Ab-initio scattering using Nuclear lattice EFT

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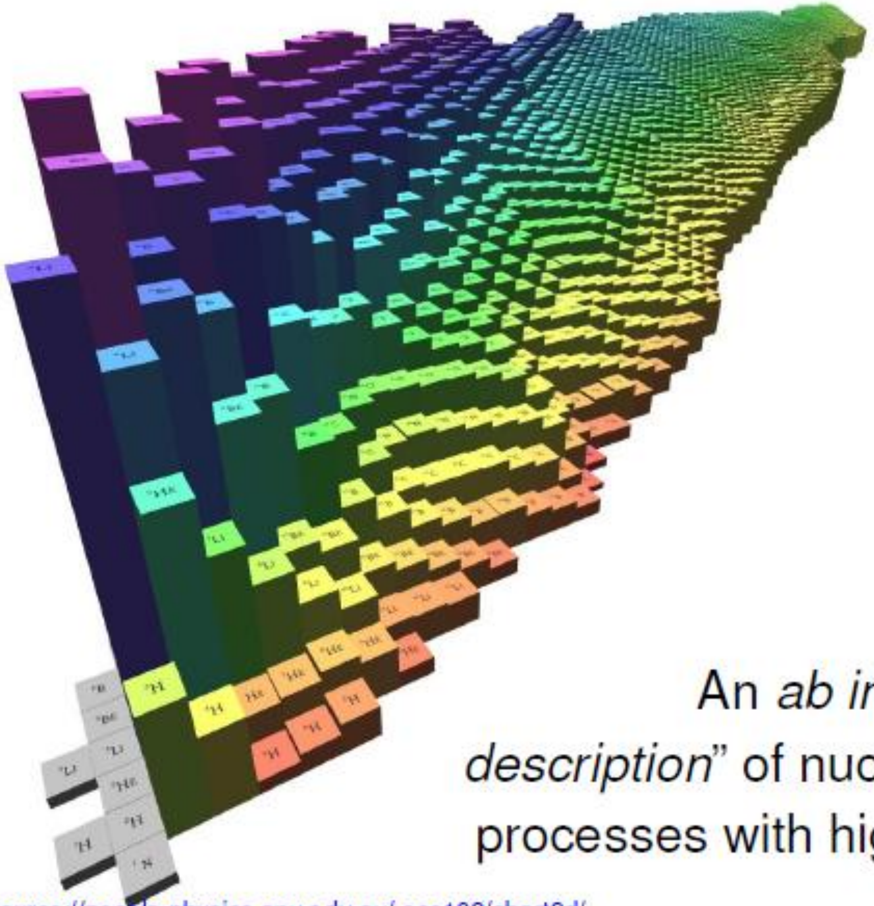
NLEFT Collaboration

Outline

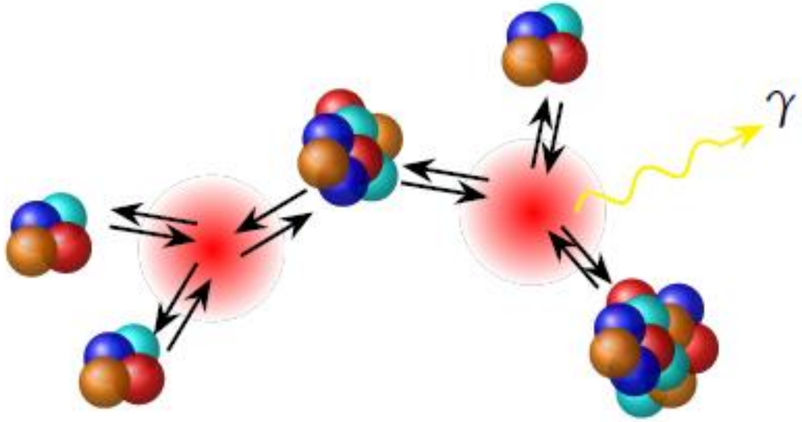
- Introduction to NLEFT
- Adiabatic Projection Method
- Alpha-alpha scattering
- Summary

Ab initio nuclear theory

The aim is to predict the properties of nuclear systems from microscopic nuclear forces

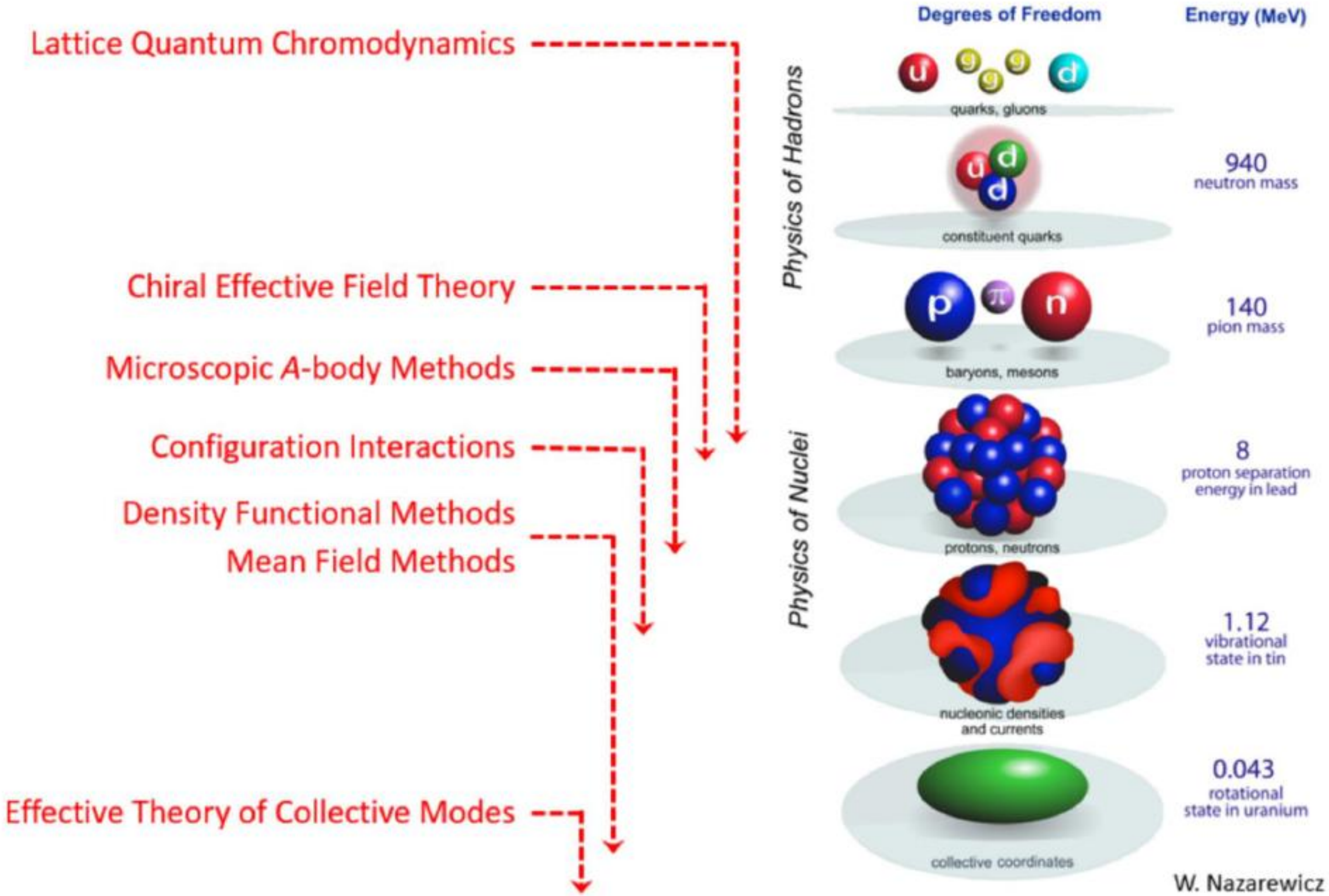


source://people.physics.anu.edu.au/ ecs103/chart3d/



An *ab initio* nuclear theory that has a "*unified description*" of nuclear structure and scattering/reaction processes with high predictive power.

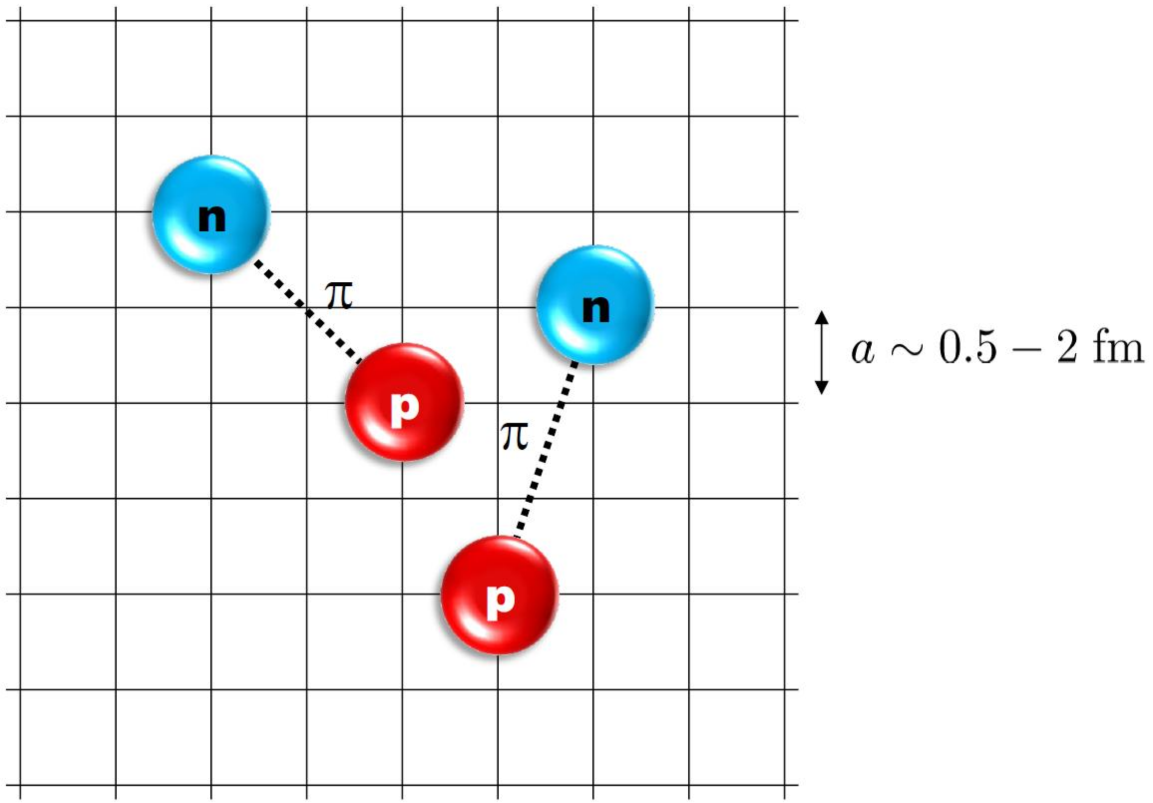
Hierarchy of Degrees of Freedom



Nuclear Lattice simulations

Lattice EFT = Chiral EFT + Lattice + Monte Carlo

- Discretized chiral nuclear force
- Lattice spacing $a \sim 1$ fm
- Protons and neutrons interacting via short-range and long-range pion exchange interactions
- Exact method, polynomial scaling ($\sim A^2$)



[D.L, Prog. Part. Nucl. Phys. 63 117-154 (2009)]

[Lähde, Meißner, Nuclear Lattice Effective Field Theory (2019), Springer]

Chiral Effective Field Theory

In Chiral EFT, the nuclear forces are systematically constructed order by order.

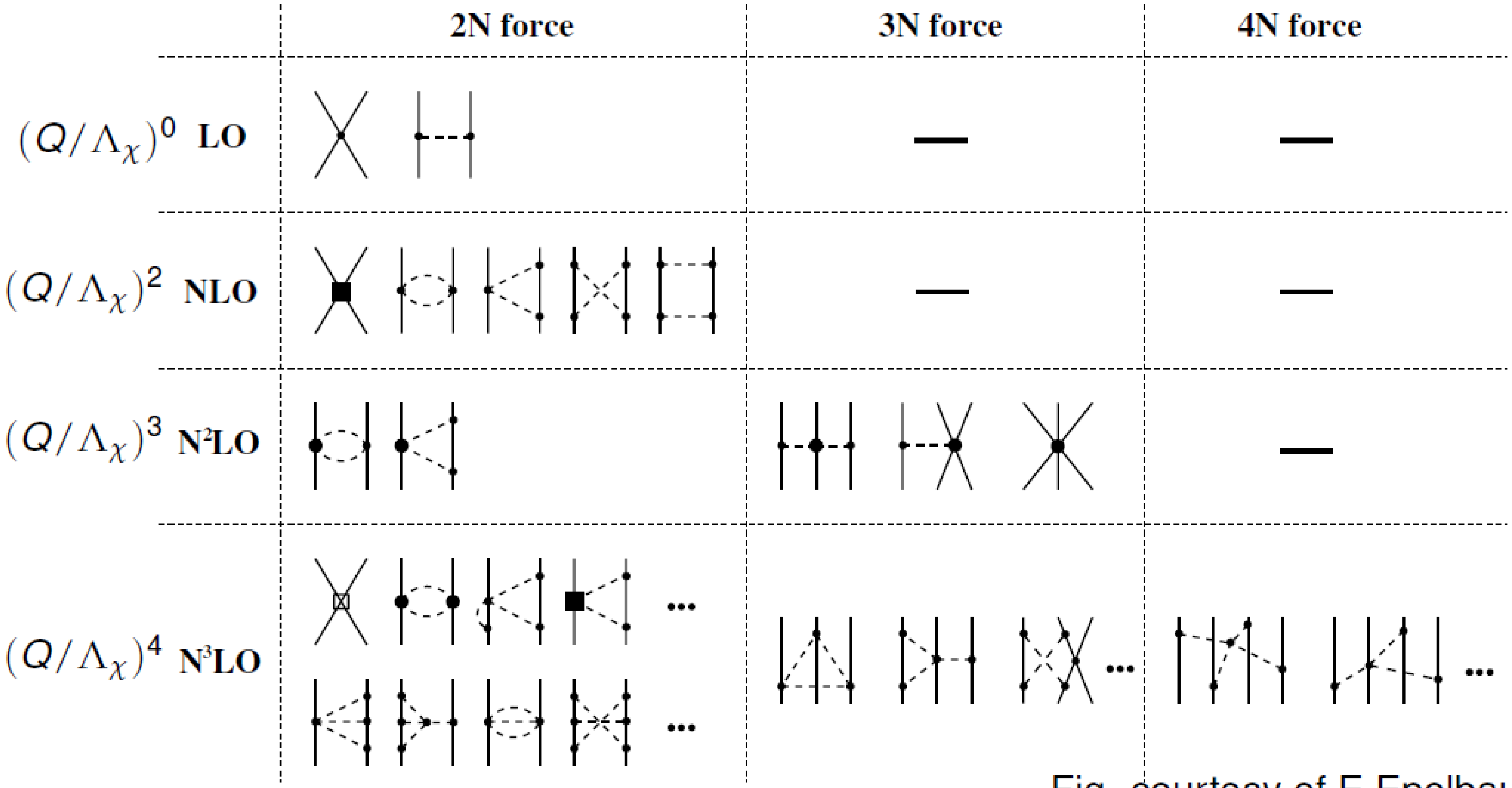


Fig. courtesy of E.Epelbaum

Euclidean Time Evolution

$$\exp\left[-\widetilde{H}t\right]|\psi\rangle = \exp\left[-\widetilde{H}t\right]\sum_i c_i|\psi\rangle_i$$

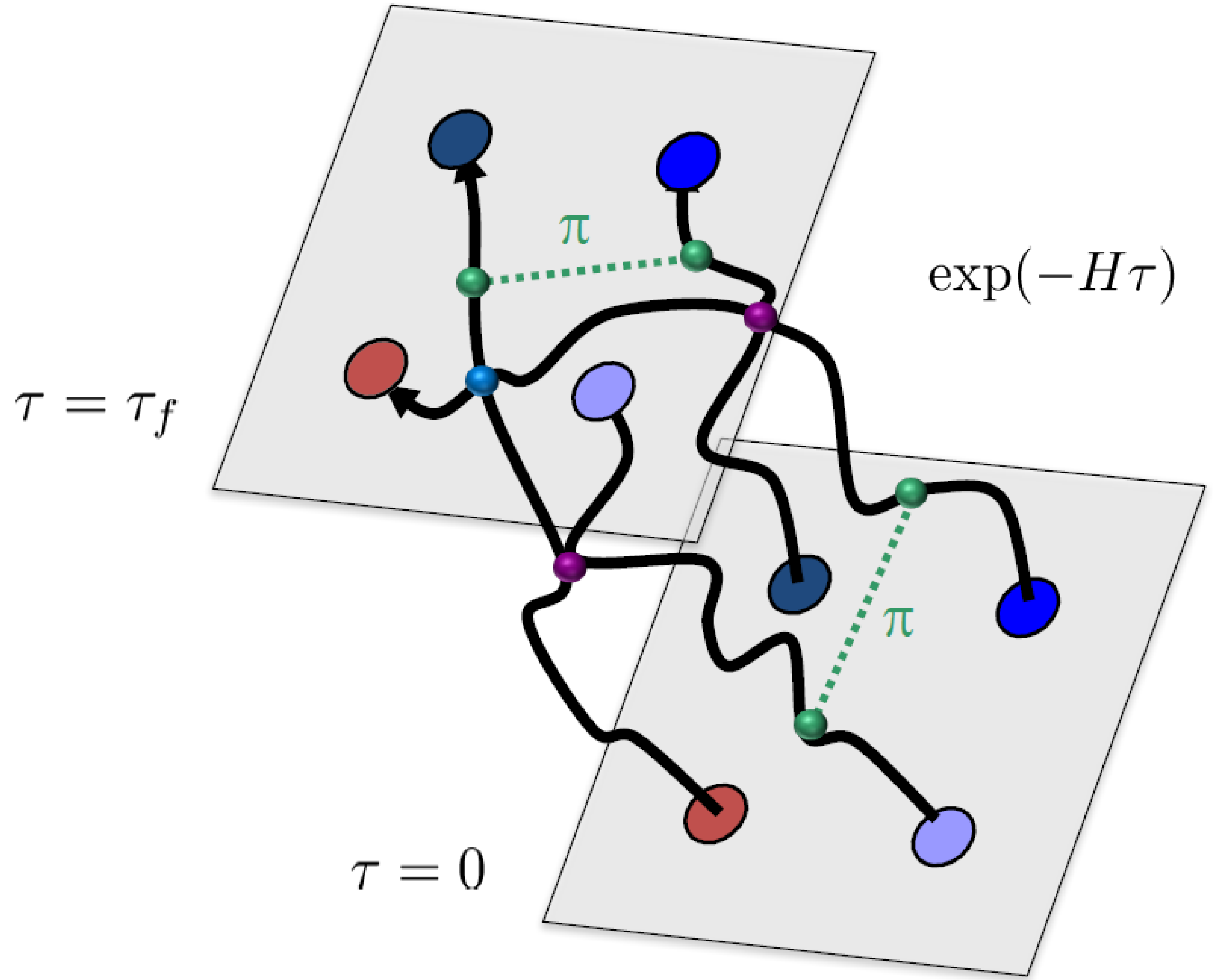
$$= \exp[-H_0 t]c_0|\psi\rangle_0 + \sum_{i>0}\exp[-H_i t]\, c_i|\psi\rangle_i$$

$$= \exp[-H_0 t]c_0\left[|\psi\rangle_0 + \sum_{i>0}\exp[-(H_i - H_0)t]\, c_i|\psi\rangle_i\right]$$

$$\lim_{t\to\infty}\exp\left[-\widetilde{H}t\right]|\psi\rangle \approx c_0\exp[-H_0 t]\,|\psi\rangle_0$$

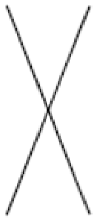
- Low energy filter.
- Excited states can be estimated using multi-channel calculations.

Euclidean time projection



Auxiliary field method

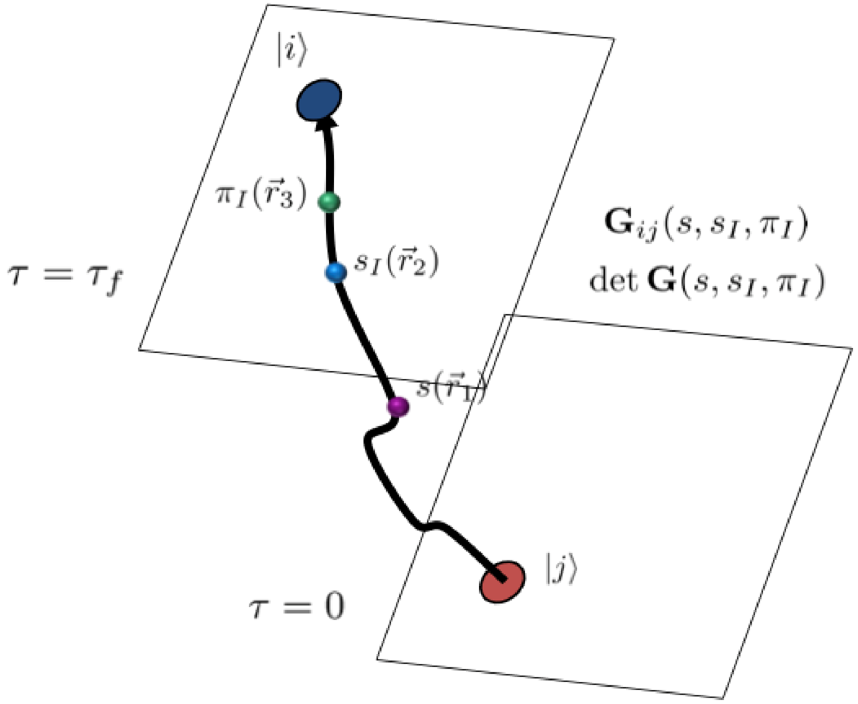
$$\exp\left[-\frac{C}{2}\left(N^\dagger N\right)^2\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} ds\, \exp\left[-\frac{1}{2}s^2 + \sqrt{-C}s\left(N^\dagger N\right)\right]$$



$(N^\dagger N)^2$

$sN^\dagger N$

Nucleon-nucleon interaction → Nucleon interacting with background field



Many-body amplitude for A-body = Determinant of *A x A* matrix of single nucleon amplitudes

[1] - B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, U.-G Meißner, Eur. Phys. J. A 31, 105 (2007)
[2] - S. Elhatisari, E. Epelbaum, H. Krebs, et al., Phys. Rev. Lett. 119, 222505 (2017)

Quantum Monte Carlo

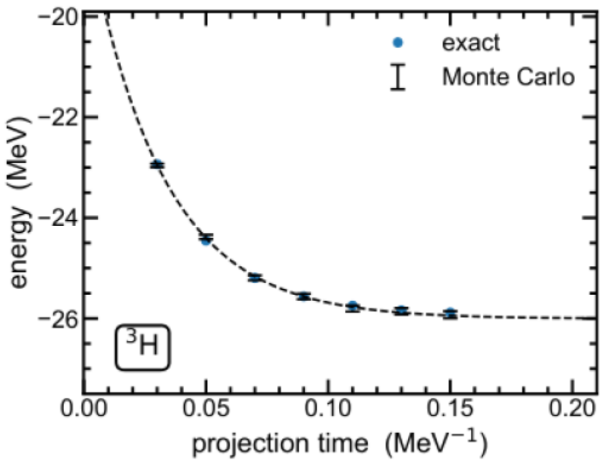
$$\hat{O}_{Lt} = \frac{\sum \frac{\langle \psi_{init} | e^{-H\delta t} | \dots | \hat{O} | \dots | e^{-H\delta t} | \psi_{init} \rangle}{|\langle \psi_{init} | e^{-H\delta t} | \dots | e^{-H\delta t} | \psi_{init} \rangle|}}{\sum \frac{\langle \psi_{init} | e^{-H\delta t} | \dots | e^{-H\delta t} | \psi_{init} \rangle}{|\langle \psi_{init} | e^{-H\delta t} | \dots | e^{-H\delta t} | \psi_{init} \rangle|}}$$

Lt steps for Euclidean Time Evolution (numerator)

Importance sampling (denominator)

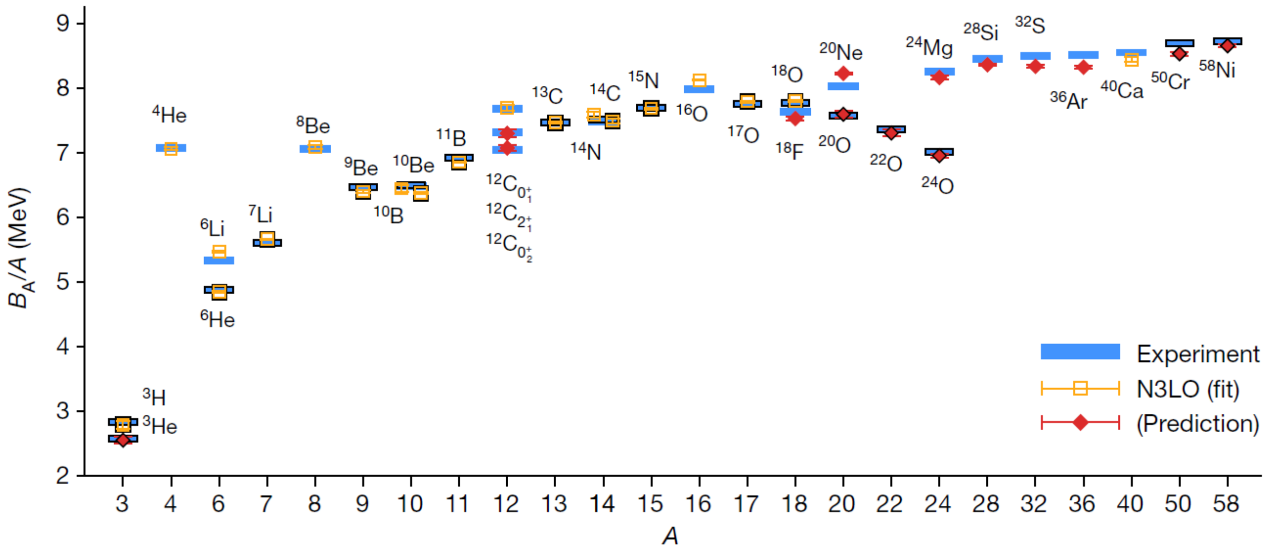
$$\bar{O} = \lim_{Lt \to \infty} \hat{O}_{Lt}$$

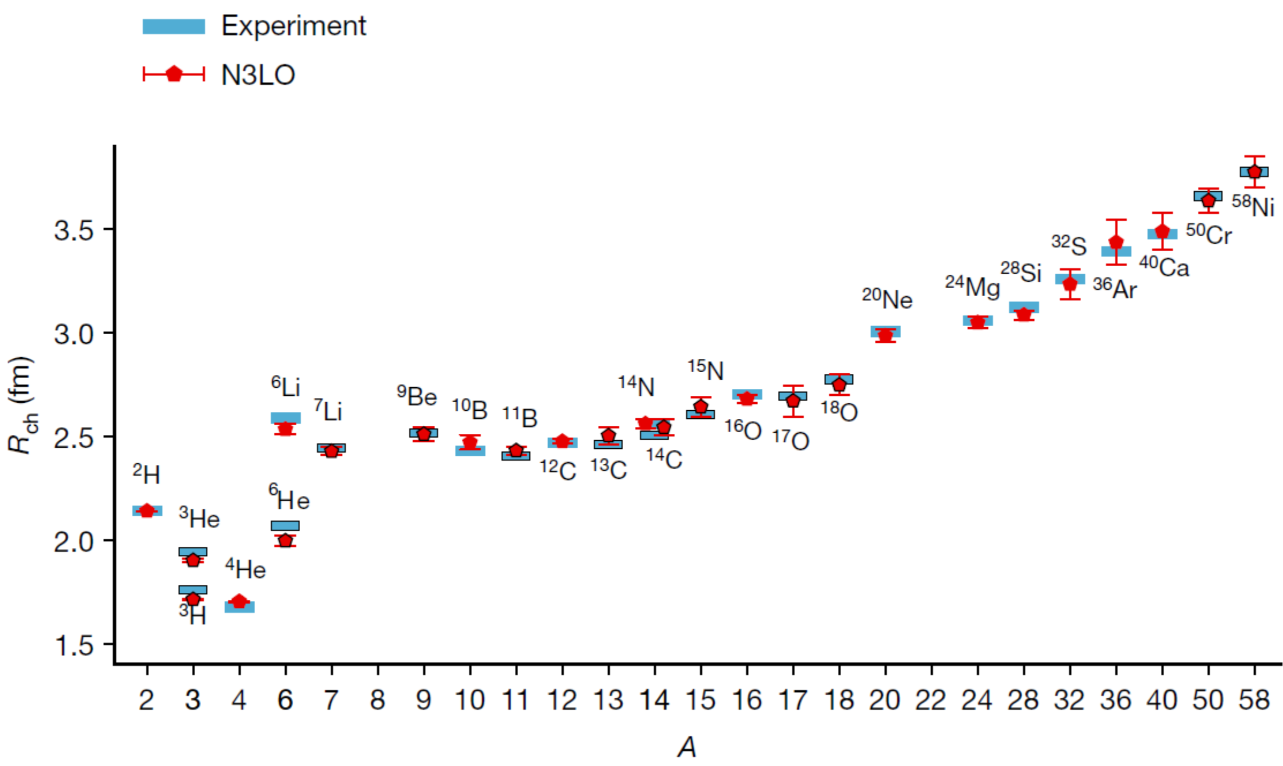
Extrapolated from different Lt data



- Matrix determinant should not fluctuate strongly in sign or complex phase - Sign problem!

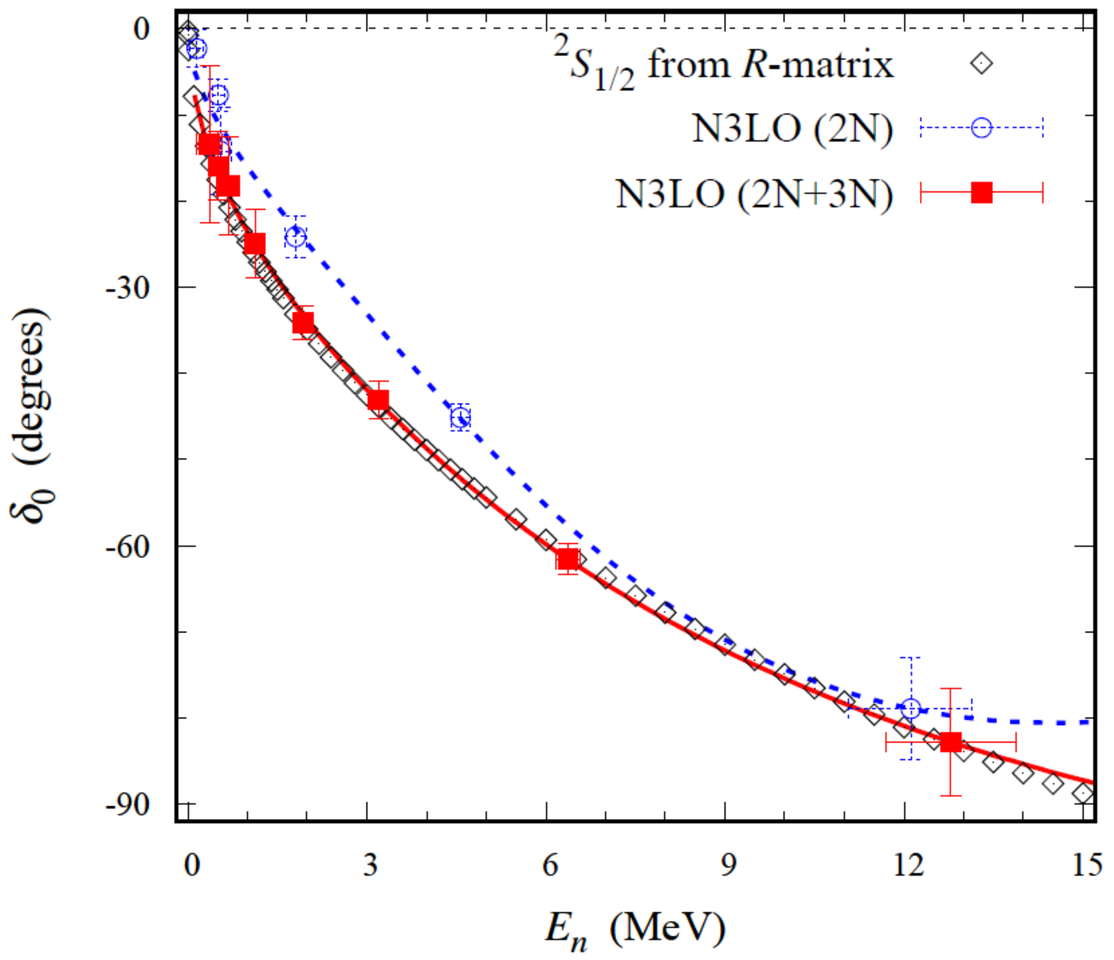
Binding energies and charge radii

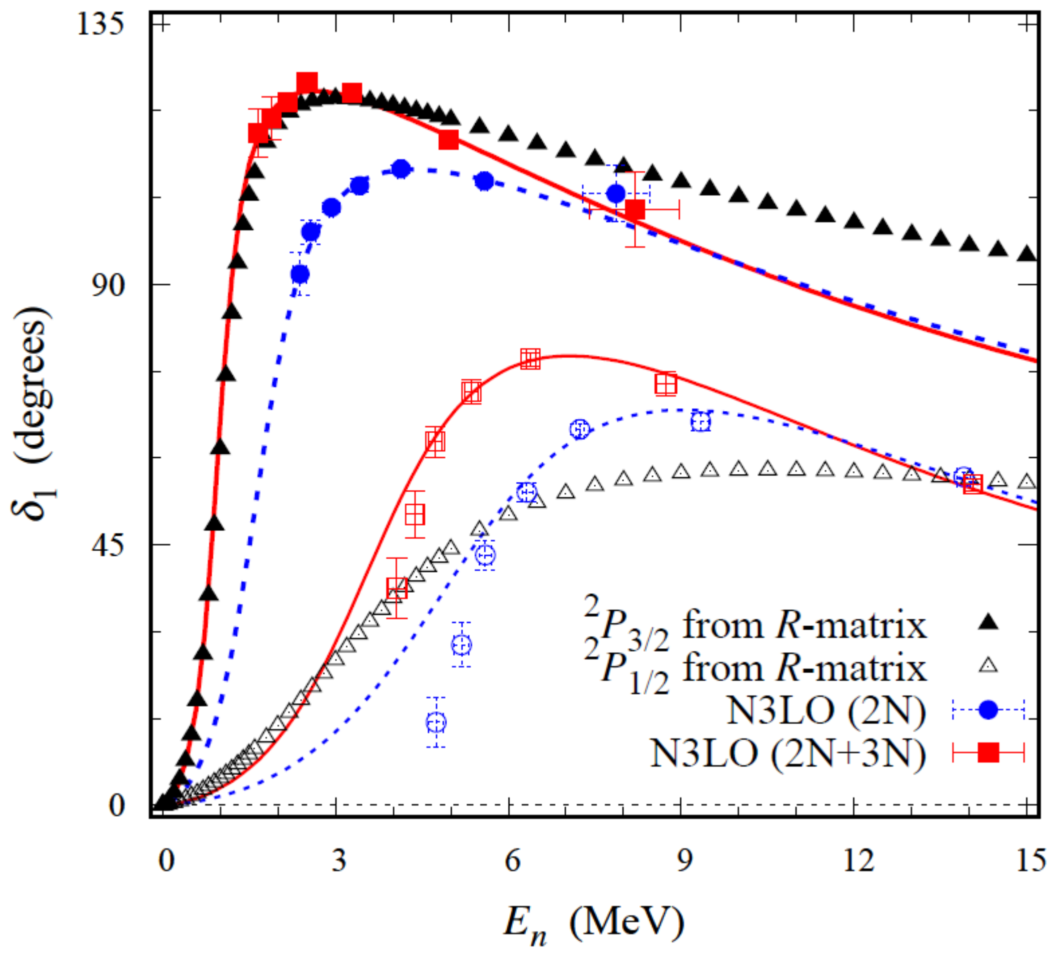




[1] – S. Elhatisari *et al.*, *Nature* **630**, 59–63 (2024)

n-alpha phase shifts from Lüscher formalism





[1] – S. Elhatisari *et al.*, *Nature* **630**, 59–63 (2024)

Adiabatic Projection Method

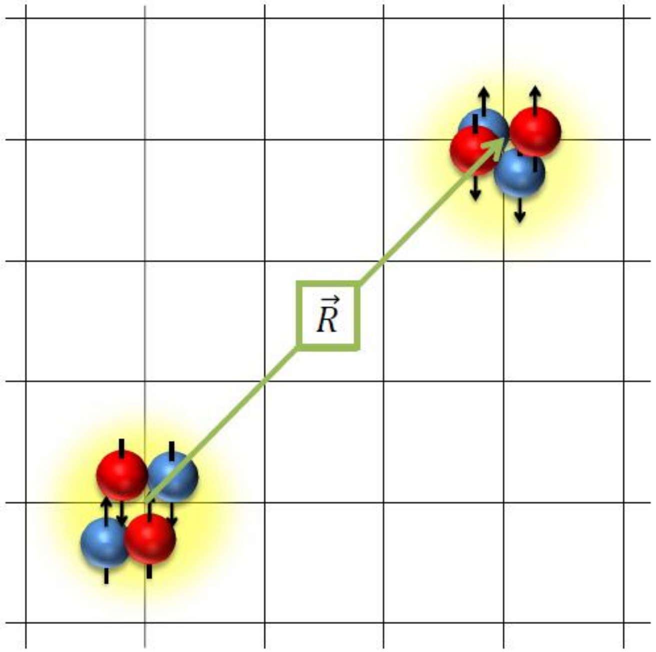
- Initial cluster states with relative separation $\vec{R}$

$$|\vec{R}\rangle = \sum_{\vec{r}} |\vec{r} + \vec{R}\rangle \otimes |\vec{r}\rangle$$

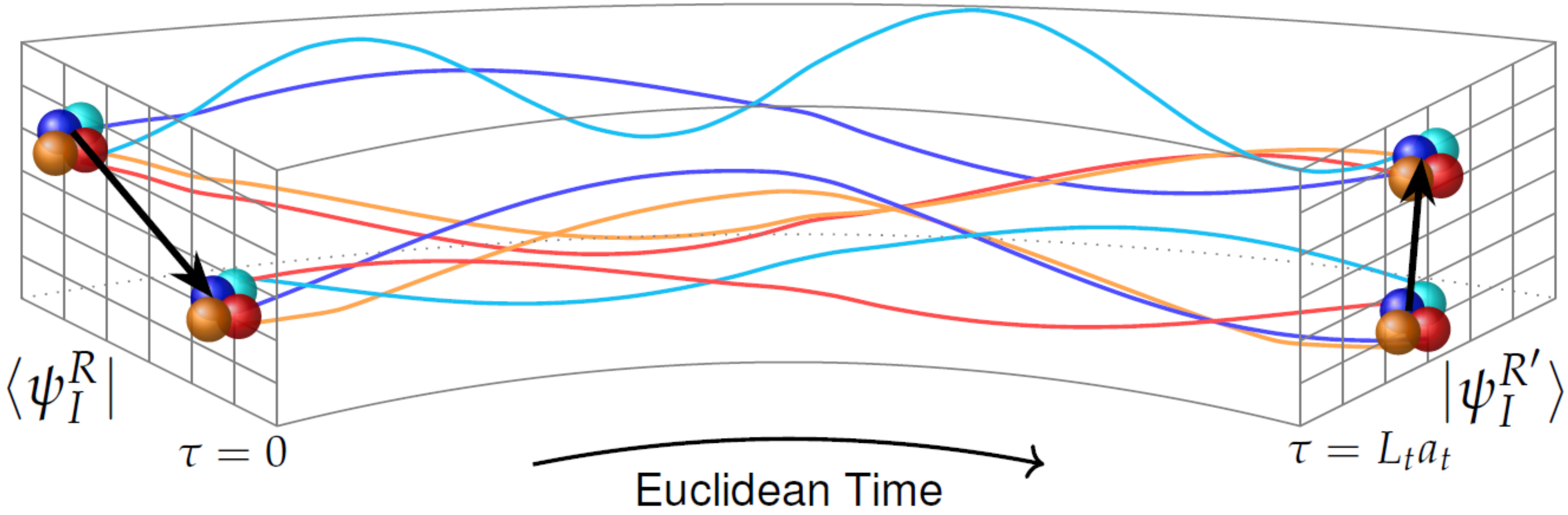
- Time evolved 'dressed cluster states'

$$|\vec{R}\rangle_\tau = \exp(-H\tau)|\vec{R}\rangle$$

  - H is full A-body interacting microscopic Hamiltonian
  - $\tau = L_t a_t$
  - $\lim_{\tau\to\infty} \{|\vec{R}\rangle_\tau\}$ spans low-energy two-cluster subspace



[1] M. Pine, D. Lee, G. Rupak, *Eur. Phys. J. A* 49, 151 (2013)
[2] S. Elhatisari, D. Lee, Phys. Rev. C 90, 064001
[3] S. Elhatisari, D. Lee, G. Rupak, E. Epelbaum, H. Krebs, T. Lähde, T. Luu, & U-G. Meißner. Nature 528, 111-114 (2015).
[4] S. Elhatisari, D. Lee, U-G. Meißner, G. Rupak, Eur. Phys. J. A 52, 174 (2016)
[5] S. Elhatisari, T. Lähde, D. Lee, U-G. Meißner, T. Vonk, *J. High Energ. Phys.* 2022



- Projected Hamiltonian

$$[H_\tau]_{\vec{R},\vec{R}'} = {}_\tau\langle \vec{R} | H | \vec{R}' \rangle_\tau$$

Norm

$$[N_\tau]_{\vec{R},\vec{R}'} = {}_\tau\langle \vec{R} | \vec{R}' \rangle_\tau$$

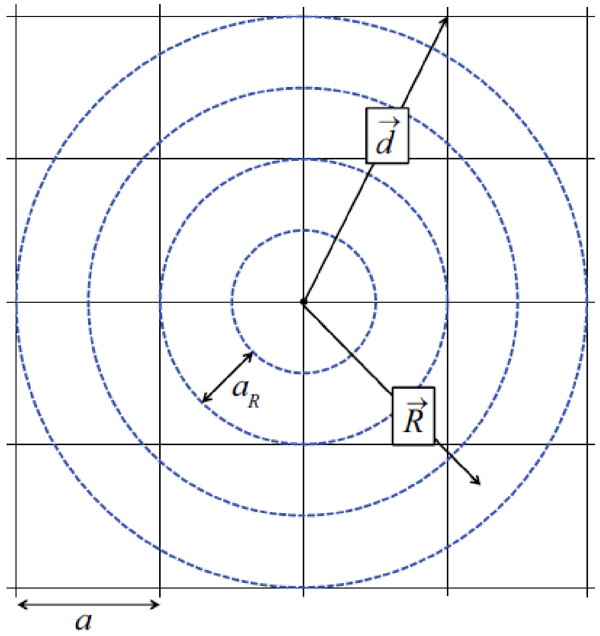
- Adiabatic Hamiltonian

$$[H_\tau^a]_{\vec{R},\vec{R}'} = \left[N_\tau^{-1/2} H_\tau N_\tau^{-1/2} \right]_{\vec{R},\vec{R}'}$$

Adiabatic Hamiltonian

- 10x10 matrix → 100 elements to be computed → multi-channel calculations.
- Lowest eigenvalue → direct one-channel calculation (after infinite Euclidean time propagation).
- $E_{8_{Be}}(Lt = 20, N3LO) = -57.34 \pm 0.05$
  $E_{APM}(Lt = 20, N3LO) = -59.34 \pm 0.74$
- Information of the clusters at different distances → can extract phase shifts with spherical wall.
- Adiabatic projection method is better than Lüscher’s method for NLEFT

Radial binning



L	$[M^{a}_{L_t}]_{\vec{d},\vec{d}'}$	$[M^{a}_{L_t}]^{0,0}_{d,d'}$	$[M^{a}_{L_t}]^{0,0}_{R,R'}$	
			$a_R = 0.125$ l.u.	$a_R = 0.250$ l.u.
10	$10^3 \times 10^3$	22×22	21×21	14×14
20	$20^3 \times 20^3$	85×85	58×58	34×34
30	$30^3 \times 30^3$	189×189	97×97	54×54
40	$40^3 \times 40^3$	335×335	137×137	74×74
50	$50^3 \times 50^3$	522×522	177×177	94×94
60	$60^3 \times 60^3$	752×752	217×217	114×114

- Project initial states on spherical harmonics

$$|\vec{d}\,\rangle = \sum_{\vec{n}} |\vec{n} + \vec{d}\,\rangle_1 \otimes |\vec{n}\,\rangle_2 \quad \Rightarrow \quad |d\rangle^{\ell,\ell_z} = \sum_{\vec{d}\,'} \mathrm{Y}_{\ell,\ell_z}(\hat{d}')\delta_{d,|\vec{d}'|}\,|\vec{d}'\rangle$$

and bin the points together

$$|R\rangle^{\ell,\ell_z} = \sum_{|d-R|<a_R/2} |d\rangle^{\ell,\ell_z}$$

[1] S. Elhatisari, D. Lee, U-G. Meißner, G. Rupak, Eur. Phys. J. A 52, 174 (2016)

Constructing large box Adiabatic Hamiltonian

- Low energy phase shifts $\rightarrow$ large lattice box size $\rightarrow$ full interacting calculation too expensive
- Two separate calculations – 1. Full interacting microscopic Hamiltonian in small box, 2. Non-interacting Hamiltonian in large box

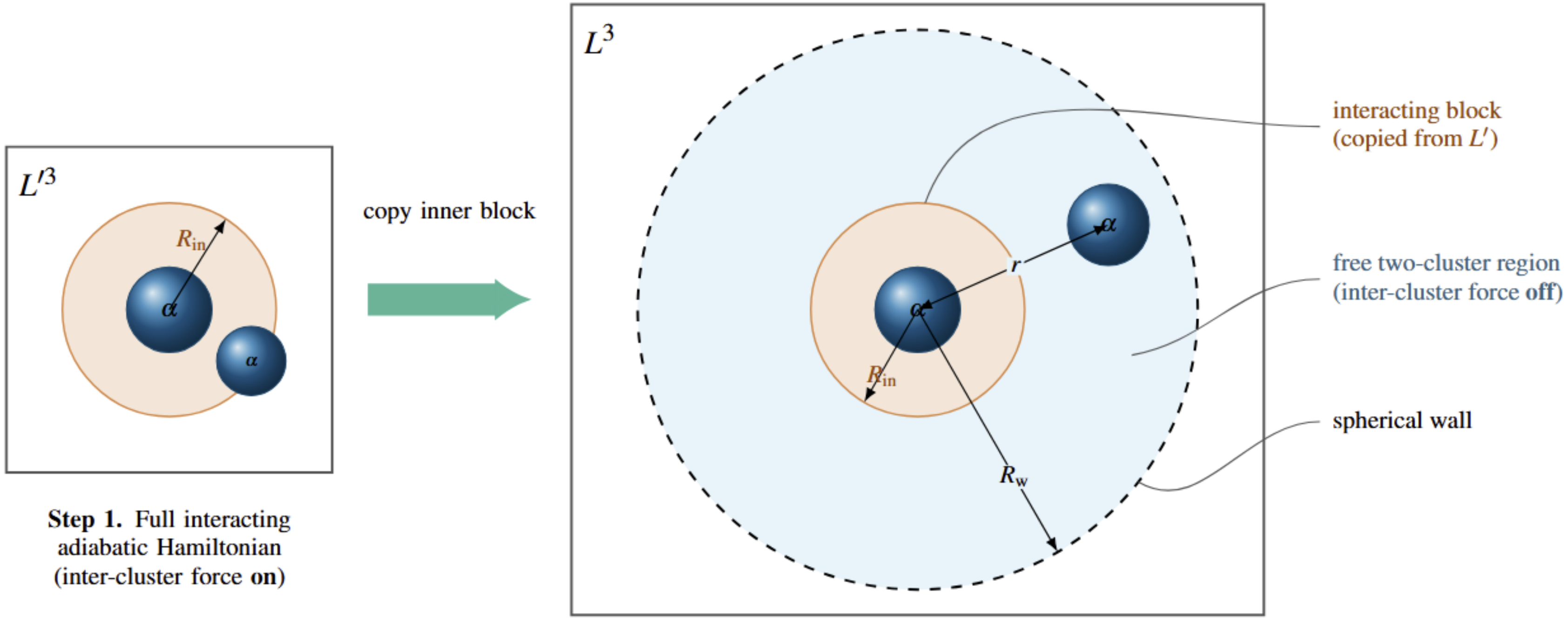
$$\underbrace{\begin{bmatrix} H_{int} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & H_{int} \end{bmatrix}}_{10\times10}$$

$$H_{interacting}$$

copy

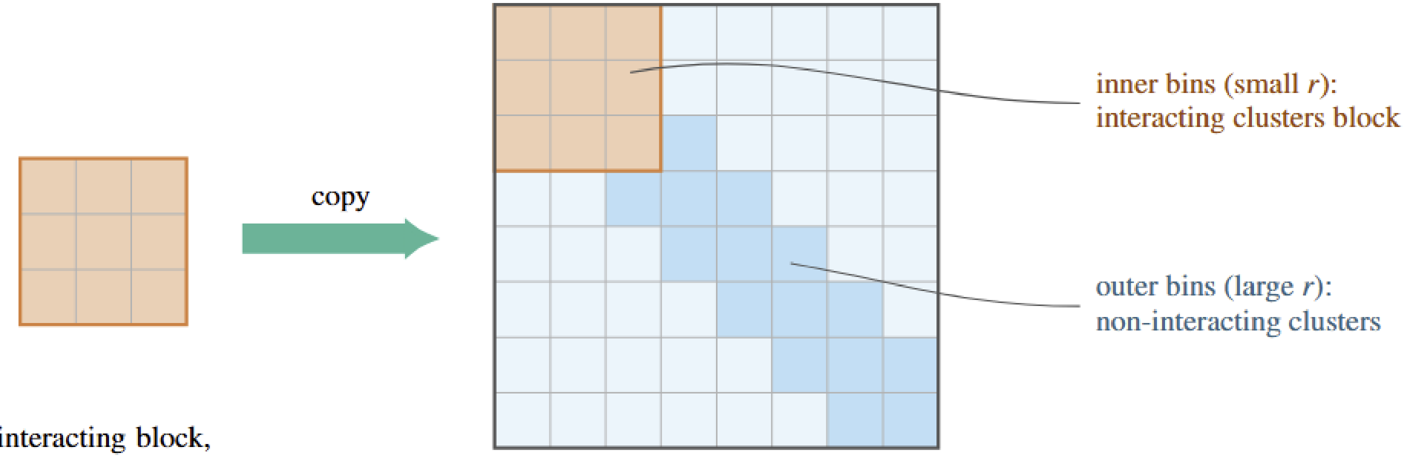
$$\underbrace{\begin{bmatrix} \begin{bmatrix} H_{int} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & H_{int} \end{bmatrix}_{10\times10} & \cdots & H_{free} \\ \vdots & \ddots & \vdots \\ H_{free} & \cdots & H_{free} \end{bmatrix}}_{50\times50}$$

$$H_{Adiabatic}$$



Step 1. Full interacting adiabatic Hamiltonian (inter-cluster force **on**)

Step 2. Free clusters out to the wall, with the interacting block inserted at short range



interacting block, small box L'

full Hamiltonian, large box L

Phase shift calculation

- Find eigenvector of Adiabatic Hamiltonian and look at asymptotic region

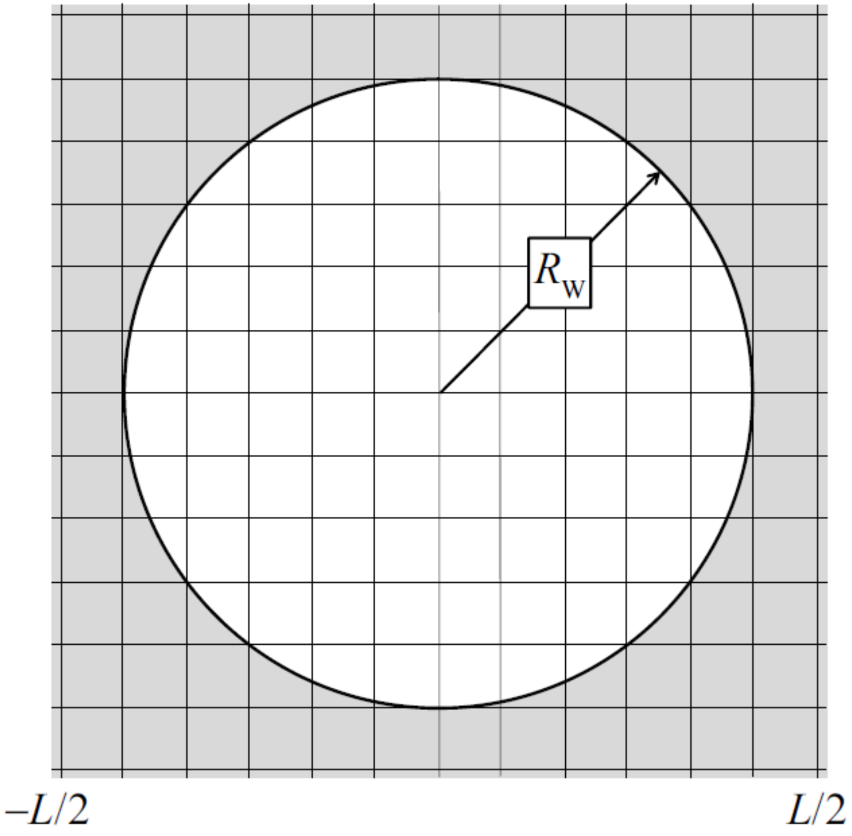
$$R_\ell^{(p)}(r) = N_\ell(p) \times \begin{cases} \cot\delta_\ell(p)\, j_\ell(p\,r) - n_\ell(p\,r), \\ \qquad \text{for any finite-range potential,} \\ \cot\delta_\ell(p)\, F_\ell(p\,r) + G_\ell(p\,r), \\ \qquad \text{for charged clusters,} \end{cases}$$

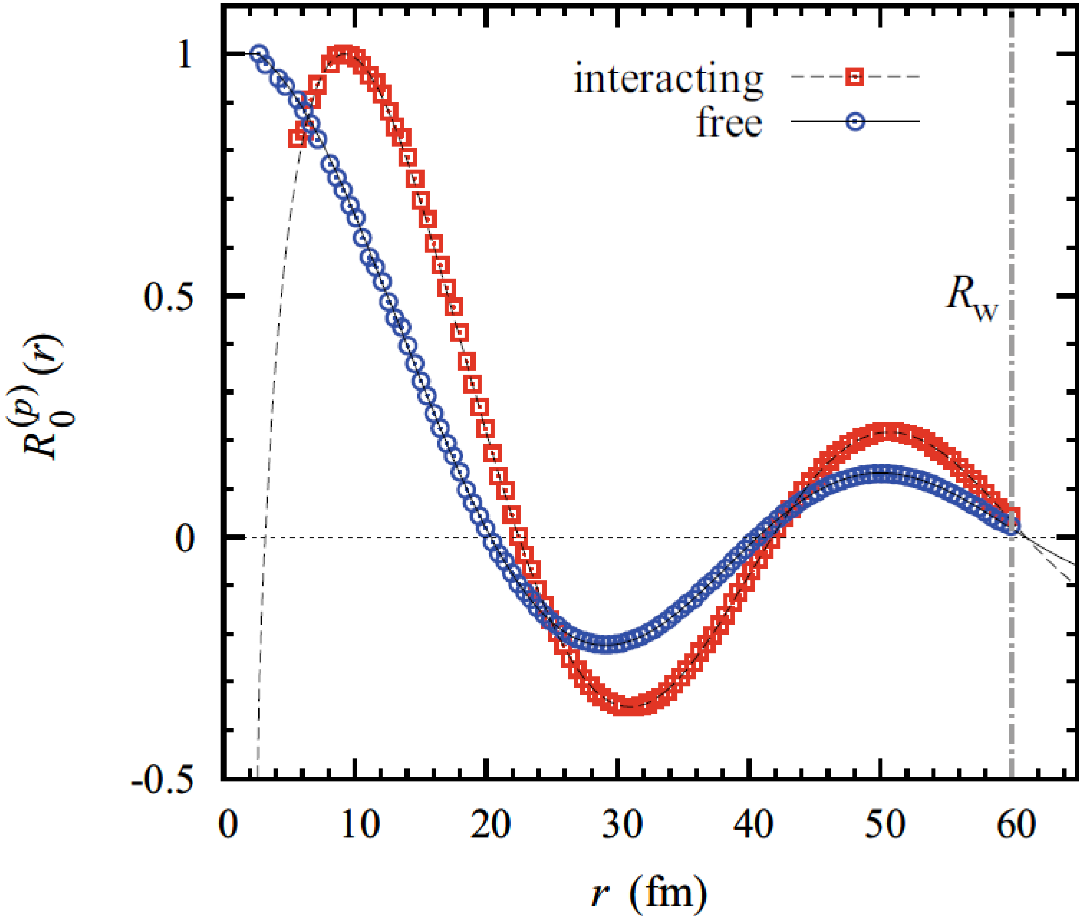
- Impose spherical hard wall and calculate phase shifts

$$\delta_\ell(p) = \begin{cases} \tan^{-1}\left[\dfrac{j_\ell(p\,R'_\mathrm{w})}{n_\ell(p\,R'_\mathrm{w})}\right], \\ \qquad \text{for any finite-range potential,} \\ -\tan^{-1}\left[\dfrac{F_\ell(p\,R'_\mathrm{w})}{G_\ell(p\,R'_\mathrm{w})}\right], \\ \qquad \text{for charged clusters.} \end{cases}$$

[1] J. Carlson, V.R. Pandharipande, R.B. Wiringa, Nucl. Phys. A 424, 47 (1984)
[2] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, U.-G. Meißner, Eur. Phys. J. A 34, 185 (2007)

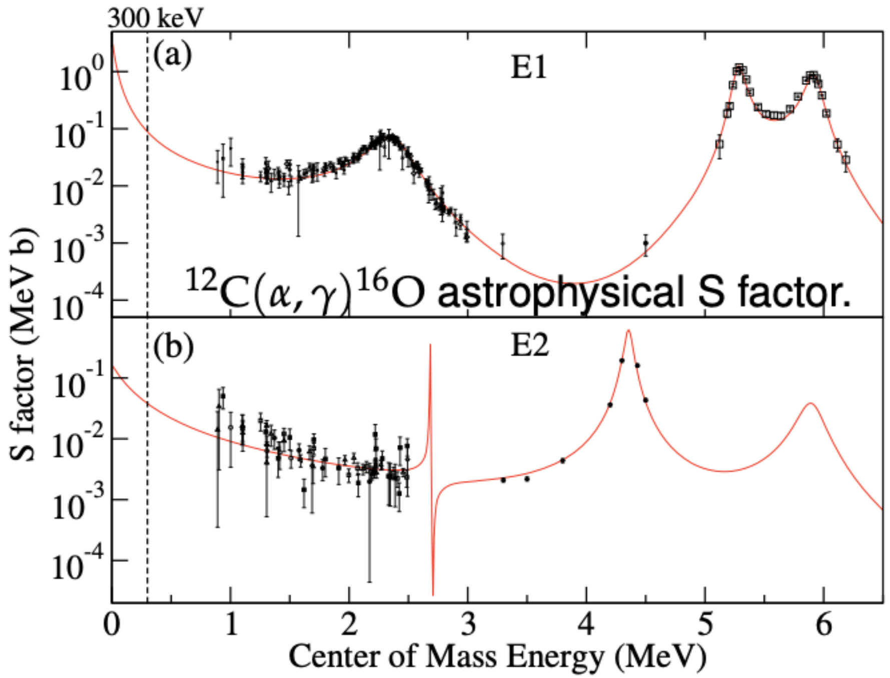
Spherical wall method



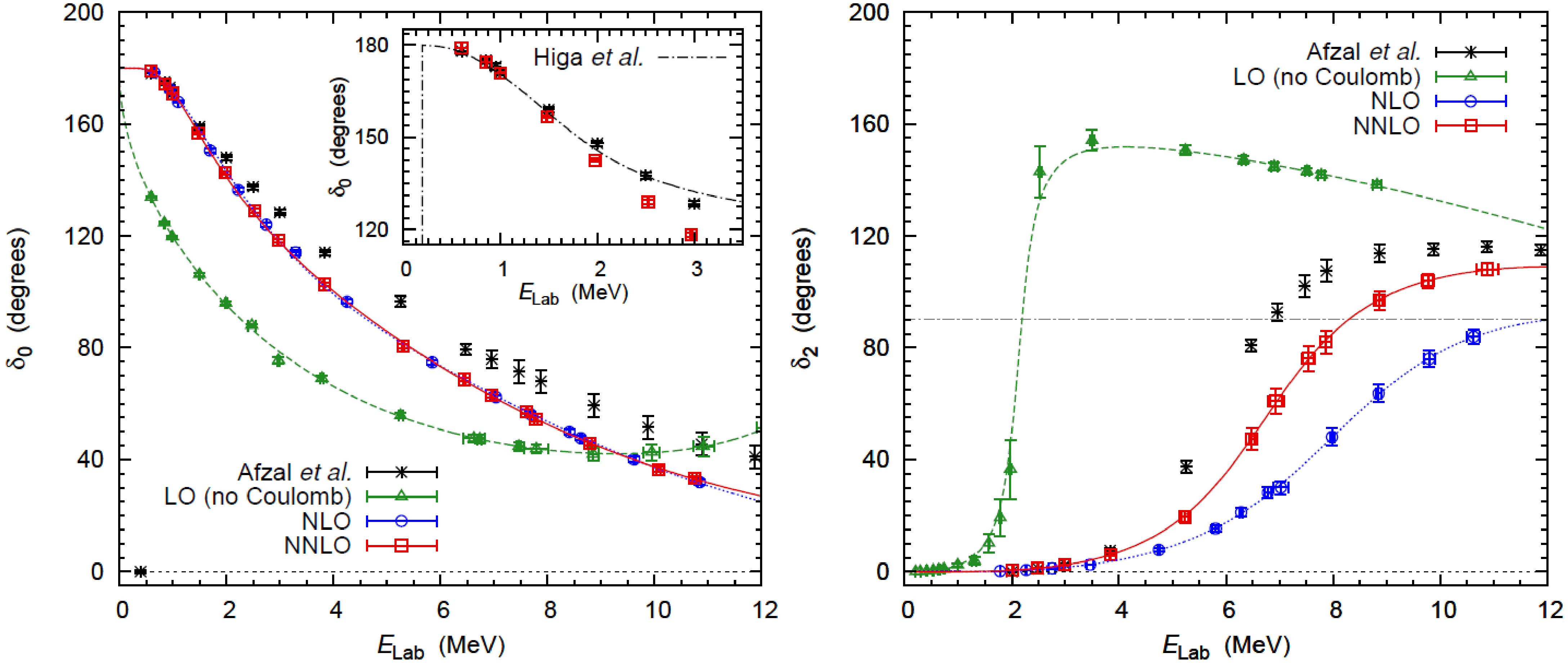


Alpha-alpha scattering

- One of the most fundamental reactions in nuclear astrophysics.
- Alpha scattering, triple-alpha reaction, alpha capture play major role in stellar nucleosynthesis.
- Direct experimental results at 300 keV is impossible due to Coulomb barrier.
- For accurate reaction rate calculation, we need cross section within energy range 0.15 – 3.4 MeV.
- Study scattering by adiabatic projection method



Previous alpha-alpha results

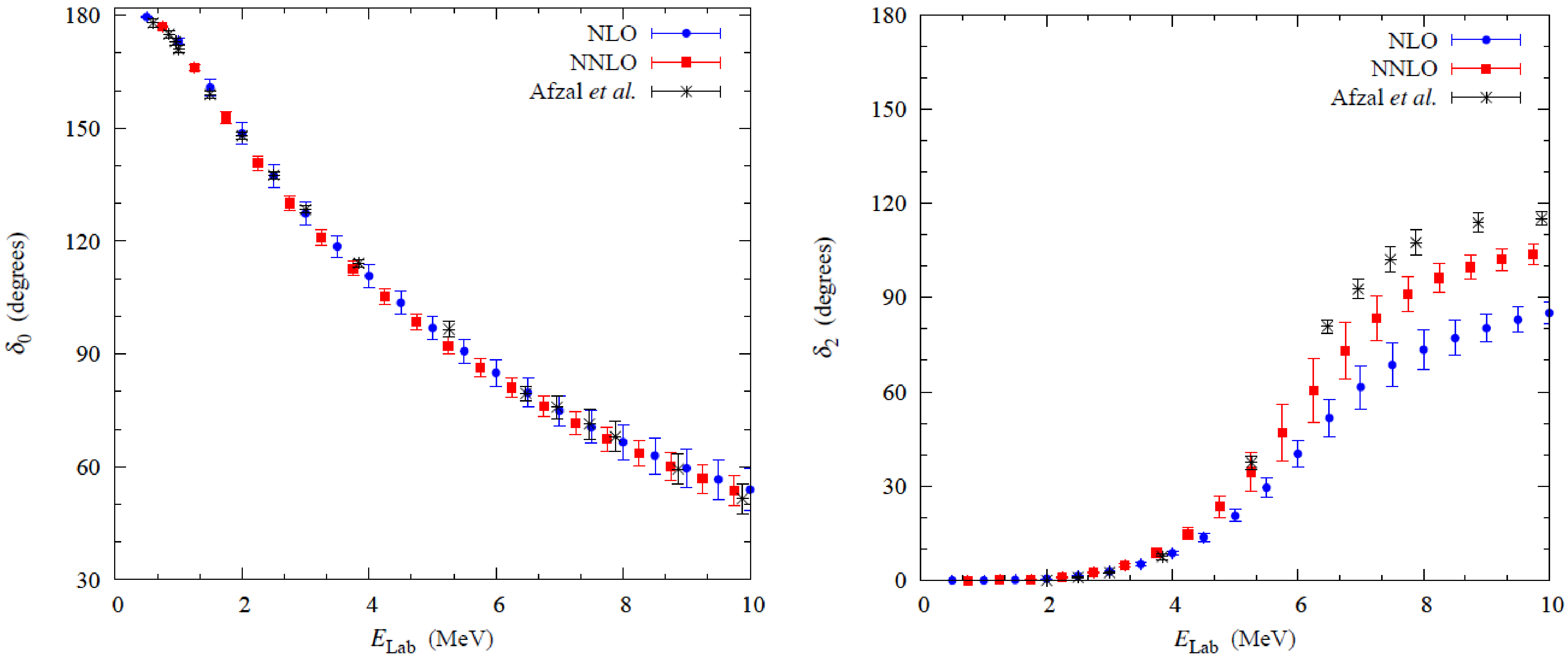


[1] Afzal, Ahmad, Ali, Rev. Mod. Phys. 41, 247, (1969)

[2] Higa, Hammer, van Kolck, Nucl.Phys. A809, 171 (2008), 0802.3426

[3] S. Elhatisari, D. Lee, G. Rupak, E. Epelbaum, H. Krebs, T. Lähde, T. Luu, & U-G. Meißner. Nature 528, 111-114 (2015).

More recent alpha-alpha results



[1] S. Elhatisari, T. Lähde, D. Lee, U-G. Meißner, T. Vonk, *J. High Energ. Phys.* 2022

What’s new?

- Finer discretization – lattice spacing a =1.97 *fm* vs 1.32 *fm*; temporal spacing $a_t = (150\ MeV)^{-1}$ vs $(1000\ MeV)^{-1}$.
- Improved interaction – NNLO vs N3LO with wavefunction matching.

Challenges

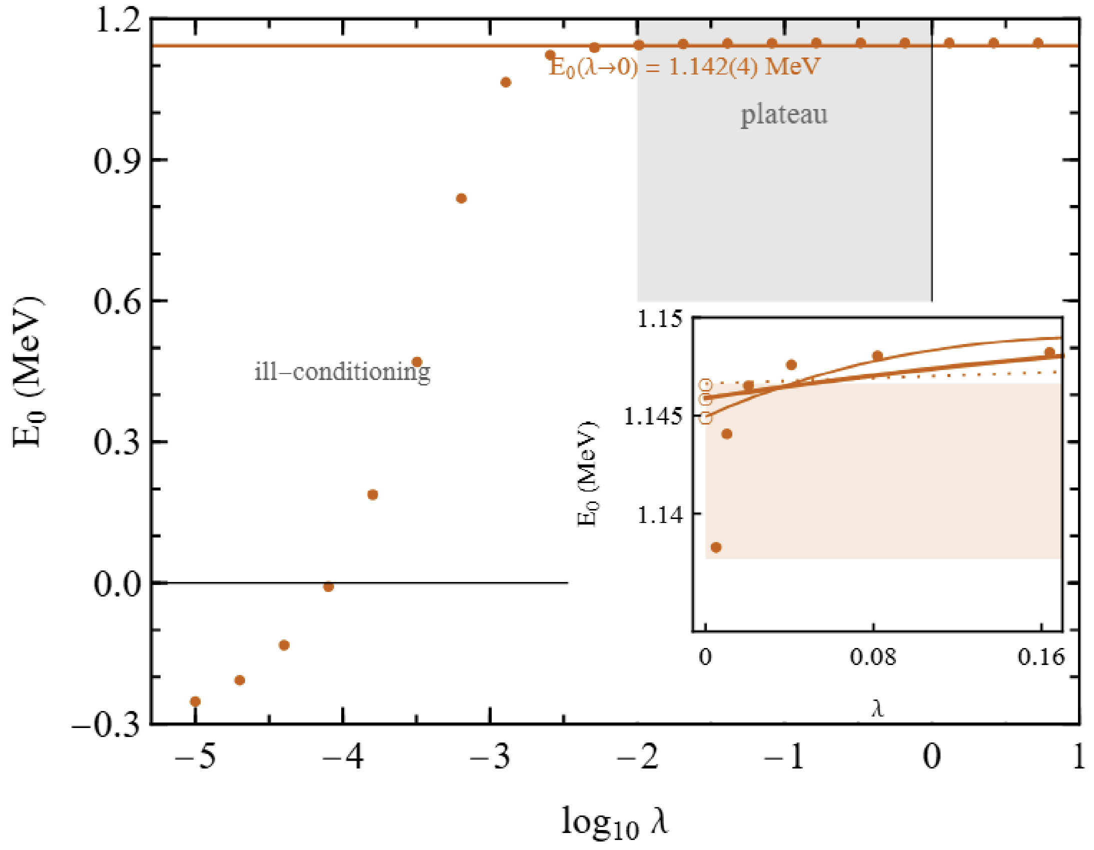
- APM needs to calculate N^{-1}. At fine lattice spacing, the two-cluster basis is difficult to finely resolve; often N is ill-conditioned.
- We need absurd amount of Monte Carlo statistics to be get meaningful results from standard APM.

Tikhonov regularization

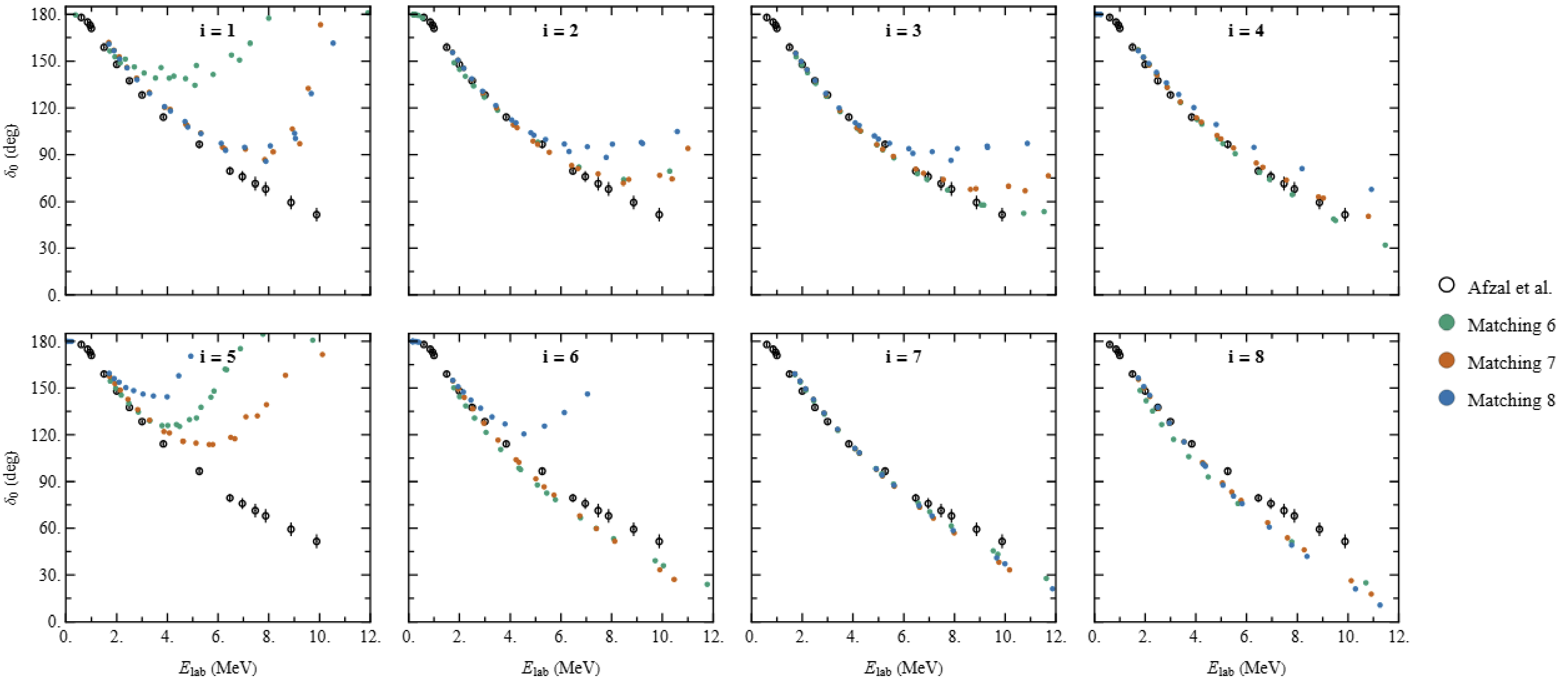
- $N \rightarrow N + \lambda B$, most common $B = I$.
- Extrapolate $\lambda \rightarrow 0$.
- Regularize selectively where needed

$$N'(\lambda) = U \begin{pmatrix} \Lambda_i + \lambda I_i & 0 \\ 0 & \Lambda_r \end{pmatrix} U^T$$

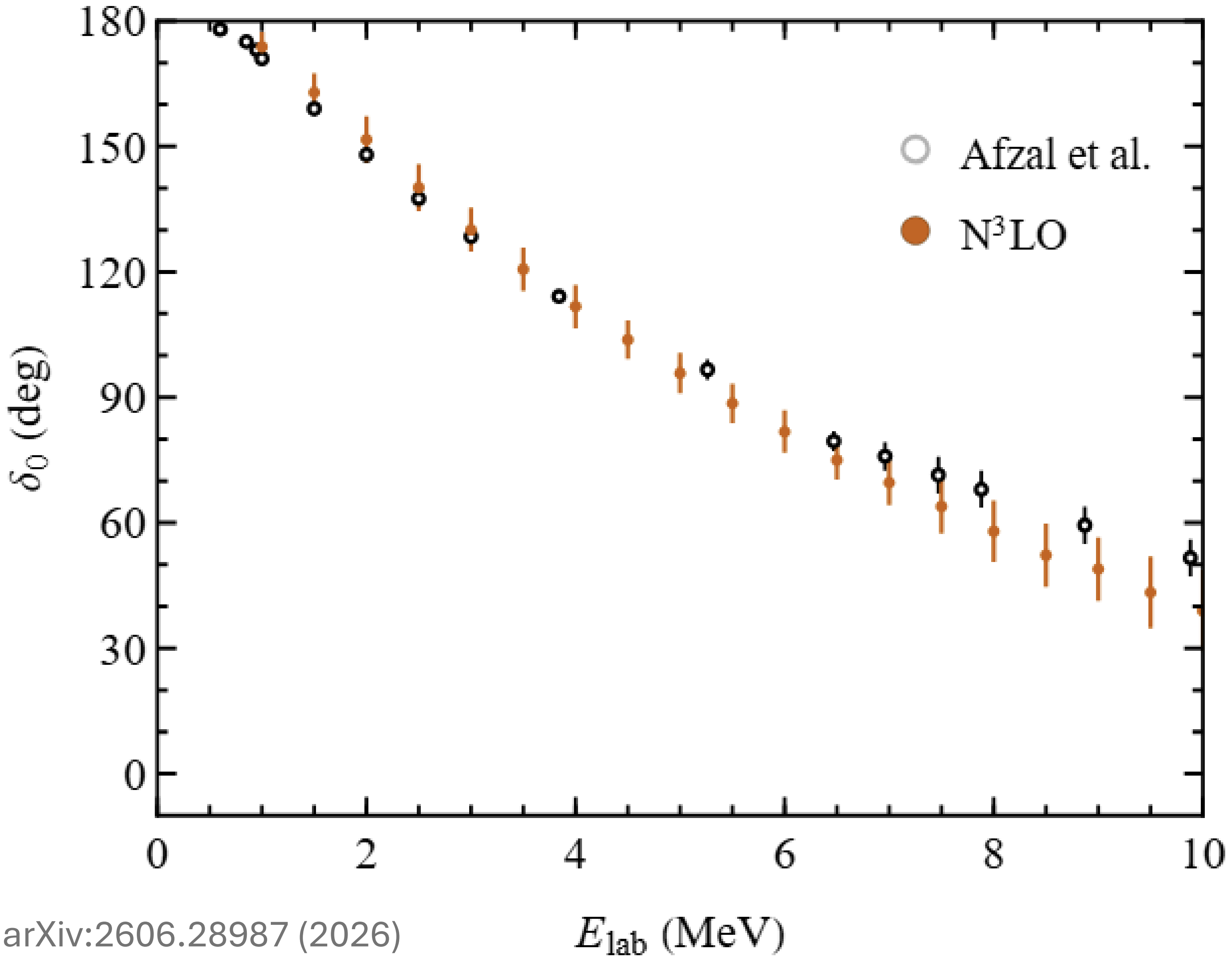
- Cross-verified with TSVD approach



Tikhonov regularization dimension

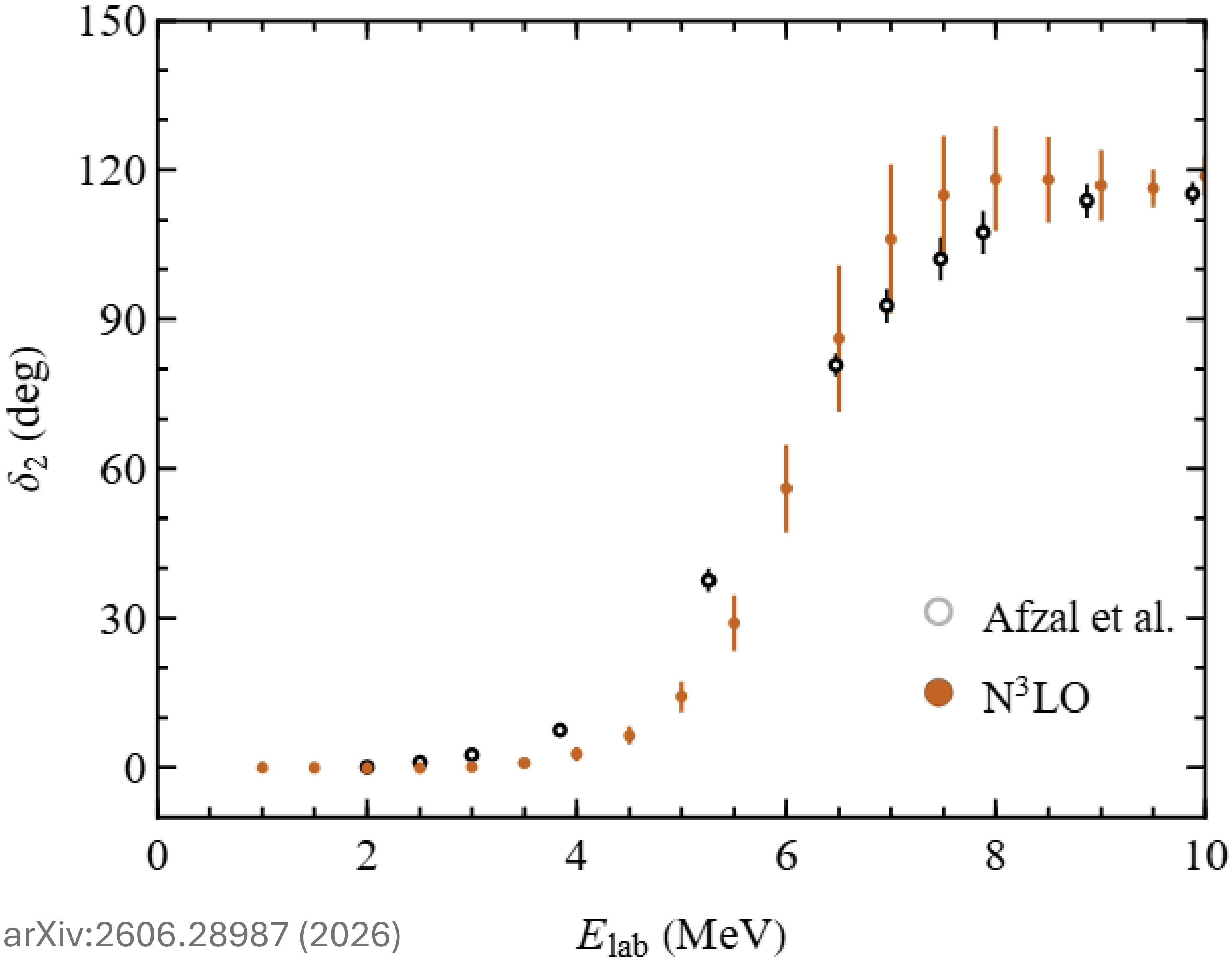


S-wave phase shifts



[1] – A. Sarkar et al., arXiv:2606.28987 (2026)

D-wave phase shifts



ERE fit:
$E_0 = 3.25 \pm 0.25\ MeV$
$\Gamma = 0.75 \pm 0.25\ MeV$

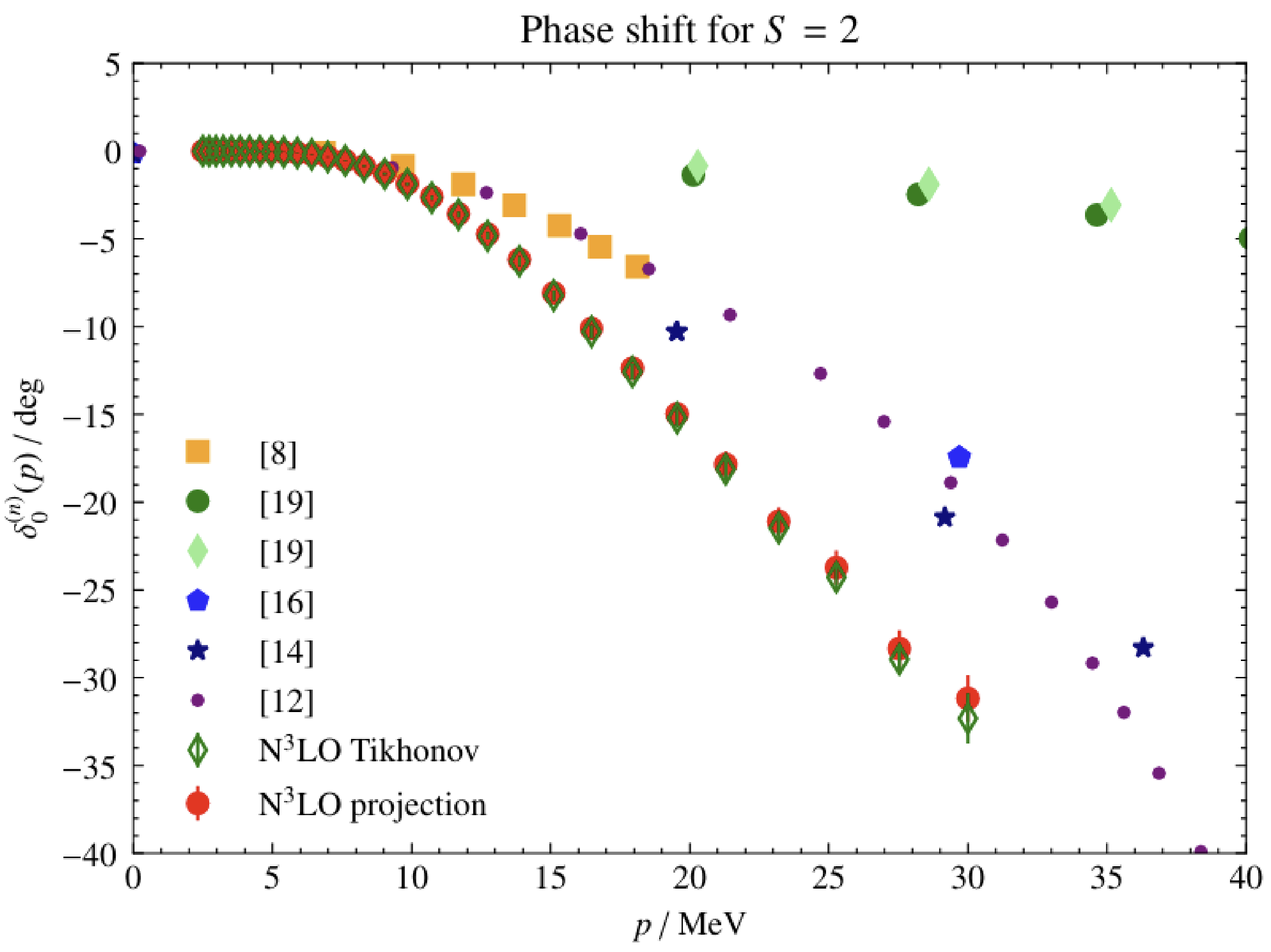
Emperical:
$E_0 = 2.92 \pm 0.18\ MeV$
$\Gamma = 1.34 \pm 0.50\ MeV$

[1] – A. Sarkar et al., arXiv:2606.28987 (2026)

Summary and Outlook

- Ab-initio phase shifts with NLEFT and APM.
- Expensive calculations and regularization of norm matrix is needed.
- Phase shifts not perfect – need to tune interaction further.
- Alpha-capture and d-d scattering (arXiv: 2607.00681) in progress.
- We can only study two-clusters with APM. We are working to construct a formalism to study resonances and break-up cross sections with two or more clusters in NLEFT – persistent state method (in preparation).

d-d scattering



Thank you