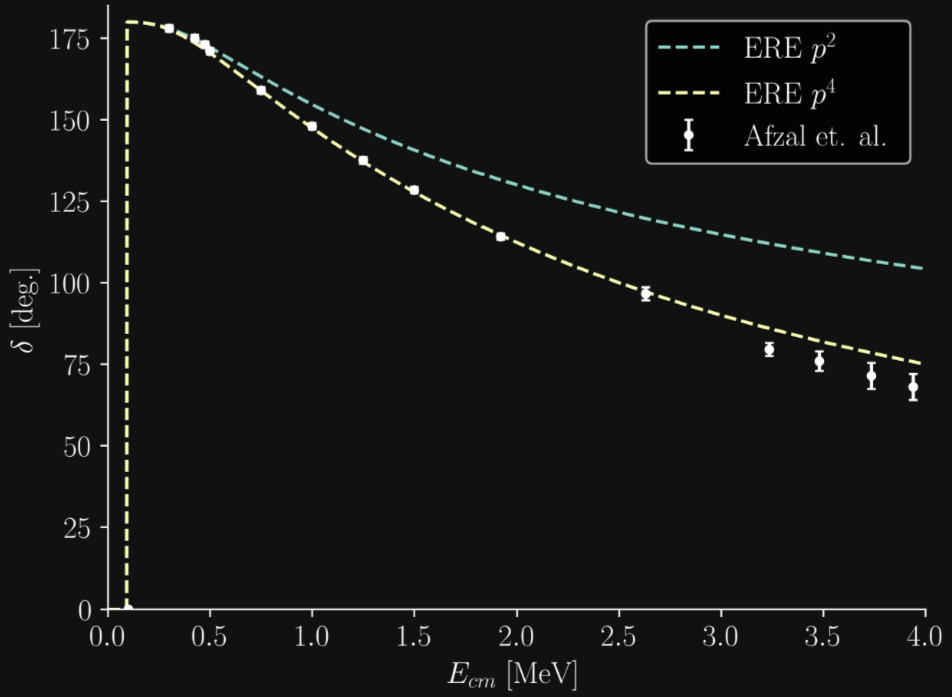

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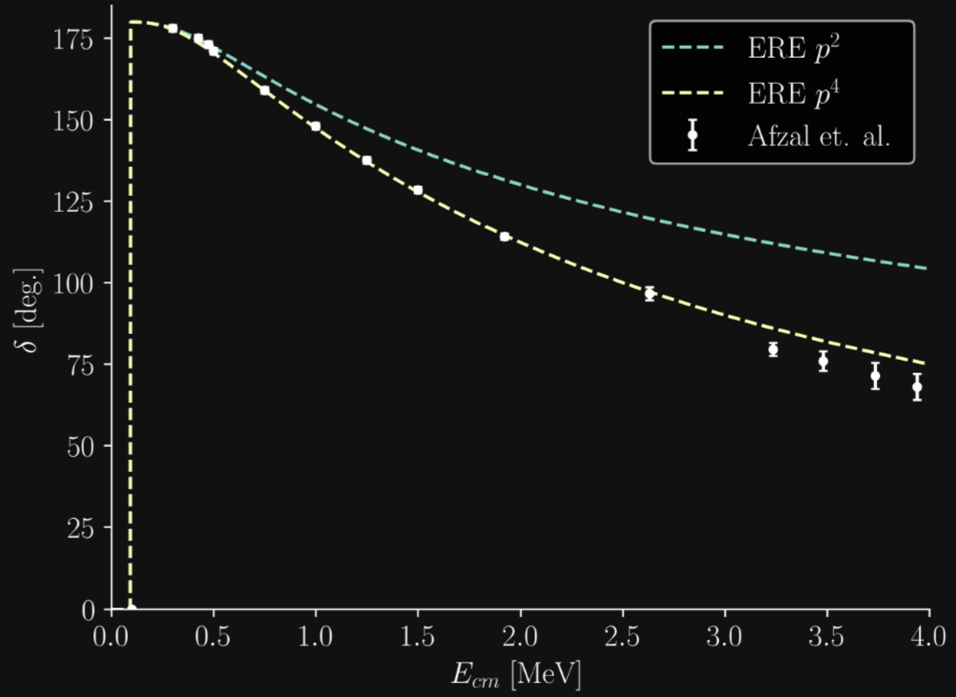

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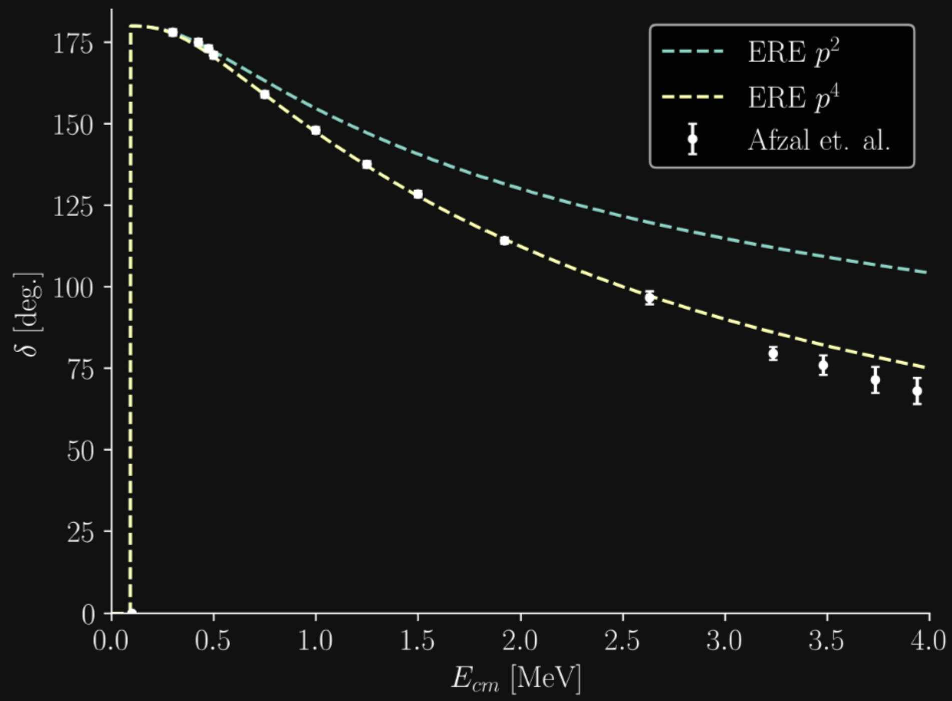

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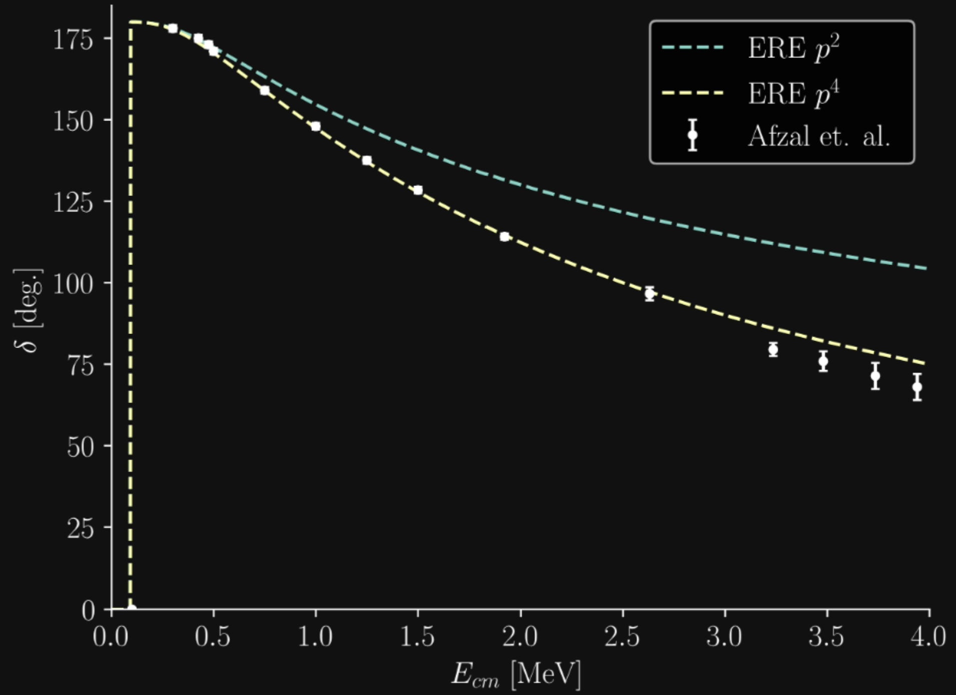

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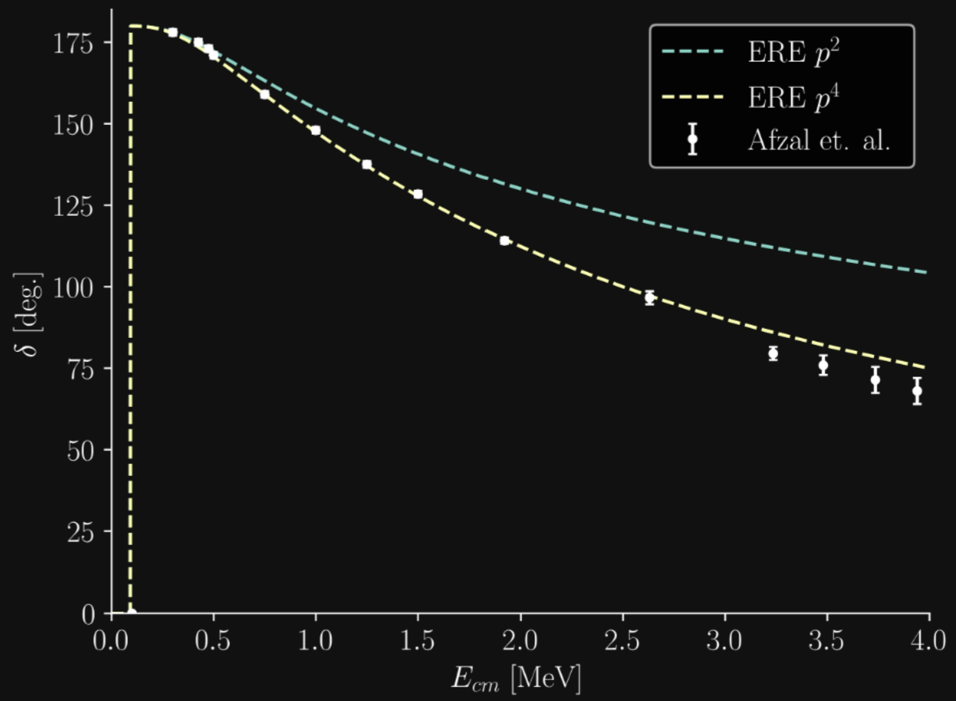

Alpha-alpha scattering

EFT Description, R. Higa et al. / Nuclear Physics A 809 (2008) 171–188

- Resonances can be described by dimeron fields

$$\mathcal{L} = \phi^\dagger \left[i\partial_0 + \frac{\vec{\nabla}^2}{2m_\alpha} \right] \phi + \sigma d^\dagger \left[i\partial_0 + \frac{\vec{\nabla}^2}{4m_\alpha} - \Delta \right] d + g \left[d^\dagger \phi\phi + (\phi\phi)^\dagger d \right] + \cdots$$

$$V(\boldsymbol{p}', \boldsymbol{p}) = \frac{\sigma g^2}{E - \Delta} + \cdots$$

Alpha-alpha scattering

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$$T(k) = -\frac{2\pi}{\mu}\, C_\eta^2 e^{2i\sigma_0} \left[\frac{1}{\tilde{r}_0(k^2 - k_R^2)/2 - ikC_\eta^2} + \frac{\tilde{\mathcal{P}}_0}{4} \frac{k^4 - k_R^4}{\left(\tilde{r}_0(k^2 - k_R^2)/2 - ikC_\eta^2\right)^2} + \cdots \right]$$

$$\tilde{r}_0 = r_0 - \frac{1}{6\alpha m_\alpha} \qquad \tilde{\mathcal{P}}_0 = \mathcal{P}_0 - \frac{1}{30(\alpha m_\alpha)^3} \qquad k_R^2 = \frac{2}{a_0 \tilde{r}_0}\left(1 - \frac{\mathcal{P}_0}{a_0 \tilde{r}_0^2} + \cdots\right)$$

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Beane, S. R., & Farrell, R. C. (2022). Few-Body Systems, 63(2), 45.

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- Allows positive effective range
- Produces 2 LO poles

Beane, S. R., & Farrell, R. C. (2022). Few-Body Systems, 63(2), 45.

The LO Square Root

$$V(p,p') = -\frac{2\pi}{\mu}\frac{\lambda}{\sqrt{p^2+\gamma}\sqrt{p'^2+\gamma}} \longrightarrow T(k) = -\frac{1}{2\mu}\left[-\frac{1}{a}+\frac{1}{2}rk^2-ik\right]^{-1}$$

- Allows positive effective range
- Produces 2 LO poles
- Essentially a resummation of all s-wave contact operators

Beane, S. R., & Farrell, R. C. (2022). Few-Body Systems, 63(2), 45.

The NLO Square Root

Genuine NLO:

$$V^{(1)}(p,p') = \frac{2\pi g}{\mu}\frac{p^2+p'^2}{\sqrt{p^2+\gamma}\sqrt{p'^2+\gamma}}$$

NLO Counterterm:

$$V_{\text{ct}}^{(1)} = \frac{1}{\sqrt{p^2+\gamma^{(0)}}\sqrt{p'^2+\gamma^{(0)}}}\left[\lambda^{(1)} - 2\lambda^{(0)}\gamma^{(1)}\left(\frac{1}{p^2+\gamma^{(0)}}+\frac{1}{p'^2+\gamma^{(0)}}\right)\right]$$

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Alpha-alpha scattering

Description based on $\sqrt{\mathrm{EFT}}$

- $e^{-p'^2/\Lambda^2} \langle \boldsymbol{p}' | V | \boldsymbol{p} \rangle e^{-p^2/\Lambda^2}$

- $\left[-\frac{1}{a_c} + \frac{1}{2} r_c k^2\right]$

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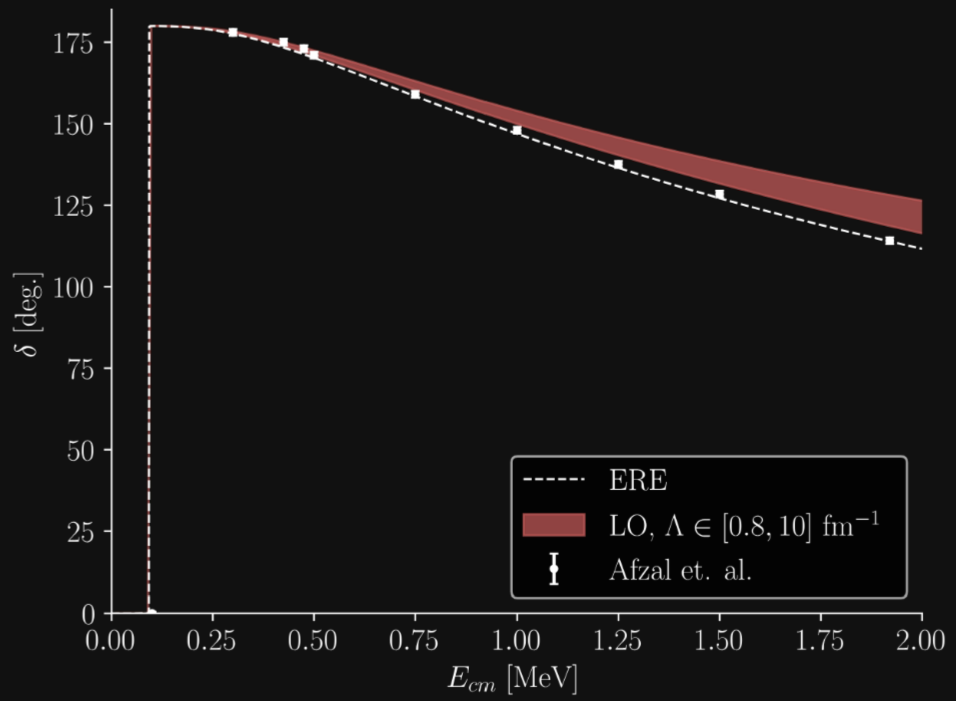

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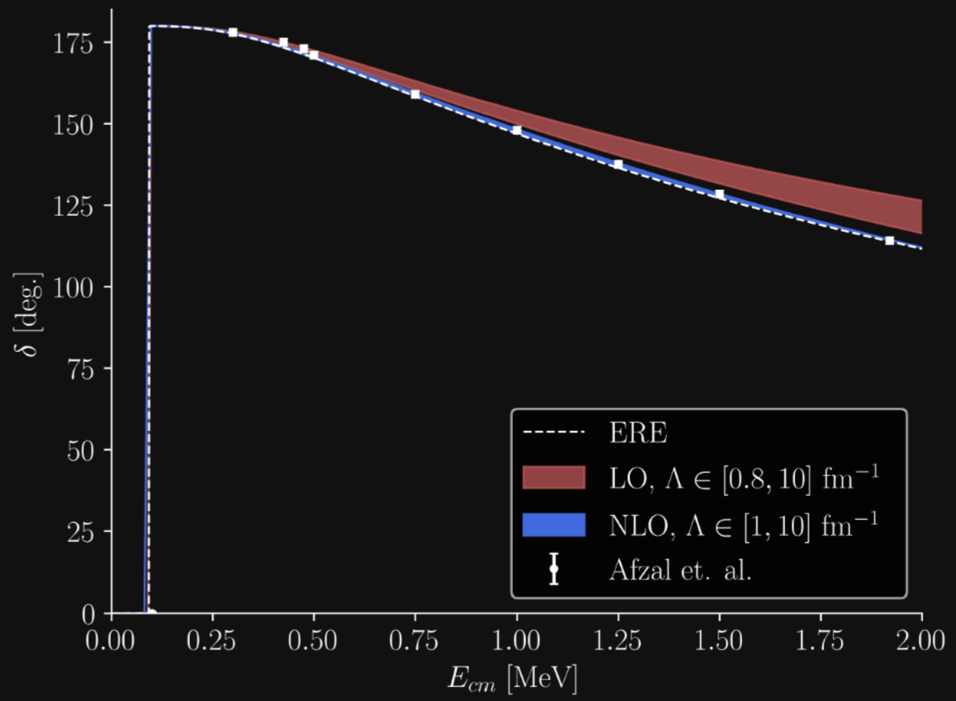

Correlated Gaussian Basis

- Spatial basis functions:

$$F_{(L_1L_2)LM}(\boldsymbol{u}_1, \boldsymbol{u}_2, A, \boldsymbol{x}) = \exp\left(-\frac{1}{2}\boldsymbol{x}^\dagger A\boldsymbol{x}\right)\left[\mathcal{Y}_{L_1}(\boldsymbol{u}_1^\dagger\boldsymbol{x})\mathcal{Y}_{L_2}(\boldsymbol{u}_2^\dagger\boldsymbol{x})\right]_{LM}$$

Suzuki, Y., et al. Few-Body Systems 42.1 (2008): 33-72.

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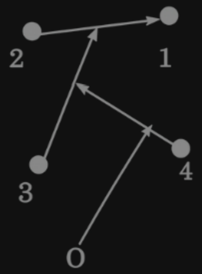

$$\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n-1})$$

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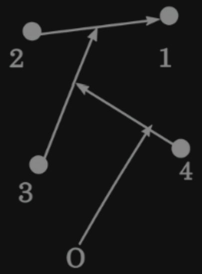

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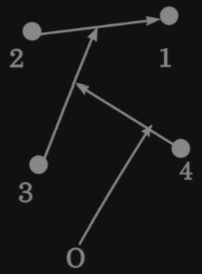

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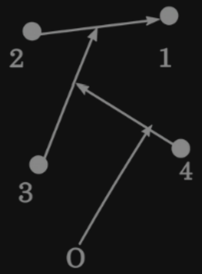

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- Total basis function:

$$\psi_{JM_JTM_T} = [\psi_L\chi_S]_{JM_J}\,\xi_{TM_T}$$

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Stochastic Variational Method

1. Generate $\{A_i\}$

2. Determine $\{E_i\}$

3. Select the lowest E_i

4. Repeat

Suzuki, Y., & Varga, K. (2002). Stochastic variational approach to quantum-mechanical few-body problems.

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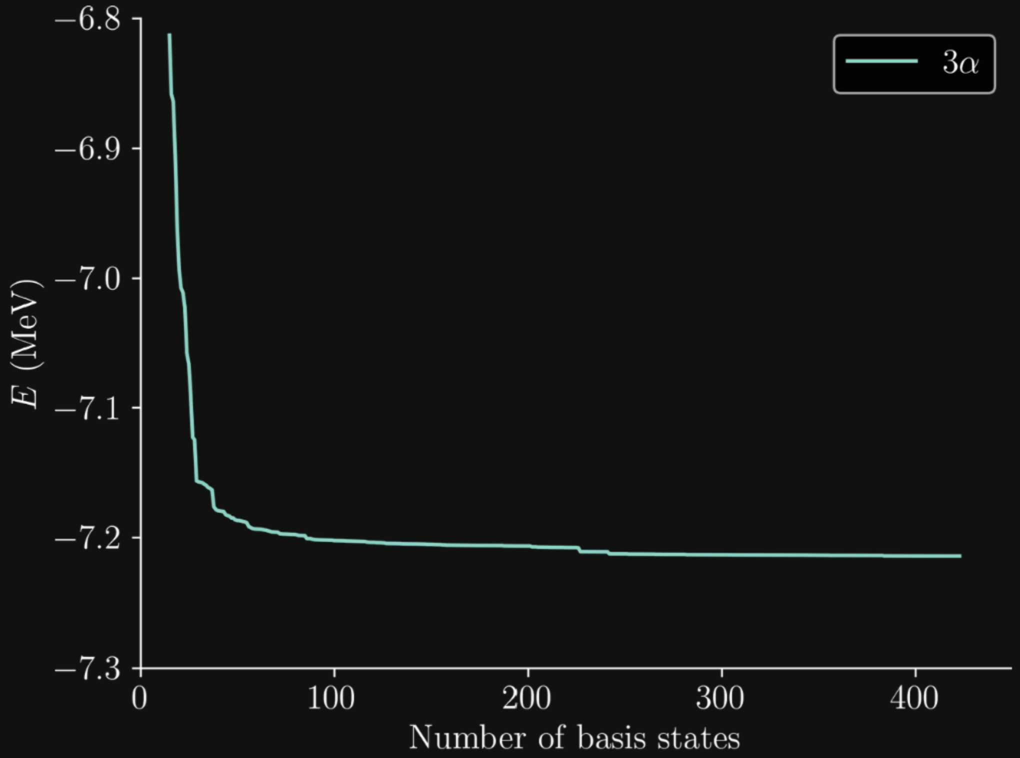

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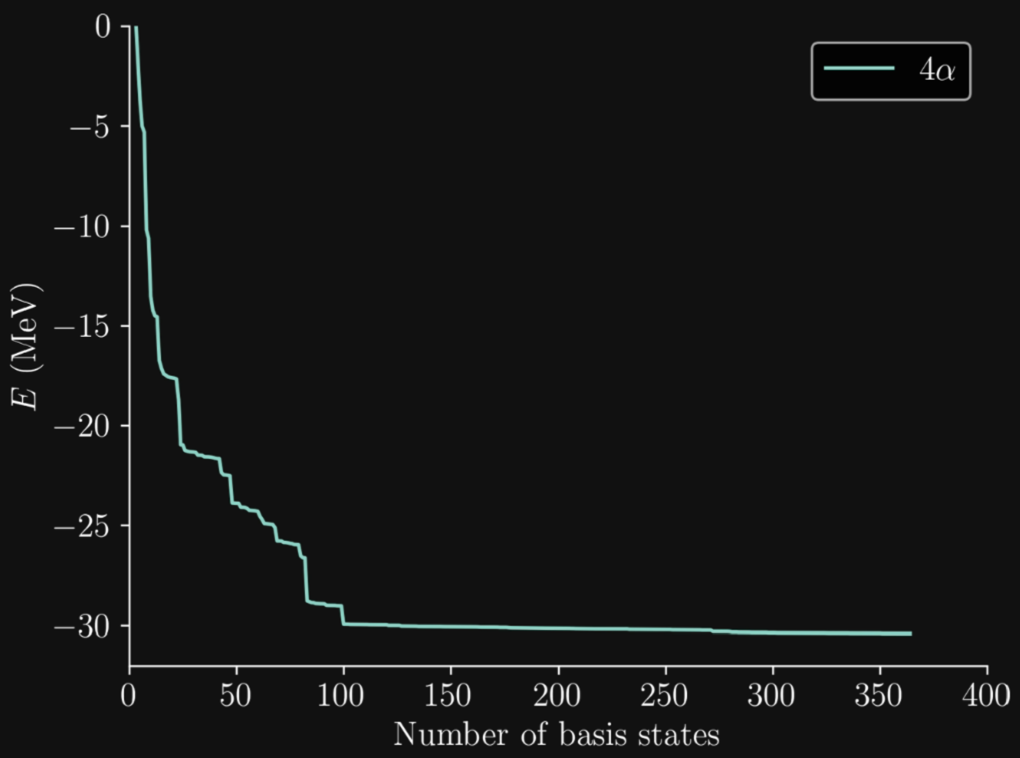

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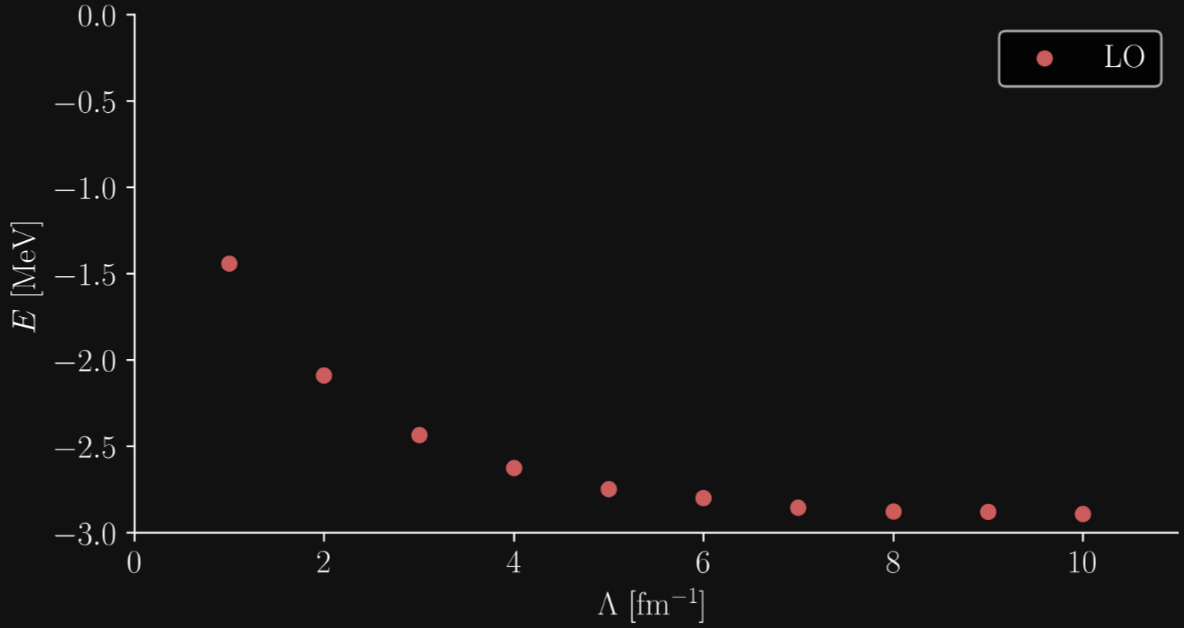
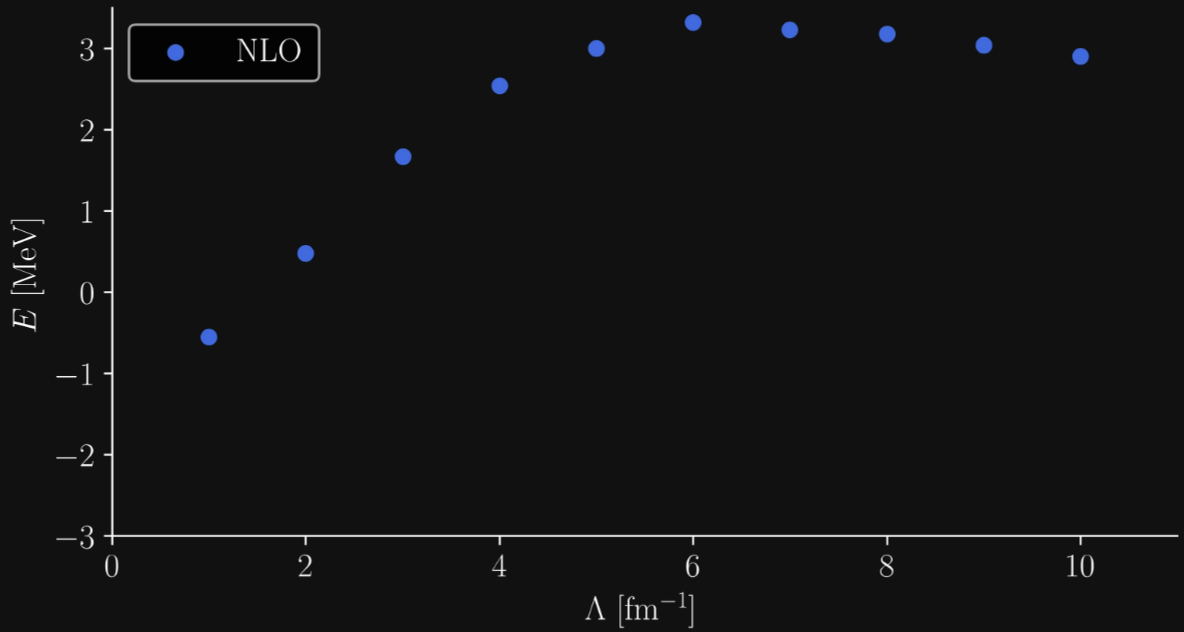
Three-alpha system

Leading order $\sqrt{\text{EFT}}$



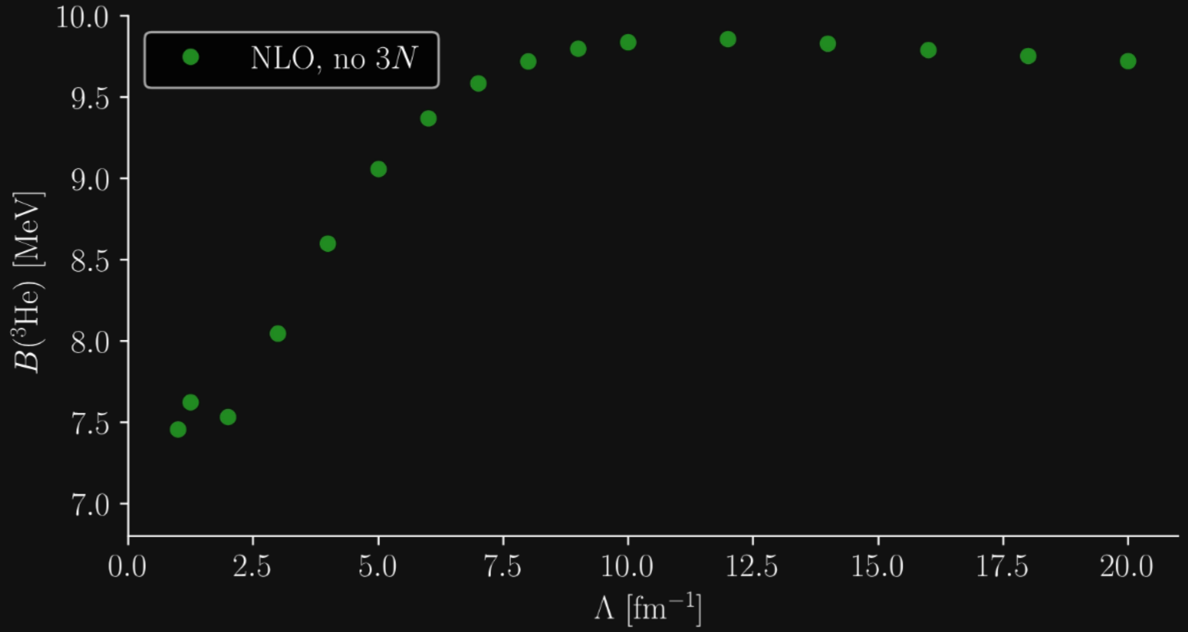
Three-alpha system

Next-to-Leading order $\sqrt{\mathrm{EFT}}$



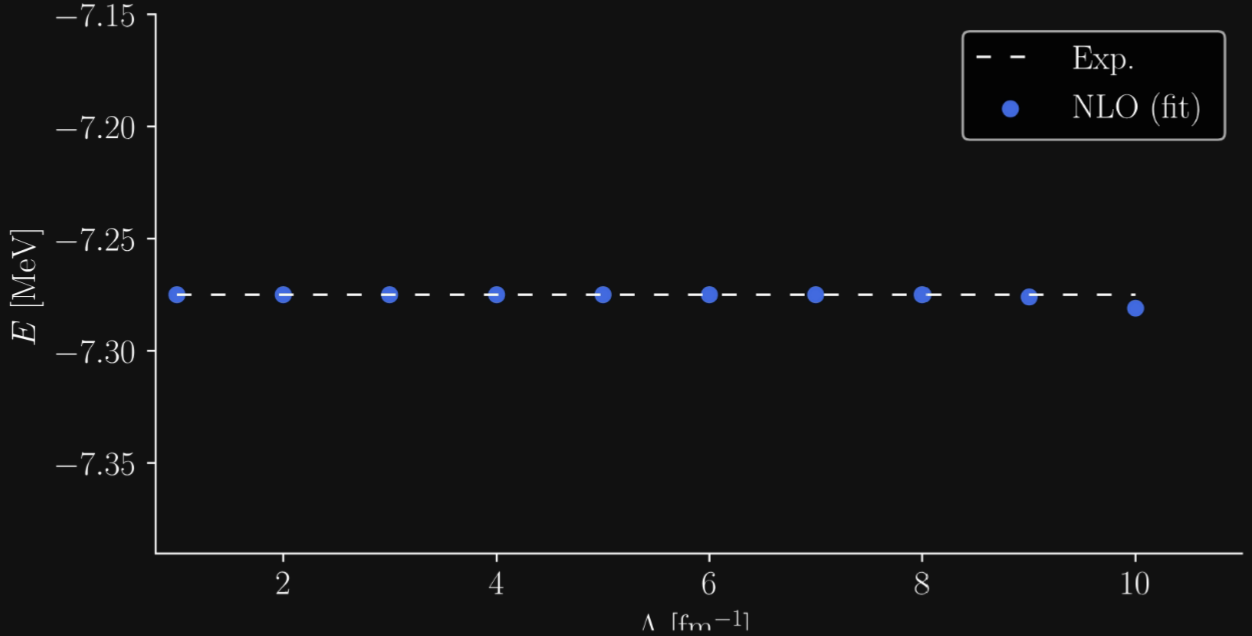
Three-alpha system and helion

Next-to-Leading order



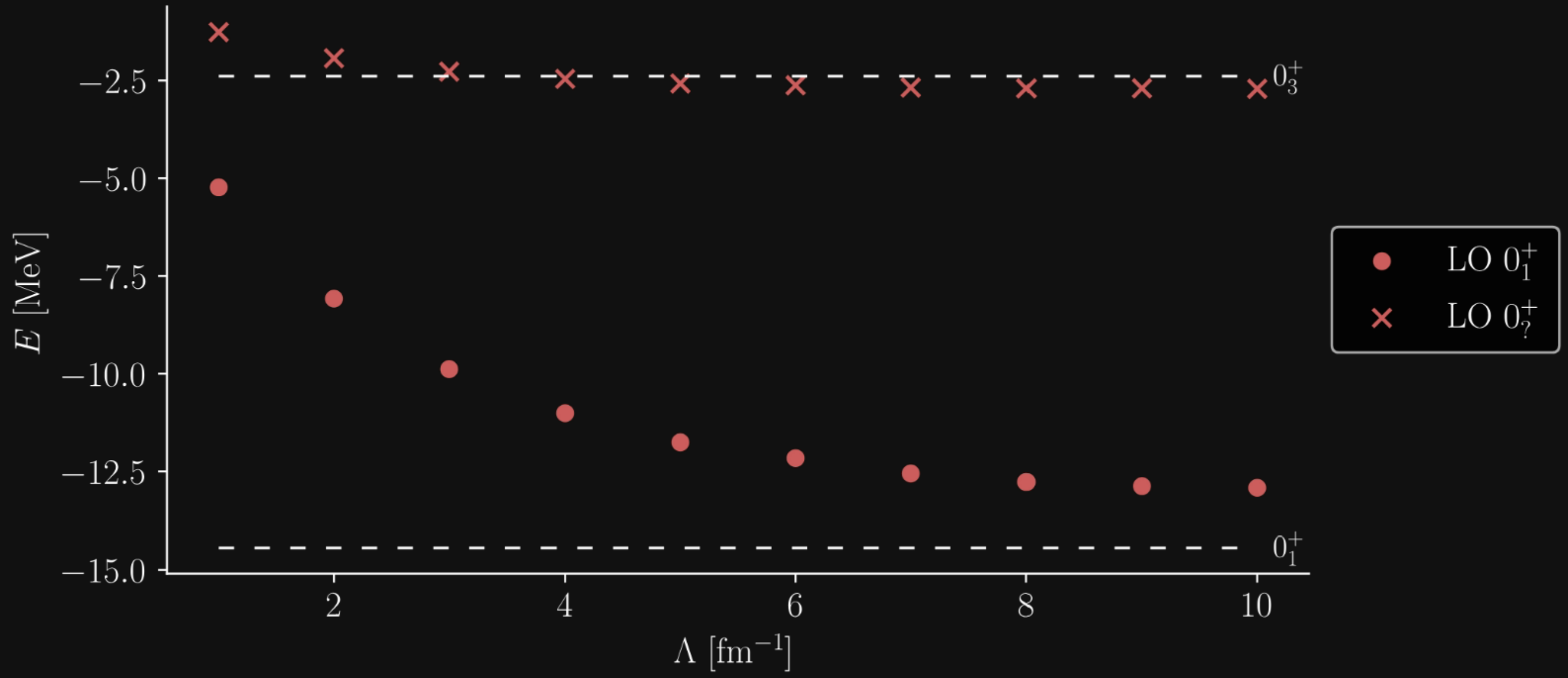
Three-alpha system fit

Next-to-Leading order



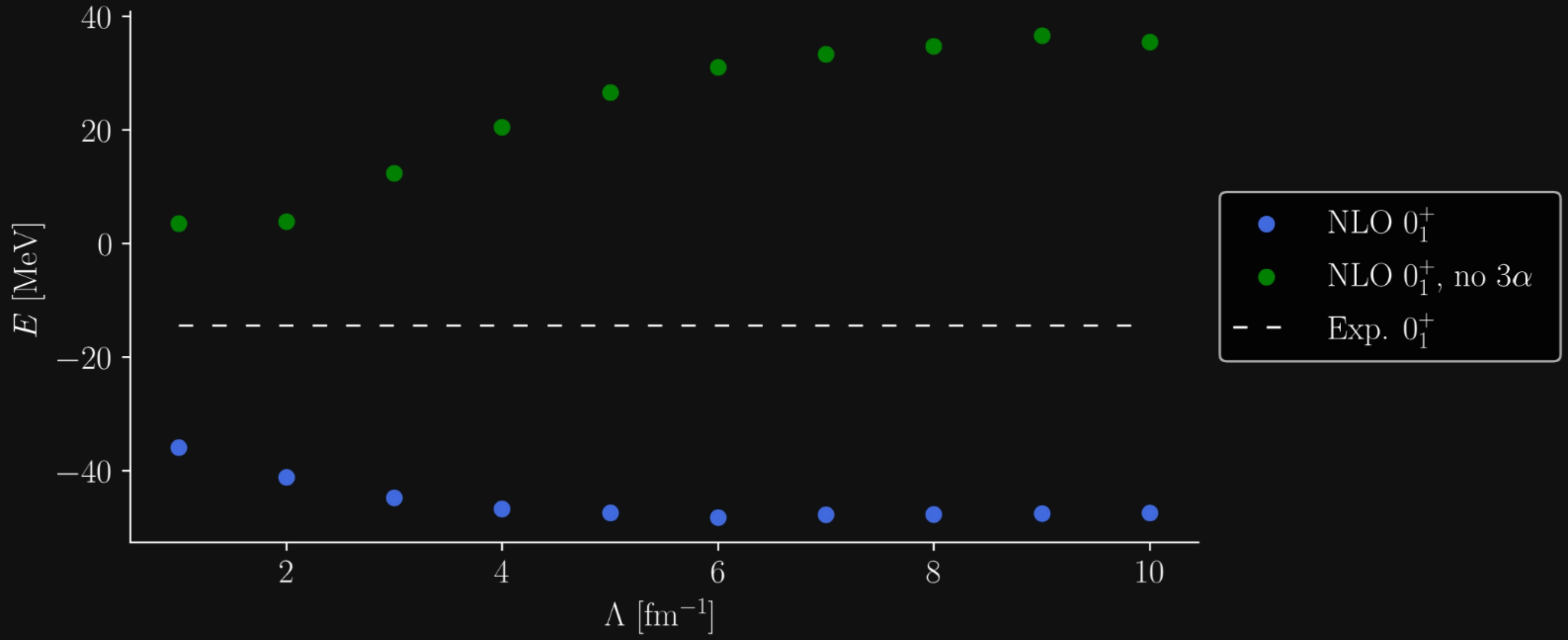
Four-alpha system

Leading Order



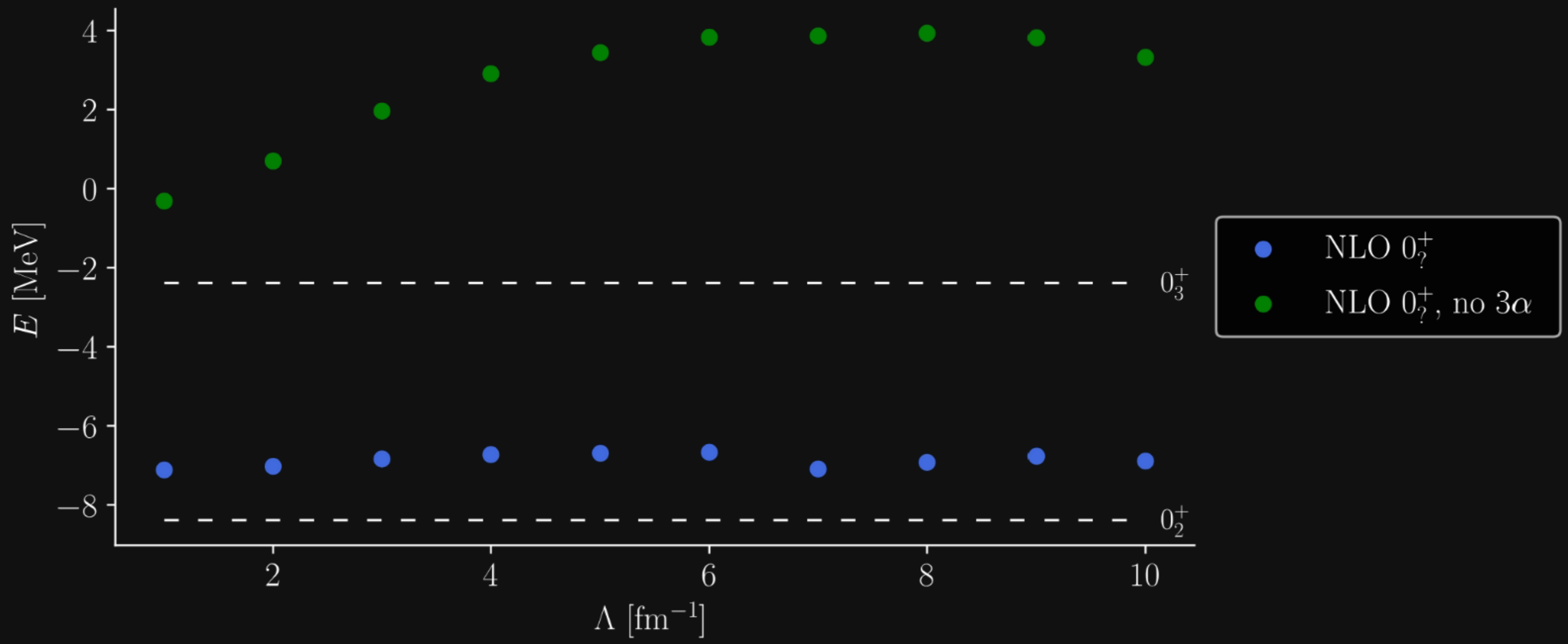
Four-alpha system

Next-to-Leading Order



Four-alpha system

Next-to-Leading Order Excited State



Resonances and the SVM

- SVM always optimizes for the lowest energy
- As a consequence, resonances are difficult to find

n	E [MeV]
1	-2.892
2	0.058
3	0.068
4	0.074
5	0.083
6	0.087
7	0.089
8	0.092
9	0.099
10	0.106

IACCC

Inverse analytical continuation in the coupling constant

- We consider

$$H = T + V_{EFT} + \alpha V_{aux}$$

- Then

$$\alpha(k) \approx \frac{P_M(k)}{Q_N(k)} = \frac{\sum_{j=0}^{M} c_j k^j}{1 + \sum_{j=1}^{N} d_j k^j}$$

- Only calculation of bound-state energies needed

Horáček, Jiří, and Lukáš Pichl. Communications in Computational Physics 21.4 (2017): 1154-1172

IACCC

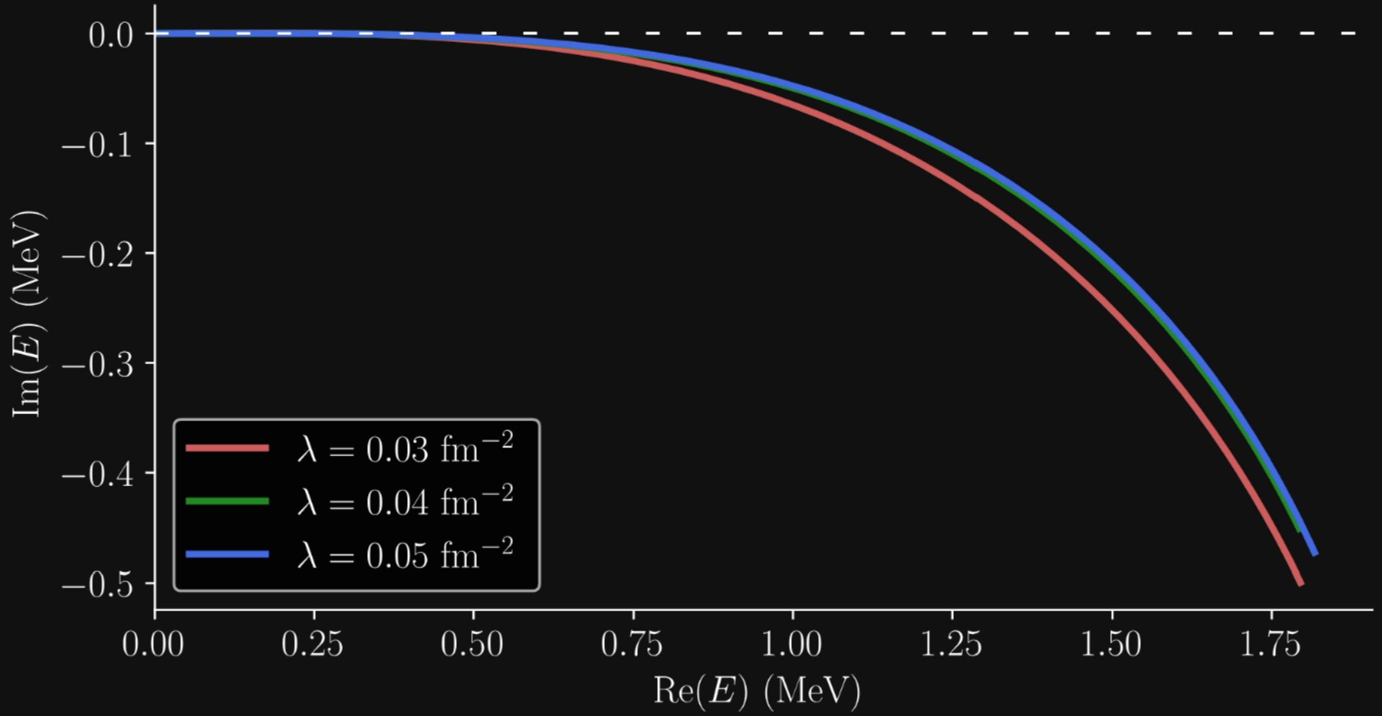
Inverse analytical continuation in the coupling constant

- In principle independent of V_{aux}, I use:

$$V_{\text{aux}} = \alpha \exp\left(-\lambda \sum_{i<j}^{3} r_{ij}^2\right)$$

- Very high precision required (~10^{-3} MeV)

Hoyle State at Leading Order



IACCC at Next-to-Leading Order

- NLO Hamiltonian of the form

$$H = T + V_{\text{EFT}}^{\text{LO}} + \xi V_{\text{EFT}}^{\text{NLO}} + \alpha V_{\text{ACCC}}$$

- We then find

$$E(\alpha, \xi) = E_{\text{LO}} + \xi E_{\text{NLO}} + \xi^2 E + \cdots$$

- and by a Padé fit

$$E(\alpha, \xi) = E_r^{\text{LO}} + \xi E_r^{\text{NLO}}$$

LO Helion and Alpha

Leading order, with three-body force

Cutoff [fm^{-1}]	**^{3}H/^{3}He [MeV]**	**^{4}He [MeV]**
2.0	-8.488	-28.77
4.0	-8.489	-29.87
6.0	-8.490	-30.41
8.0	-8.494	-30.54
10.0	-8.484	-30.49
Exp.	-8.482	-28.296

Conclusions

- The $\sqrt{\mathrm{EFT}}$ allows usage of dimeron interactions in few/many-body methods
- Ground states of ^{12}C and ^{16}O difficult to describe
- The Hoyle state could be corrected at NLO
- Calculating resonances with perturbative corrections is not easy

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Outlook

- Application to Beryllium isotopes
- Investigate alpha-condensate states in ^{16}O, ^{20}Ne ...
- Application to reactions with alpha clusters
- Application to EFTs with nucleonic degrees of freedom

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