Multi-channel few-body reactions: momentum-space treatment

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Scattering: wave function vs transition operator

- Schrödinger equation

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+ impose asymptotic boundary conditions explicitly

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wave function $$|\psi\rangle = |\mathbf{k}\rangle + G_0 v|\psi\rangle$$

$$G_0 = (E + i0 - H_0)^{-1}$$

transition matrix $$T|\mathbf{k}\rangle = v|\psi\rangle$$

$$T = v + vG_0 T$$

$$|\psi\rangle = |\mathbf{k}\rangle + G_0 T|\mathbf{k}\rangle$$

Outline

- Momentum-space description of few-body scattering: screening and renormalization for Coulomb
  [Taylor, Alt, Sandhas, ...]
  [AD, Fonseca, Sauer]

- S&R variations and other methods

- Applications: 3N, 4N, nuclear reactions, ...

Screened Coulomb

$$w_R(r) = w_C(r)\,e^{-\left(\frac{r}{R}\right)^n}$$

- standard scattering theory

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  physical observables insensitive to screening,
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- nature: Coulomb is screened at large distances
- large R:
  physical observables insensitive to screening,
  screened and full Coulomb physically indistinguishable
- in the $R \to \infty$ limit physical results are recovered

Screened and full Coulomb physically indistinguishable

$$\langle \mathbf{p}' | T_R | \mathbf{p} \rangle \quad \overset{?}{\underset{R \to \infty}{\longrightarrow}} \quad \langle \mathbf{p}' | T_C | \mathbf{p} \rangle$$

Screened and full Coulomb physically indistinguishable

$$e^{2i\phi_R} \langle \mathbf{p}' | T_R | \mathbf{p} \rangle \quad \overset{?}{\underset{R \to \infty}{\longrightarrow}} \quad \langle \mathbf{p}' | T_C | \mathbf{p} \rangle$$

Screened and full Coulomb physically indistinguishable

initial physical state: wave packet $\varphi_{\rm in}(\mathbf{p})$

outgoing wave packet

$$\varphi_{\rm out}(\mathbf{p}') = \int d^3\mathbf{p}\, \langle \mathbf{p}'|S|\mathbf{p}\rangle \varphi_{\rm in}(\mathbf{p})$$

$$\sim \int d^2\hat{\mathbf{p}}\, e^{2i\phi_R} \langle \mathbf{p}'|T_R|\mathbf{p}\rangle \varphi_{\rm in}(\mathbf{p}) \xrightarrow[R\to\infty]{} \int d^2\hat{\mathbf{p}}\, \langle \mathbf{p}'|T_C|\mathbf{p}\rangle \varphi_{\rm in}(\mathbf{p})$$

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$p' = p:$ $\quad e^{2i\phi_R} \langle \mathbf{p}'|T_R|\mathbf{p}\rangle \xrightarrow[R\to\infty]{} \langle \mathbf{p}'|T_C|\mathbf{p}\rangle$ as distribution

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$$p' = p: \qquad e^{2i\phi_R}\langle \mathbf{p}'|T_R|\mathbf{p}\rangle \xrightarrow[R\to\infty]{} \langle \mathbf{p}'|T_C|\mathbf{p}\rangle \quad \text{as distribution}$$

$$\phi_R \xrightarrow[R\to\infty]{} [\sigma_L - \eta_{LR}] \xrightarrow[R\to\infty]{} \alpha_e M/p\,[\ln(2pR) - C/n]$$

[*J. R. Taylor, Nuovo Cimento* **B23**, *313 (1974)*]

Screened and full Coulomb wave functions

$$r < R: \qquad w_R(r) \approx w_C(r)$$

$$\Downarrow$$

$$e^{i\phi_{LR}} \langle r | \psi_{LR}^{(+)}(p) \rangle \approx \langle r | \psi_{LC}^{(+)}(p) \rangle$$

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$$\Downarrow$$

$$e^{i\phi_{LR}} \langle r | \psi_{LR}^{(+)}(p) \rangle \approx \langle r | \psi_{LC}^{(+)}(p) \rangle$$

$$e^{i\phi_R} | \psi_R^{(+)}(\mathbf{p}) \rangle \xrightarrow[R\to\infty]{} | \psi_C^{(+)}(\mathbf{p}) \rangle$$

[*V. G. Gorshkov, Sov. Phys.-JETP* **13**, *1037 (1961)*]

Screening and renormalization

Renormalization of the on-shell screened Coulomb transition matrix $T_R = w_R + w_R G_0 T_R$ and wave function in the limit $R \to \infty$ yields Coulomb amplitude and Coulomb wave function

$$T_R \, z_R^{-1} \xrightarrow[R\to\infty]{} T_C \qquad \text{as distribution}$$

$$(1 + G_0 T_R)|\mathbf{p}\rangle z_R^{-1/2} \xrightarrow[R\to\infty]{} |\psi_C^{(+)}(\mathbf{p})\rangle$$

$$z_R = e^{-2i\phi_R}$$

Two-particle scattering

transition matrix

$$T^{(R)} = v + w_R + (v + w_R) G_0 T^{(R)}$$

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with long-range and Coulomb-distorted short-range parts

$$T^{(R)} = T_R + (1 + T_R G_0) \tilde{T}^{(R)} (1 + G_0 T_R)$$

$$\tilde{T}^{(R)} = v + v G_R \tilde{T}^{(R)}$$

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Renormalized amplitude:

$$T^{(R)} z_R^{-1} \xrightarrow[R \to \infty]{} T = T_C + \langle \psi_C^{(-)} | \tilde{T}^{(C)} | \psi_C^{(+)} \rangle$$

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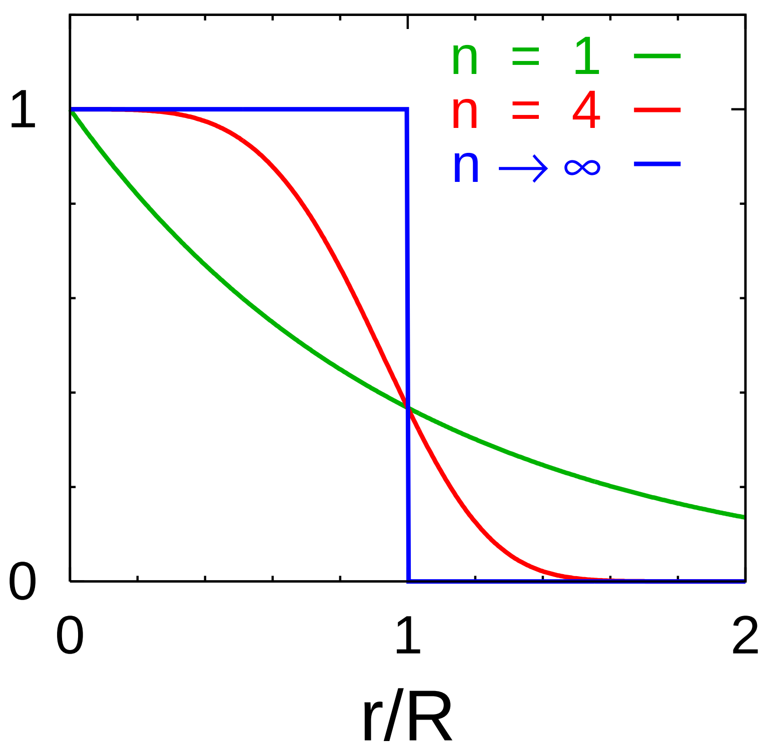
$$T^{(R)} z_R^{-1} \xrightarrow[R\to\infty]{} T = T_C + \langle \psi_C^{(-)} | \tilde{T}^{(C)} | \psi_C^{(+)} \rangle$$

$$= T_C + \lim_{R\to\infty} z_R^{-\frac{1}{2}} [T^{(R)} - T_R] z_R^{-\frac{1}{2}}$$

short-range part: fast convergence with R

Test: convergence with R in pp scattering

$$\frac{w_R(r)}{w_C(r)} = e^{-\left(\frac{r}{R}\right)^{\mathrm{n}}}$$



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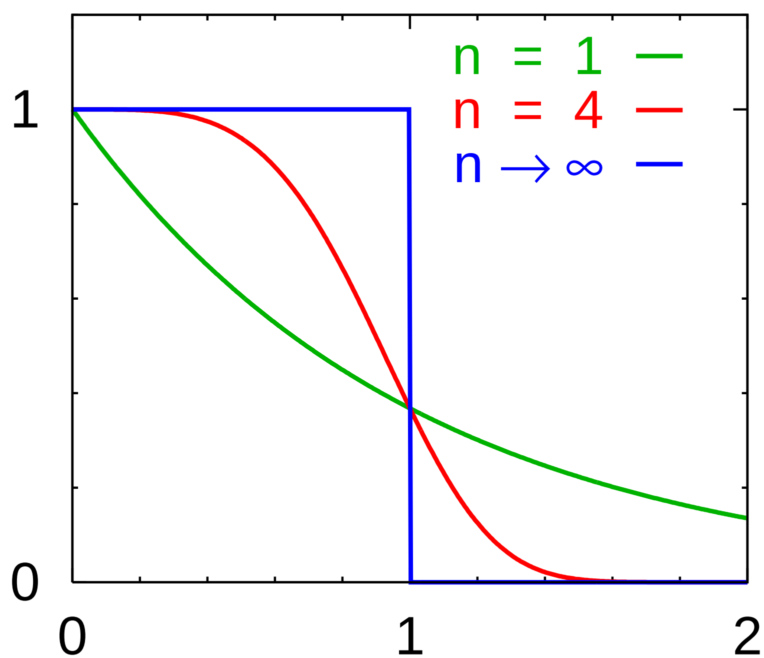

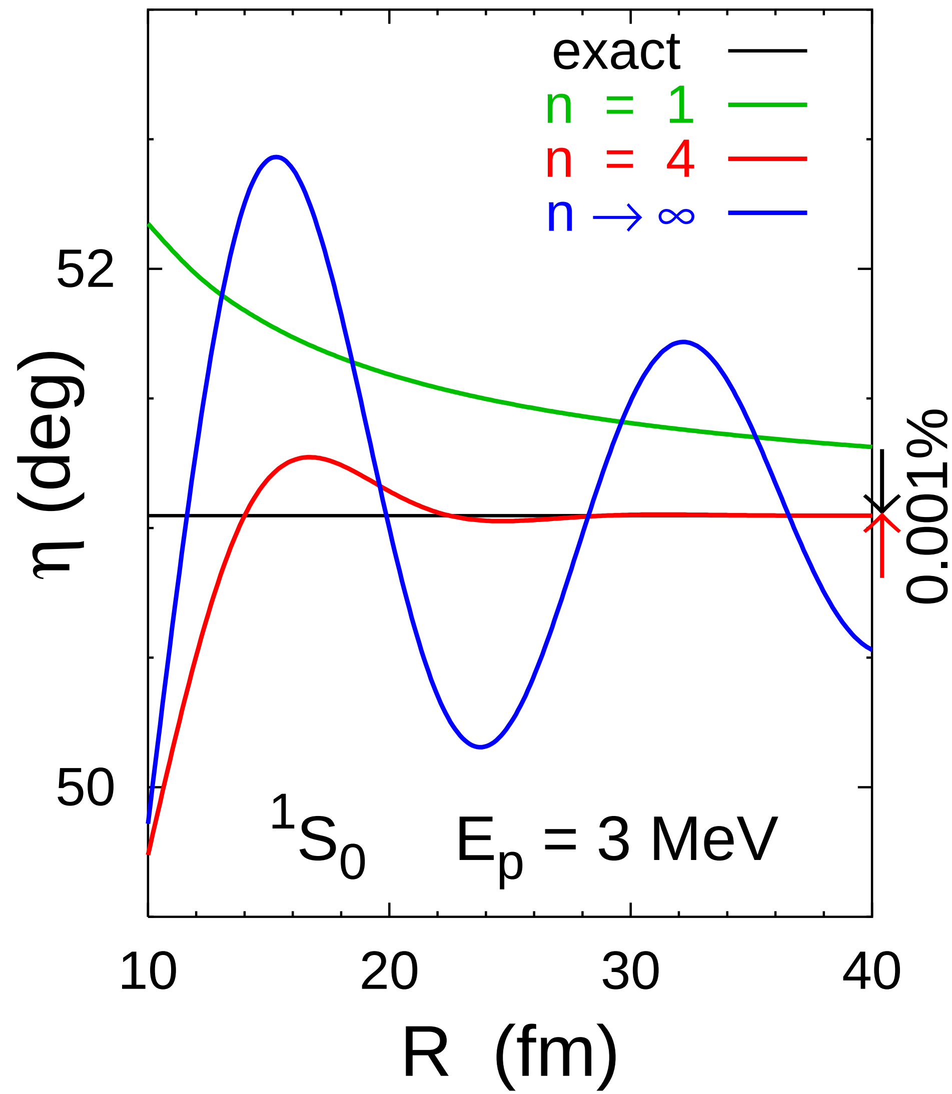

optimal choice: $3 \leq \mathrm{n} \leq 8$

Limits of practical applicability

$p \to 0$:

$\kappa = \alpha M/p$, $\sigma_L = \arg\Gamma(1 + L + i\kappa)$, and z_R diverge,

renormalization procedure ill-defined

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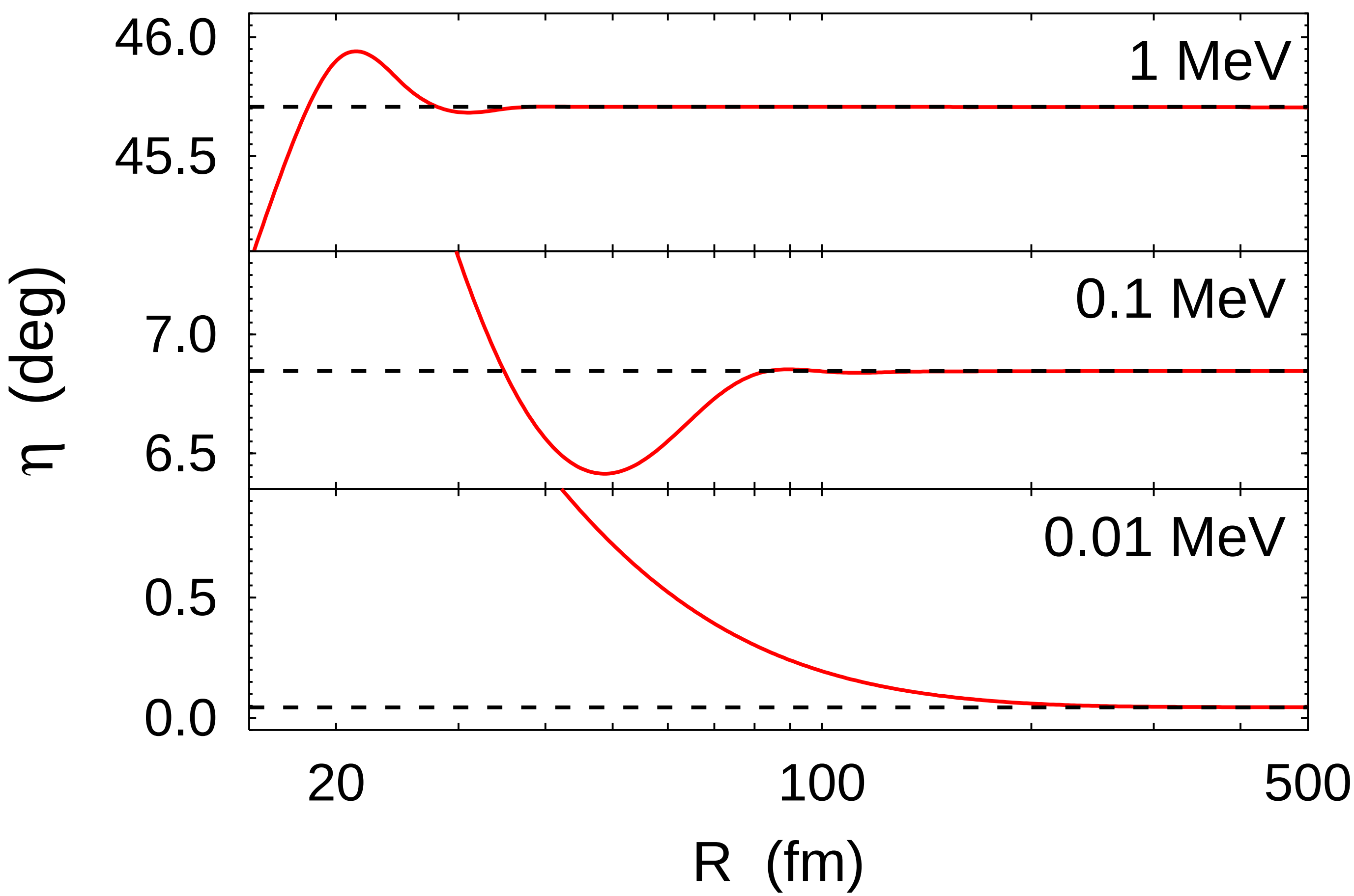
$\Rightarrow$ slow convergence with R at low relative energies



Three-particle scattering: short-range forces

- Faddeev / Alt, Grassberger, and Sandhas equations

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\sigma \bar{\delta}_{\beta\sigma} T_\sigma G_0 U_{\sigma\alpha}$$

$$U_{0\alpha} = G_0^{-1} + \sum_\sigma T_\sigma G_0 U_{\sigma\alpha}$$

$$T_\sigma = v_\sigma + v_\sigma G_0 T_\sigma$$

$$G_0 = (E + i0 - H_0)^{-1}$$

- momentum-space partial-wave representation

AGS equations with 3BF

$$V_{3BF} = \sum_{\alpha=1}^{3} u_\alpha$$

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} T_\gamma G_0 U_{\gamma\alpha} + u_\alpha + \sum_\gamma u_\gamma G_0 (1 + T_\gamma G_0) U_{\gamma\alpha}$$

Three-particle scattering: including screened Coulomb

- Faddeev / Alt, Grassberger, and Sandhas equations

$$U^{(R)}_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\sigma} \bar{\delta}_{\beta\sigma} T^{(R)}_{\sigma} G_0 U^{(R)}_{\sigma\alpha}$$

$$U^{(R)}_{0\alpha} = G_0^{-1} + \sum_{\sigma} T^{(R)}_{\sigma} G_0 U^{(R)}_{\sigma\alpha}$$

$$T^{(R)}_{\sigma} = v_{\sigma} + w_{\sigma_R} + (v_{\sigma} + w_{\sigma_R}) G_0 T^{(R)}_{\sigma}$$

$$G_0 = (E + i0 - H_0)^{-1}$$

- momentum-space partial-wave representation

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- momentum-space partial-wave representation
- Additional difficulties:
  quasi-singular nature of screened Coulomb potential
  slow partial-wave convergence

Three-particle scattering: including screened Coulomb

- Faddeev / Alt, Grassberger, and Sandhas equations

$$U_{\beta\alpha}^{(R)} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\sigma} \bar{\delta}_{\beta\sigma} T_{\sigma}^{(R)} G_0 U_{\sigma\alpha}^{(R)}$$

$$U_{0\alpha}^{(R)} = G_0^{-1} + \sum_{\sigma} T_{\sigma}^{(R)} G_0 U_{\sigma\alpha}^{(R)}$$

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- momentum-space partial-wave representation
- Additional difficulties:
  quasi-singular nature of screened Coulomb potential
  slow partial-wave convergence
- $R \to \infty$ limit?

Three-particle scattering: $R \to \infty$ limit

long-range part

$$T^{\text{c.m.}}_{\alpha R} = W^{\text{c.m.}}_{\alpha R} + W^{\text{c.m.}}_{\alpha R} G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}$$

Three-particle scattering: $R \to \infty$ limit

Split into long-range part

$$T^{\text{c.m.}}_{\alpha R} = W^{\text{c.m.}}_{\alpha R} + W^{\text{c.m.}}_{\alpha R} G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}$$

and Coulomb-distorted short-range part

$$U^{(R)}_{\beta\alpha} = \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha R} + [1 + T^{\text{c.m.}}_{\beta R} G^{(R)}_{\beta}] \tilde{U}^{(R)}_{\beta\alpha} [1 + G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}]$$

$$U^{(R)}_{0\alpha} = [1 + T_{\rho R} G_0] \tilde{U}^{(R)}_{0\alpha} [1 + G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}] \quad [\rho \text{ is neutral}]$$

Three-particle scattering: $R \to \infty$ limit

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Renormalized amplitudes:

$$U_{\beta\alpha} = \delta_{\beta\alpha} T^{\mathrm{c.m.}}_{\alpha C} + \lim_{R\to\infty} Z_{R f}^{-\frac{1}{2}} [U^{(R)}_{\beta\alpha} - \delta_{\beta\alpha} T^{\mathrm{c.m.}}_{\alpha R}] Z_{R i}^{-\frac{1}{2}}$$

$$U_{0\alpha} = \lim_{R\to\infty} z_{R}^{-\frac{1}{2}} U^{(R)}_{0\alpha} Z_{R i}^{-\frac{1}{2}}$$

Three-particle scattering: $R \to \infty$ limit

Split into long-range part

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$$U^{(R)}_{0\alpha} = [1 + T_{\rho R} G_0] \tilde{U}^{(R)}_{0\alpha} [1 + G^{(R)}_{\alpha} T^{\mathrm{c.m.}}_{\alpha R}]$$ [ρ is neutral]

Renormalized amplitudes:

$$U_{\beta\alpha} = \delta_{\beta\alpha} T^{\mathrm{c.m.}}_{\alpha C} + \lim_{R\to\infty} Z^{-\frac{1}{2}}_{R\,f} [U^{(R)}_{\beta\alpha} - \delta_{\beta\alpha} T^{\mathrm{c.m.}}_{\alpha R}] Z^{-\frac{1}{2}}_{R\,i}$$

$$U_{0\alpha} = \lim_{R\to\infty} z^{-\frac{1}{2}}_{R} U^{(R)}_{0\alpha} Z^{-\frac{1}{2}}_{R\,i}$$

short-range part: fast convergence with R

r-space methods

- Kohn VP + HH
  [Kievsky *et al*]
- differential Faddeev equations
  [Payne *et al*, Lazauskas *et al*, Suslov *et al*]
- integral Faddeev equations
  [Ishikawa]

Screening and renormalization: variations

- separable potentials, quasiparticle equations, effective two-body potentials, Coulomb distorted ffs, ...
  [Alt *et al*]
- "rigorous Coulomb treatment"
  [Oryu *et al*]
- no/different renormalization
  [Witała *et al*]
- in progress:
  separable potentials, unscreened Coulomb representation
  [Mukhamedzhanov *et al*, TORUS]

Proton-deuteron scattering

- Symmetrized Faddeev / AGS equations

$$U^{(R)} = PG_0^{-1} + PT^{(R)}G_0U^{(R)}$$

$$U_0^{(R)} = (1+P)G_0^{-1} + (1+P)T^{(R)}G_0U^{(R)}$$

$$P = P_{12}P_{23} + P_{13}P_{23}$$

- Screening function with $n = 4$
- Renormalized amplitudes:

$$U = T_C^{\text{c.m.}} + \lim_{R\to\infty} Z_R^{-1}[U^{(R)} - T_R^{\text{c.m.}}]$$

$$U_0 = \lim_{R\to\infty} z_R^{-\frac{1}{2}} U_0^{(R)} Z_R^{-\frac{1}{2}}$$

Perturbation theory for high partial waves

$$T \to T + \Delta T$$

$$(U + \Delta U) = PG_0^{-1} + P(T + \Delta T)G_0(U + \Delta U)$$

1st order in ΔT, all orders in T:

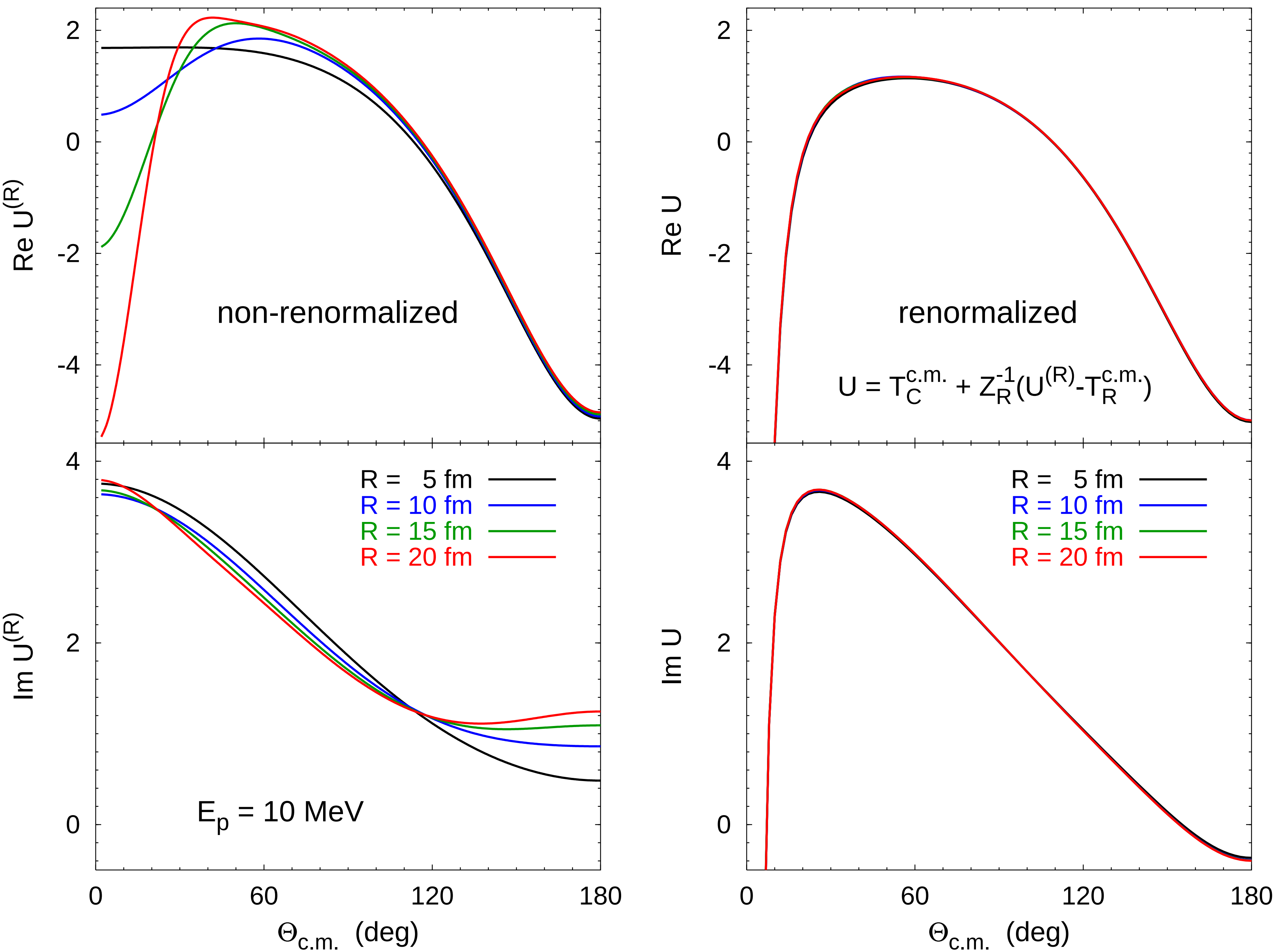
$$\begin{aligned} U &= PG_0^{-1} + PTG_0U \\ \Delta U &= P\Delta TG_0U + PTG_0\Delta U \end{aligned}$$

Exact limit:

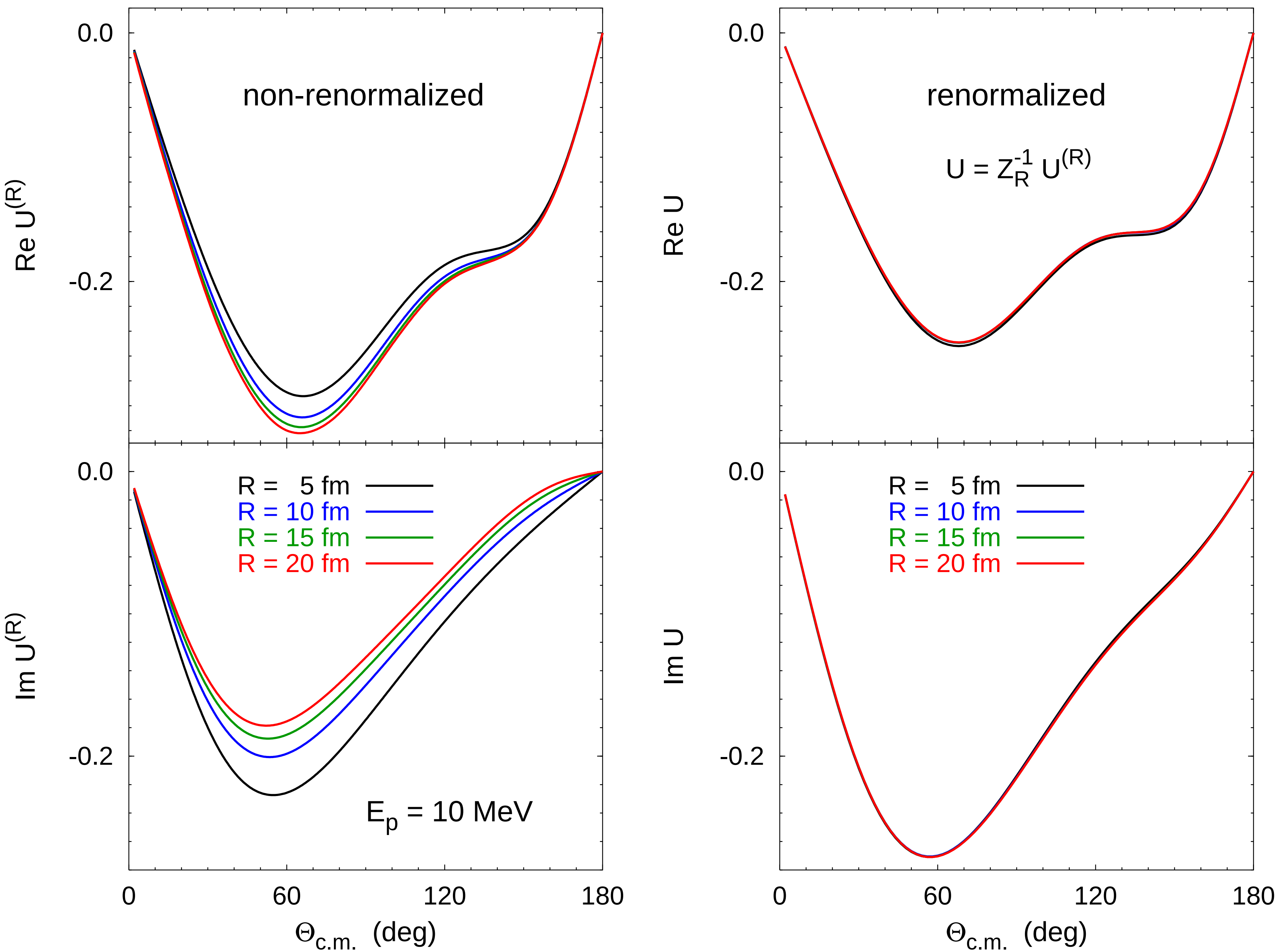
increase the number of partial waves included in T until $U + \Delta U$ becomes stable

Practical calculations: $L_T \approx L_{\Delta T}/2$

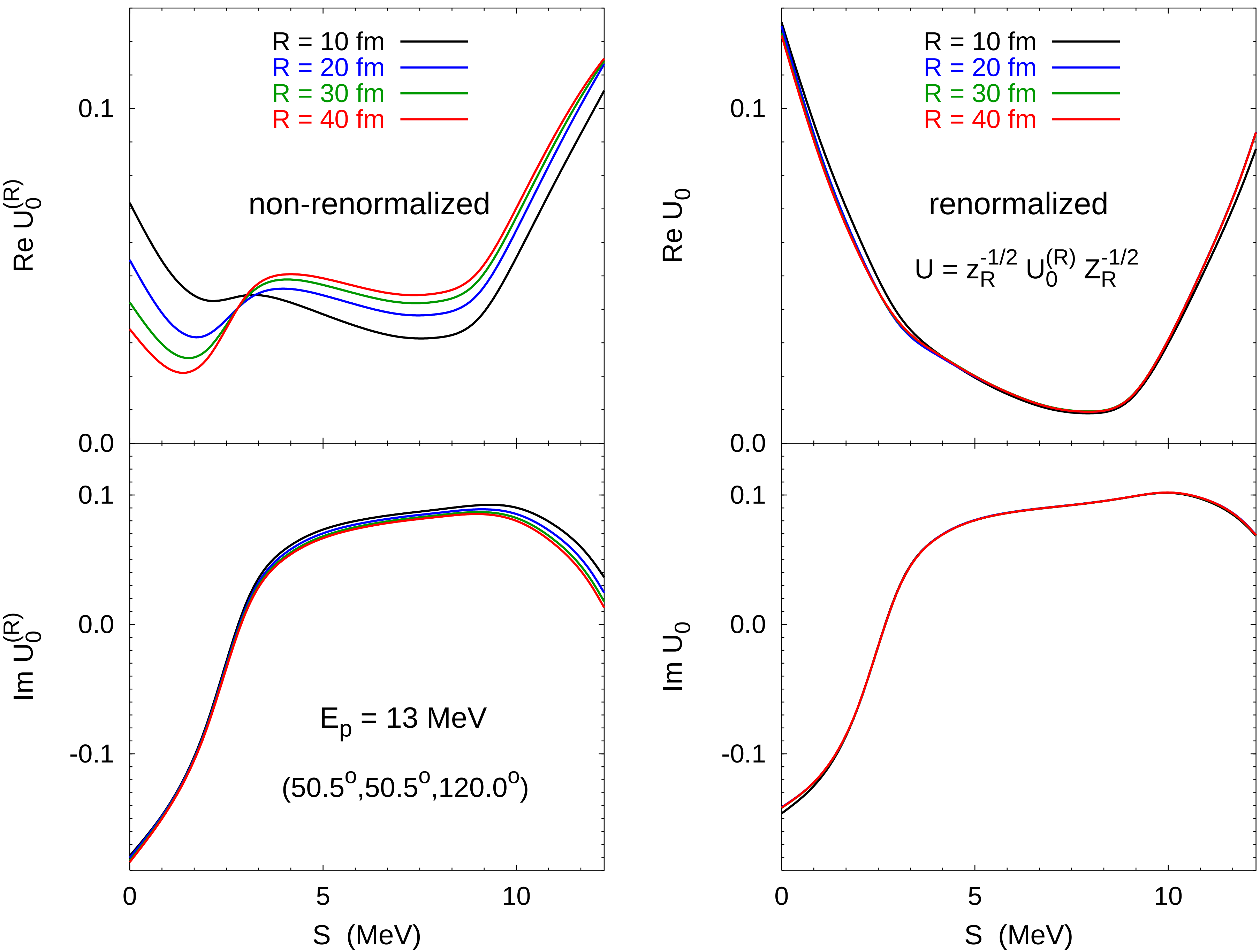
pd elastic amplitude (spin-diagonal)



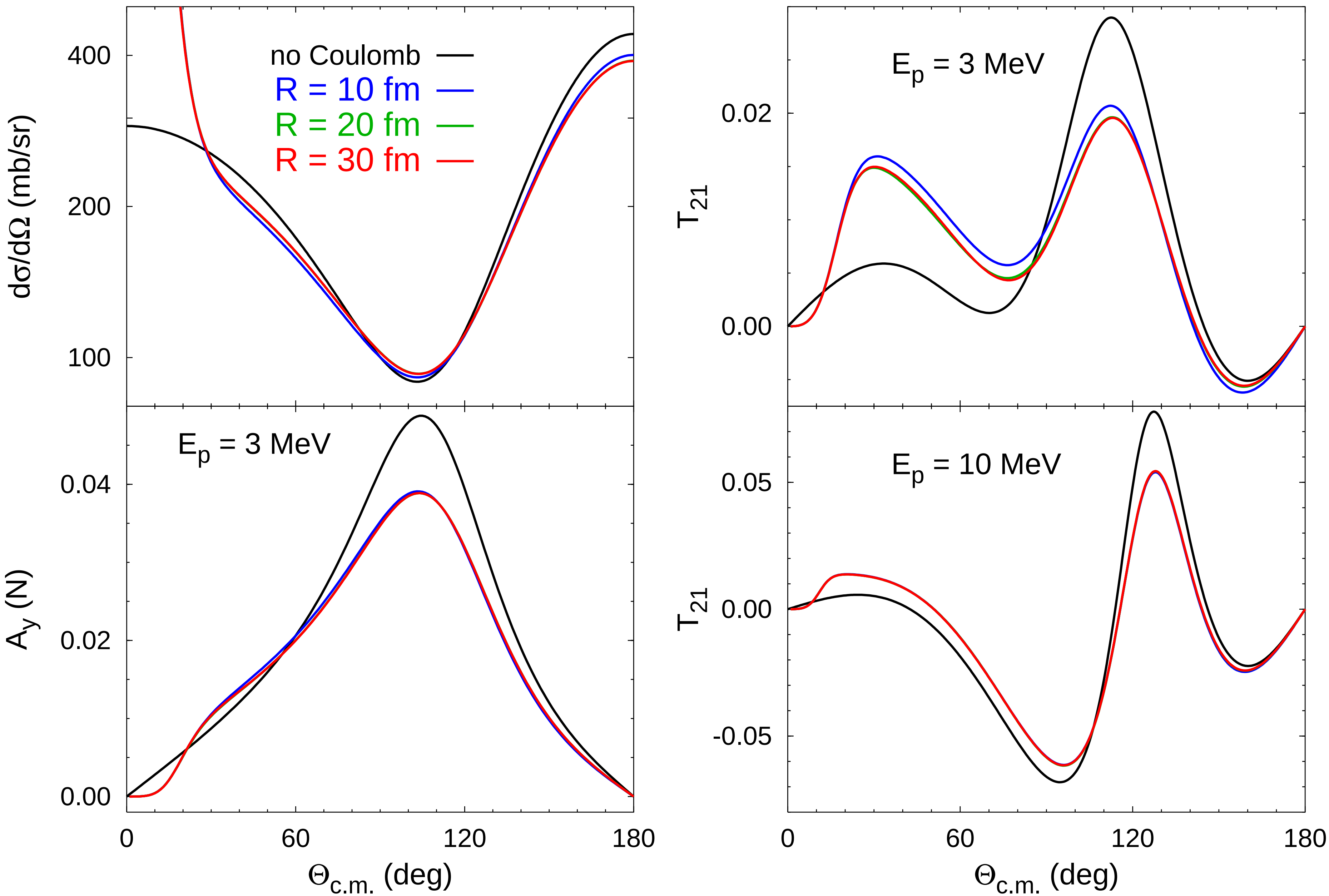
pd elastic amplitude (spin-nondiagonal)



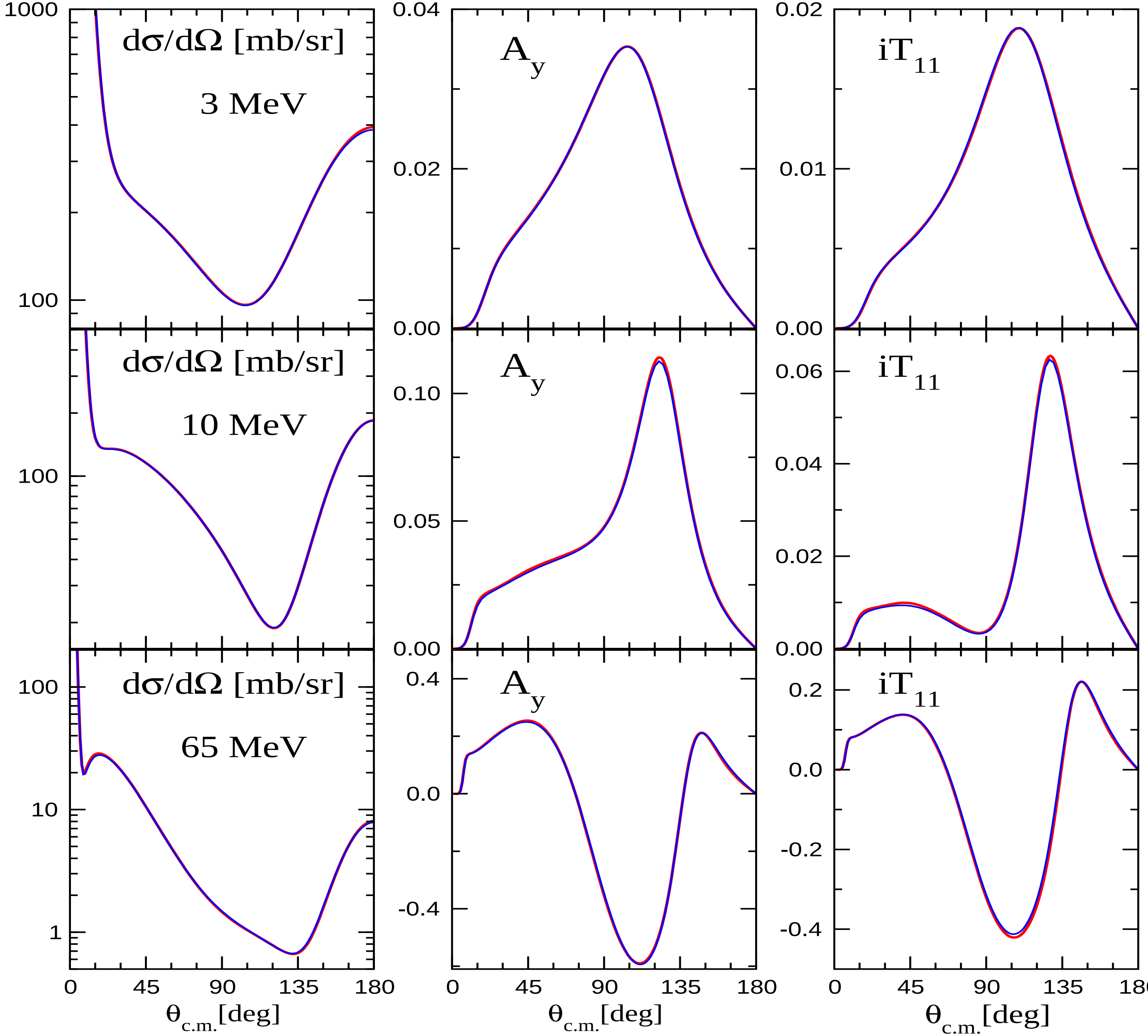
pd breakup amplitude



Convergence with *R*: *pd* elastic scattering



Comparison with configuration-space results

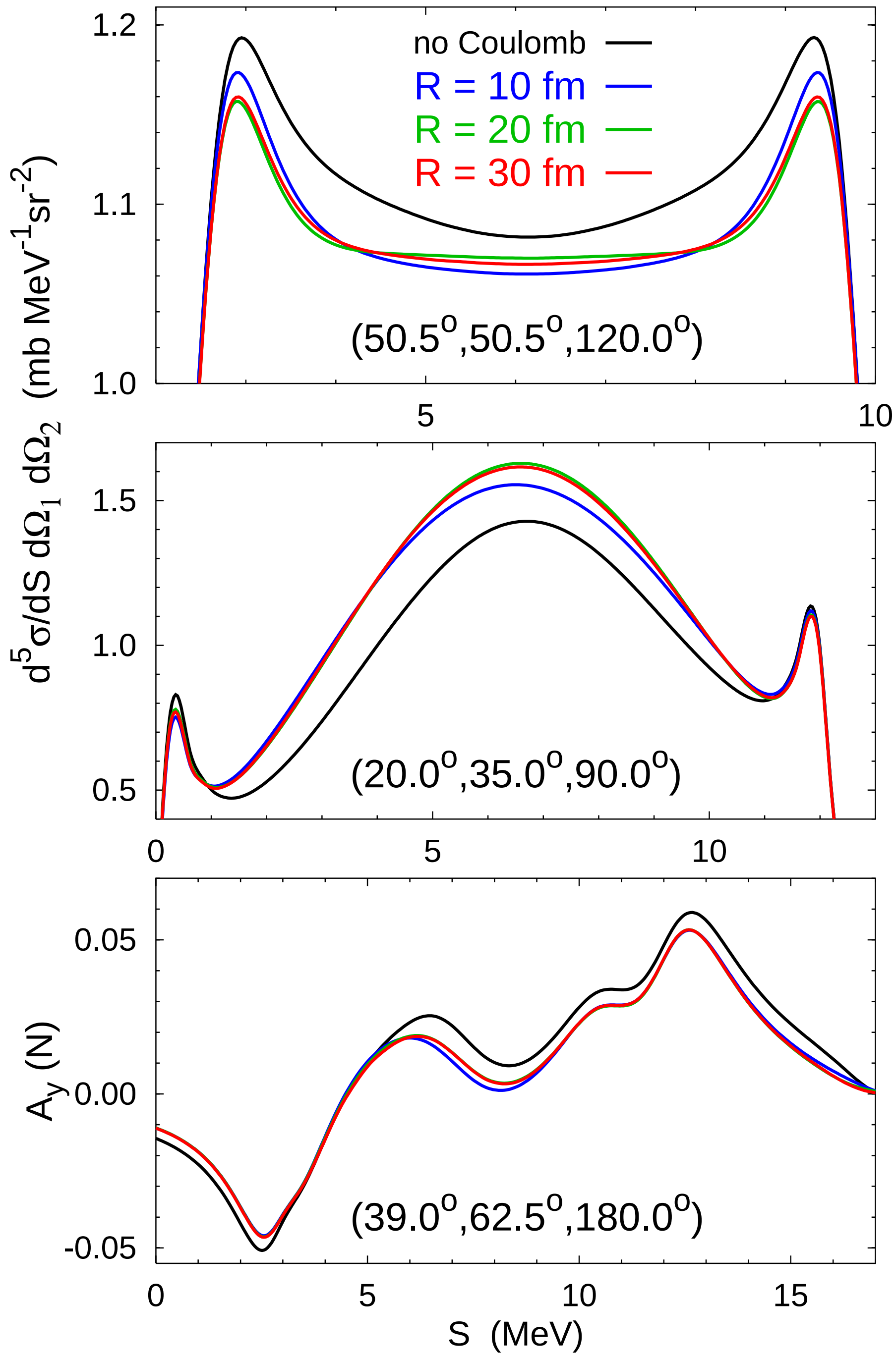


pd elastic scattering:

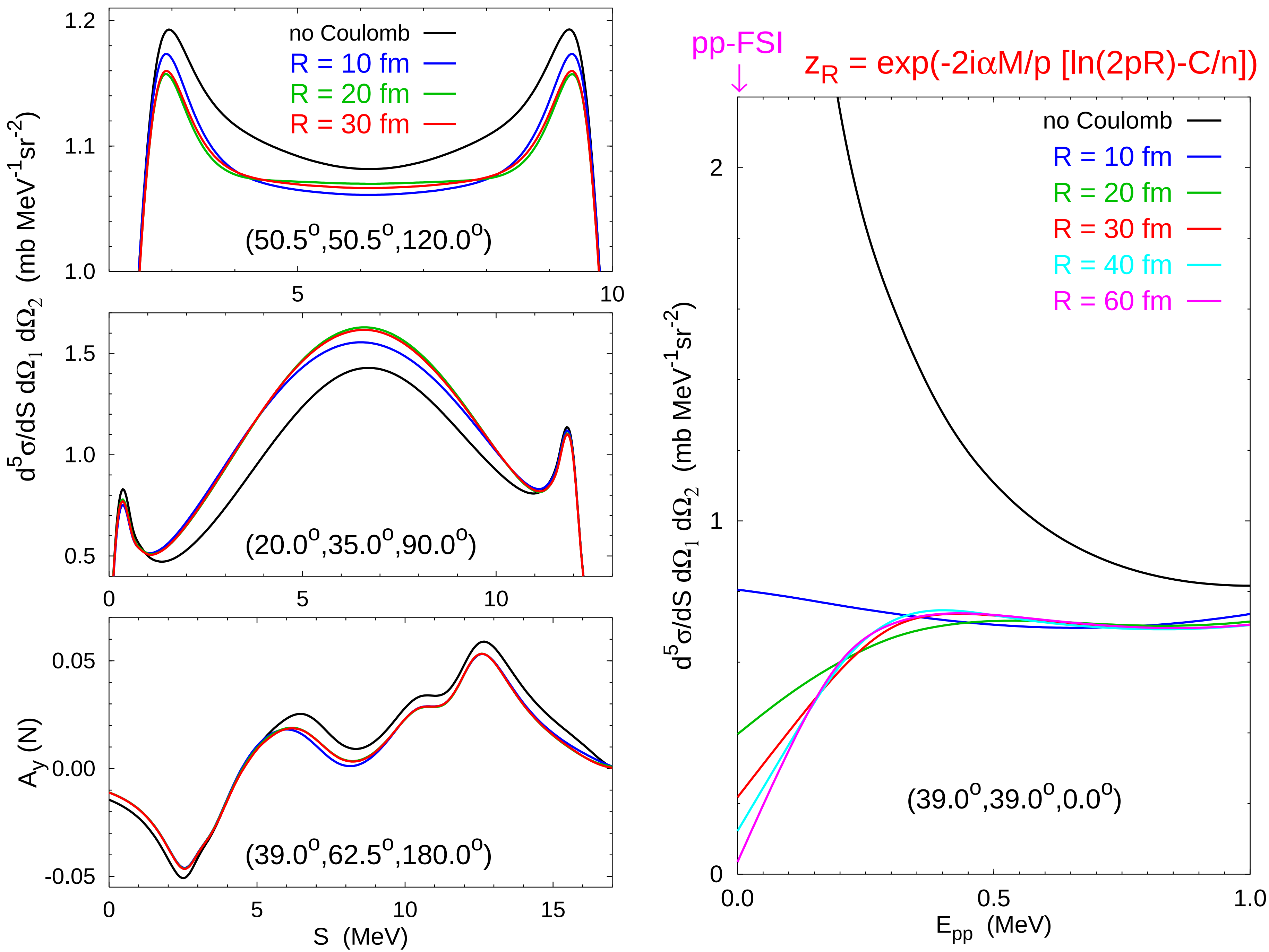
Kohn variational principle (Pisa)

screening and renormalization

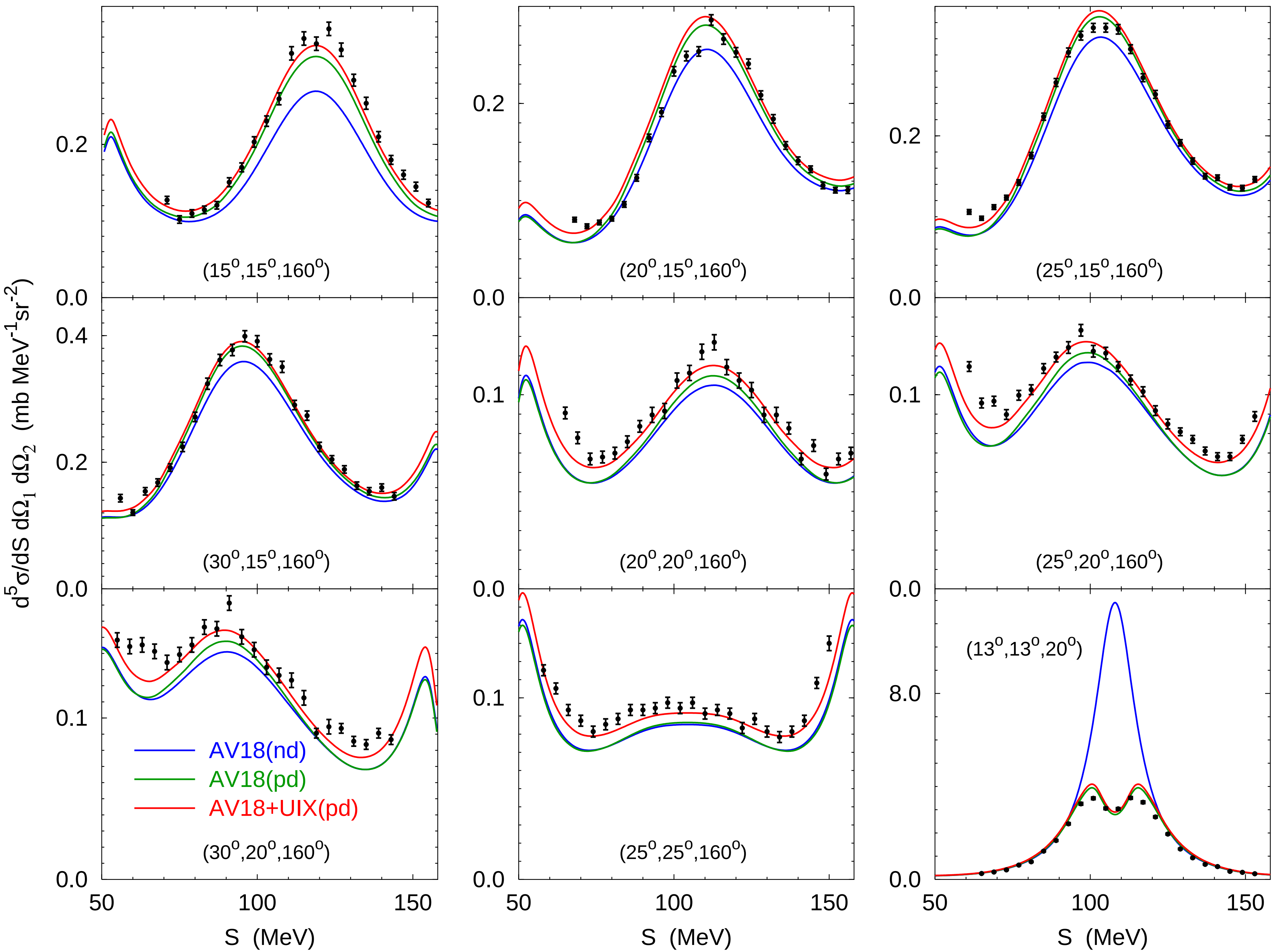
Convergence with R: pd breakup at $E_p = 13$ MeV



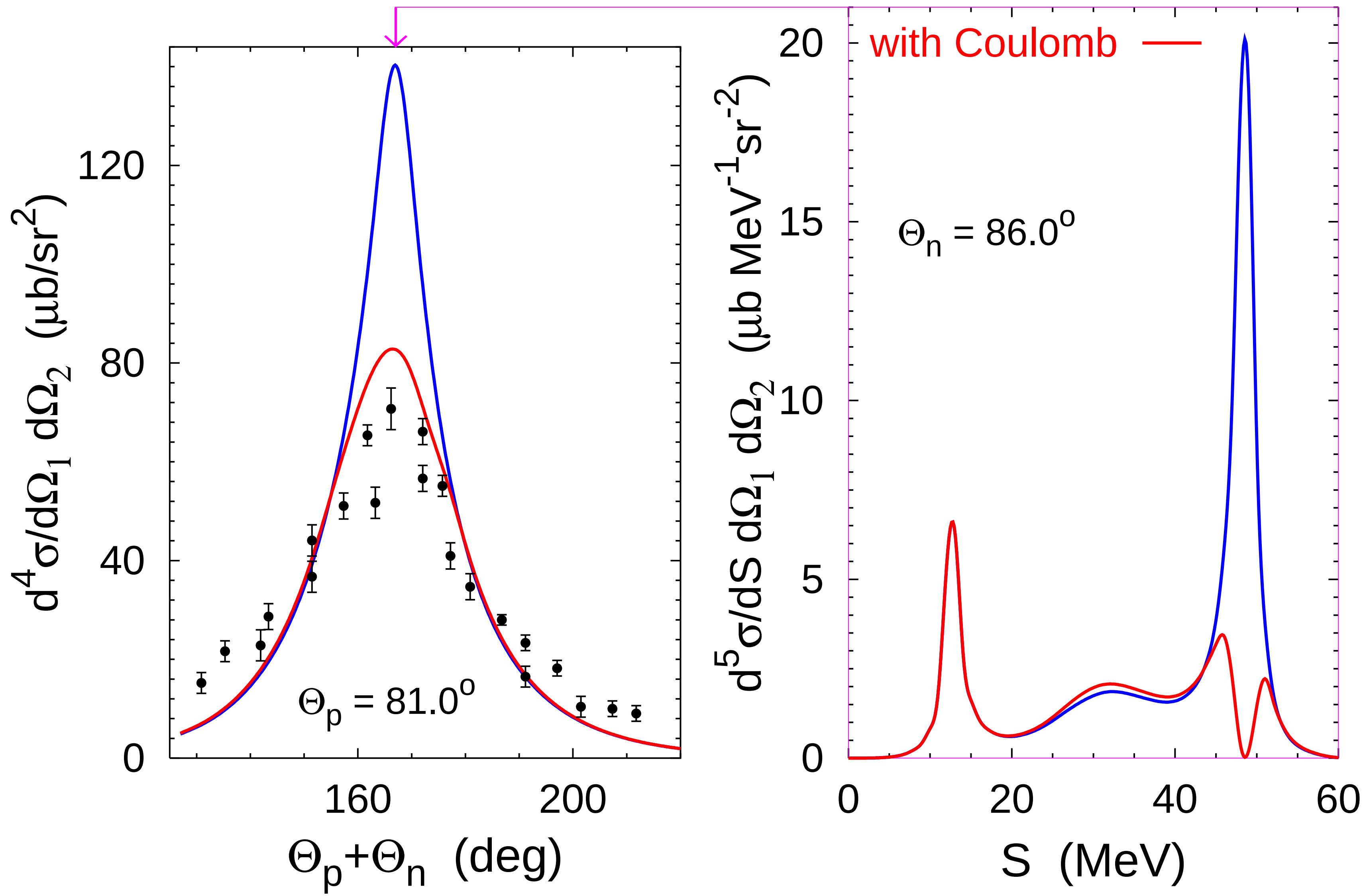
Convergence with R: pd breakup at $E_p = 13$ MeV



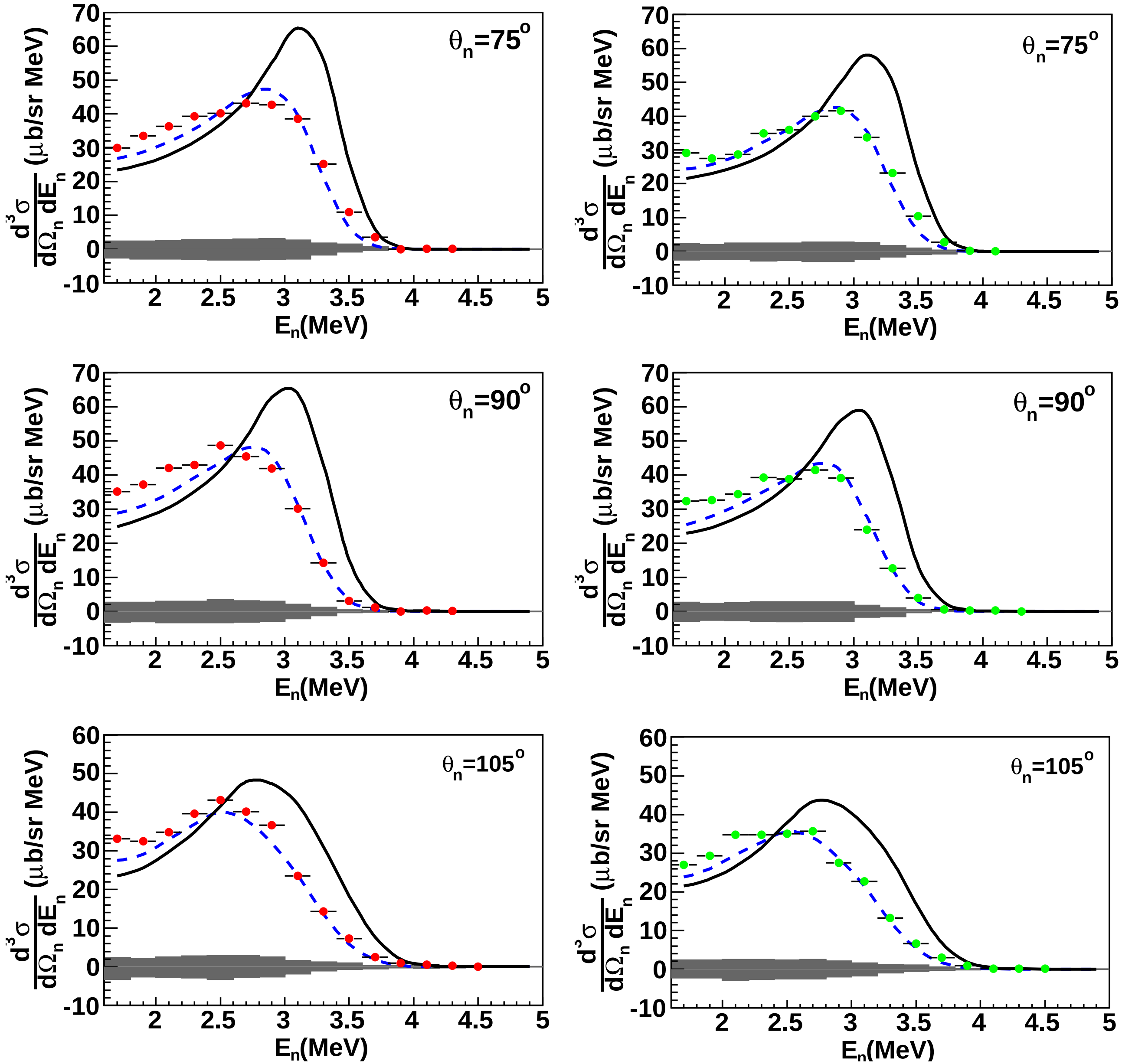
Coulomb vs 3NF: ^{1}H(d,pp)n at $E_d = 130$ MeV



$^3He(\gamma, pn)p$ at $E_\gamma = 55$ MeV



$^3\vec{\mathrm{He}}(\vec{\gamma},n)pp$ at $E_\gamma = 12.8$ MeV



parallel

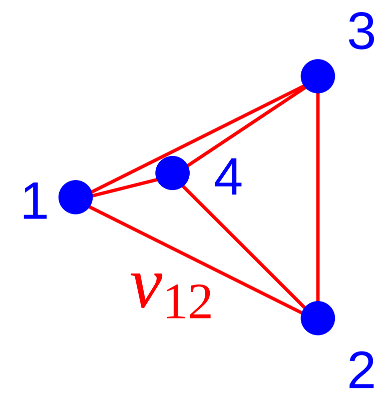
antiparallel

● ● TUNL data [PRL 110, 202501]

- - - CD Bonn + Δ (with Coulomb) (Lisbon)

— AV18 + UIX (no Coulomb) (Cracow)

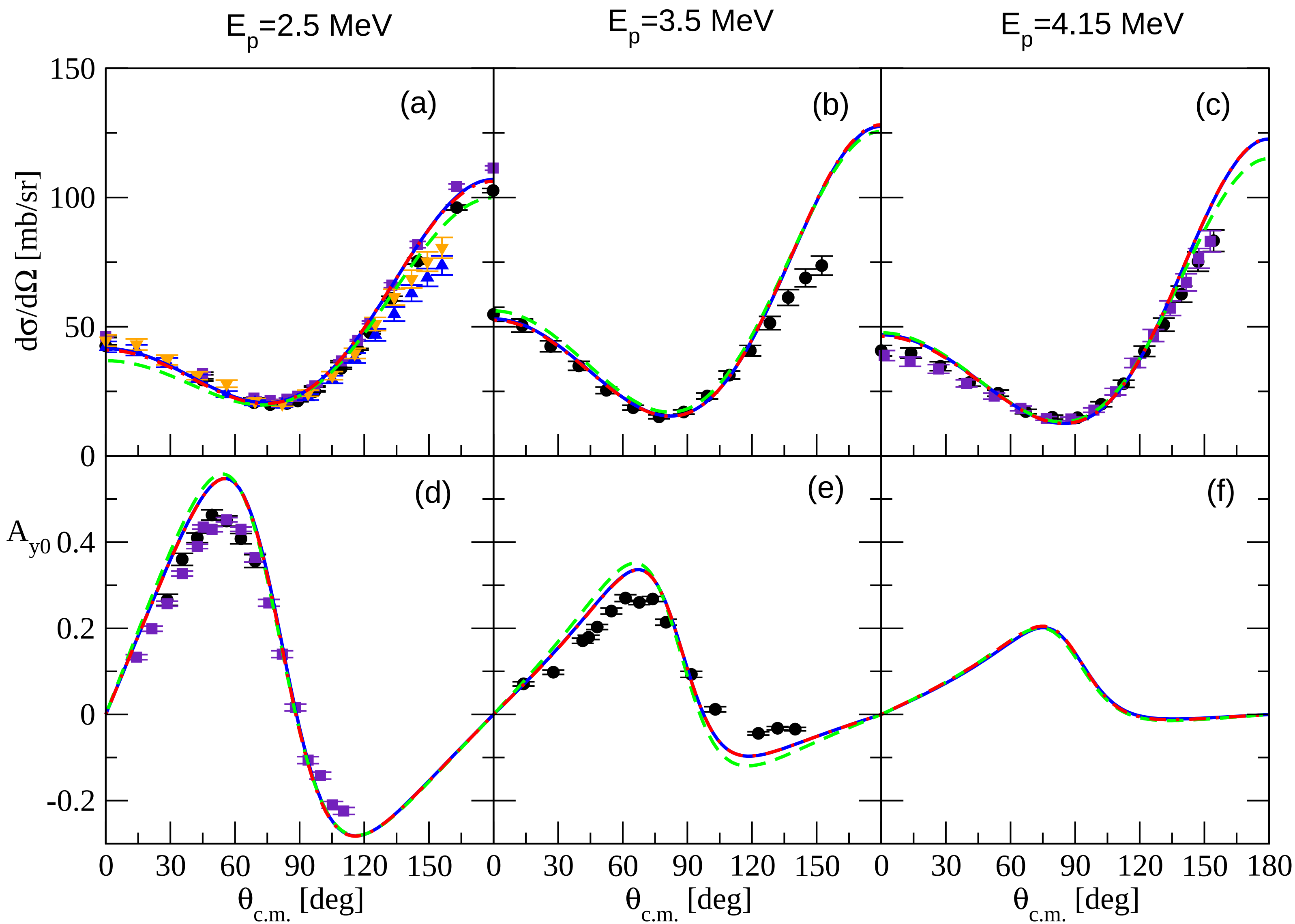
4N scattering



Hamiltonian $$H_0 + \sum_{i>j} v_{ij}$$

- Wave function:
  Schrödinger equation (HH + Kohn VP)
  [M. Viviani, A. Kievsky, L. E. Marcucci, S. Rosati, L. Girlanda]
- Wave function components:
  Faddeev-Yakubovsky equations
  [R. Lazauskas, J. Carbonell]
- Transition operators:
  Alt-Grassberger-Sandhas equations
  [AD, A. C. Fonseca]

Benchmark: $^3H(p,n)^3He$ scattering



AGS/HH/FY (Lisbon/Pisa/Strasbourg, PRC 95, 034003)

4-body scattering: AGS equations

4-body transition operators

$$t_i = v_i + v_i G_0 t_i$$

$$G_0 = (E + i0 - H_0)^{-1}$$

$$U_\gamma^{jk} = G_0^{-1} \bar{\delta}_{jk} + \sum_i \bar{\delta}_{ji} t_i G_0 U_\gamma^{ik}$$

$$\mathcal{U}_{\beta\alpha}^{ji} = (G_0 t_i G_0)^{-1} \bar{\delta}_{\beta\alpha} \delta_{ji} + \sum_{\gamma k} \bar{\delta}_{\beta\gamma} U_\gamma^{jk} G_0 t_k G_0 \, \mathcal{U}_{\gamma\alpha}^{ki}$$

i, j, k: pairs ($\equiv$ three-cluster (2+1+1) partitions)
α, β, γ: two-cluster (1+3 or 2+2) partitions

wave function

$$|\Psi_\alpha\rangle = |\Phi_\alpha\rangle + \sum_{\gamma jki} G_0 t_j G_0 U_\gamma^{jk} G_0 t_k G_0 \, \mathcal{U}_{\gamma\alpha}^{ki} |\phi_\alpha^i\rangle$$

$$|\Phi_\alpha\rangle = \sum_i |\phi_\alpha^i\rangle, \qquad |\phi_\alpha^i\rangle = G_0 \sum_j \bar{\delta}_{ij} t_j |\phi_\alpha^j\rangle$$

Symmetrized AGS equations

$$t = v + vG_0 t$$

$$G_0 = (E + i\varepsilon - H_0)^{-1}$$

$$U_j = P_j G_0^{-1} + P_j t G_0 U_j$$

$$3+1: \quad P_1 = P_{12} P_{23} + P_{13} P_{23}$$

$$2+2: \quad P_2 = P_{13} P_{24}$$

$$\mathcal{U}_{11} = (G_0 t G_0)^{-1} \zeta P_{34} + \zeta P_{34} U_1 G_0 t G_0 \, \mathcal{U}_{11} + U_2 G_0 t G_0 \, \mathcal{U}_{21}$$

$$\mathcal{U}_{21} = (G_0 t G_0)^{-1} (1 + \zeta P_{34}) + (1 + \zeta P_{34}) U_1 G_0 t G_0 \, \mathcal{U}_{11}$$

$$\mathcal{U}_{12} = (G_0 t G_0)^{-1} + \zeta P_{34} U_1 G_0 t G_0 \, \mathcal{U}_{12} + U_2 G_0 t G_0 \, \mathcal{U}_{22}$$

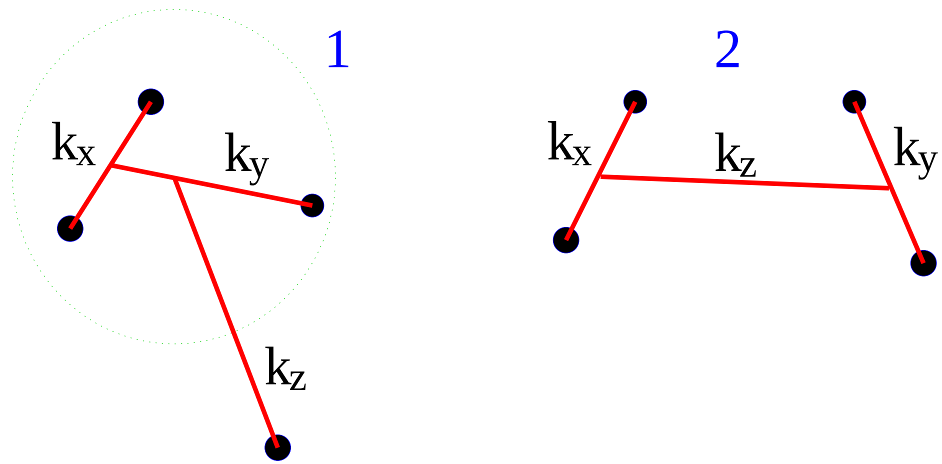
$$\mathcal{U}_{22} = (1 + \zeta P_{34}) U_1 G_0 t G_0 \, \mathcal{U}_{12}$$

$\zeta = -1 \; (+1)$ for fermions (bosons)

basis states partially symmetrized

Solution of 4N AGS equations

$$\mathcal{U}_{11}|\phi_1\rangle = -G_0^{-1}P_{34}P_1|\phi_1\rangle - P_{34}U_1 G_0\, t\, G_0\, \mathcal{U}_{11}|\phi_1\rangle + U_2 G_0\, t\, G_0\, \mathcal{U}_{21}|\phi_1\rangle$$



- momentum-space partial-wave basis
  $|k_x k_y k_z [l_z(\{l_y[(l_x S_x) j_x s_y] S_y\} J_y s_z) S_z] JM, [(T_x t_y) T_y t_z] T M_T\rangle_1$
  $|k_x k_y k_z [l_z\{(l_x S_x) j_x [l_y(s_y s_z) S_y] j_y\} S_z] JM, [T_x(t_y t_z) T_z] T M_T\rangle_2$
- large system (up to 30000) of coupled 3-variable integral equations with integrable singularities
- Coulomb interaction: screening and renormalization [PRC 75, 014005, PRL 98, 162502]

Singularities of 4N AGS equations

^{3}H, ^{3}He, or d+d bound state poles

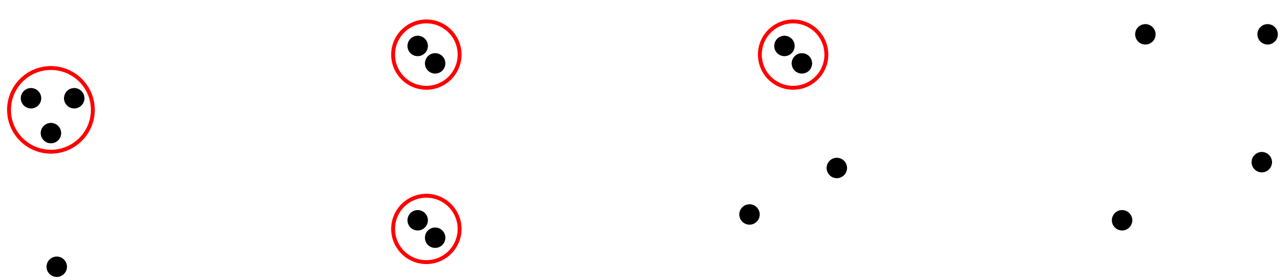
$$G_0 U_j G_0 \to \frac{P_j |\phi_j\rangle s_{jj} \langle\phi_j| P_j}{E + i\varepsilon - E_j^b - k_z^2/2\mu_j}$$

deuteron bound state poles

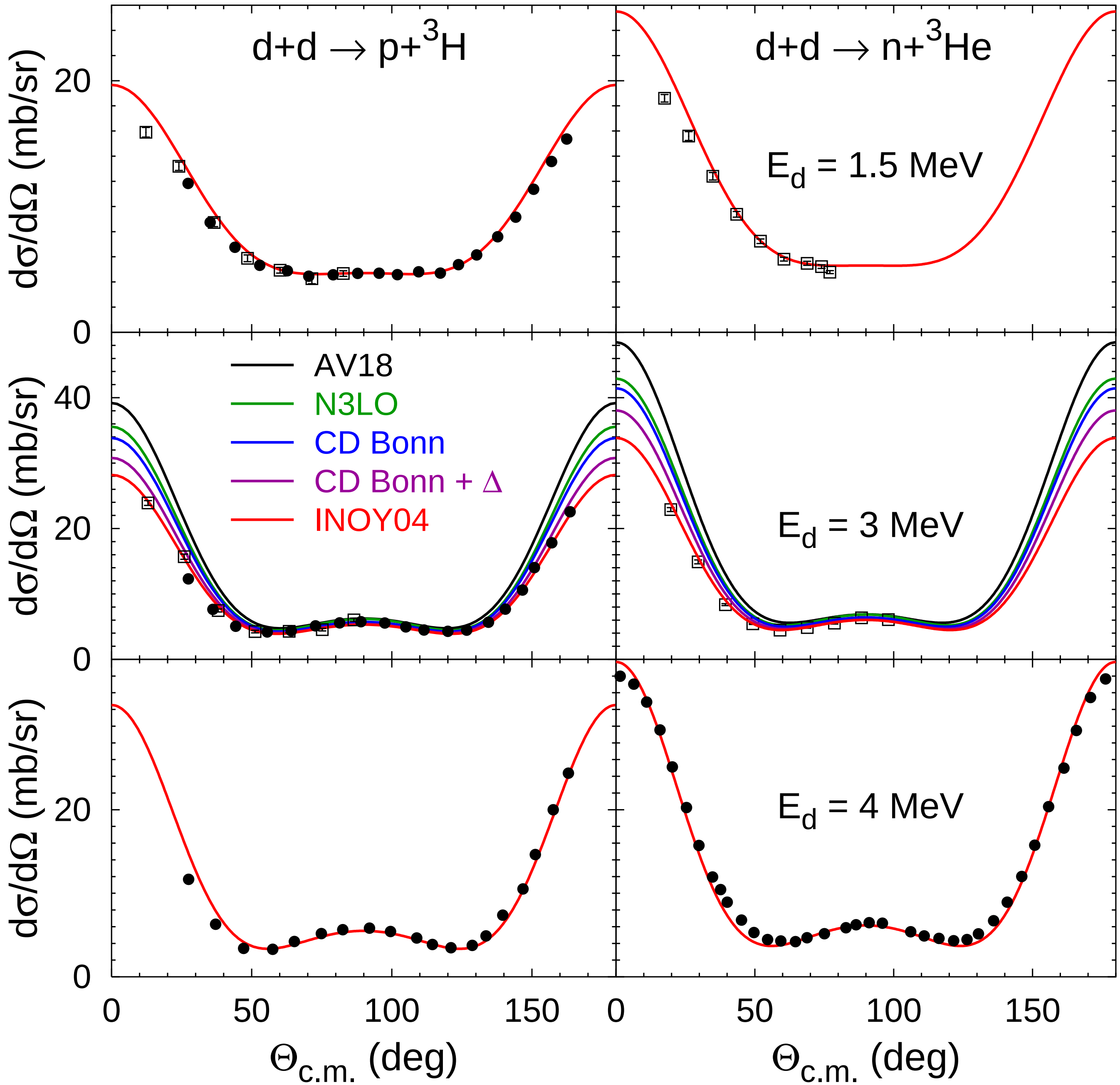
$$t \to \frac{v|\phi_d\rangle\langle\phi_d|v}{E + i\varepsilon - e_d - k_y^2/2\mu_j^y - k_z^2/2\mu_j}$$

free resolvent

$$G_0 \to \frac{1}{E + i\varepsilon - k_x^2/2\mu_j^x - k_y^2/2\mu_j^y - k_z^2/2\mu_j}$$

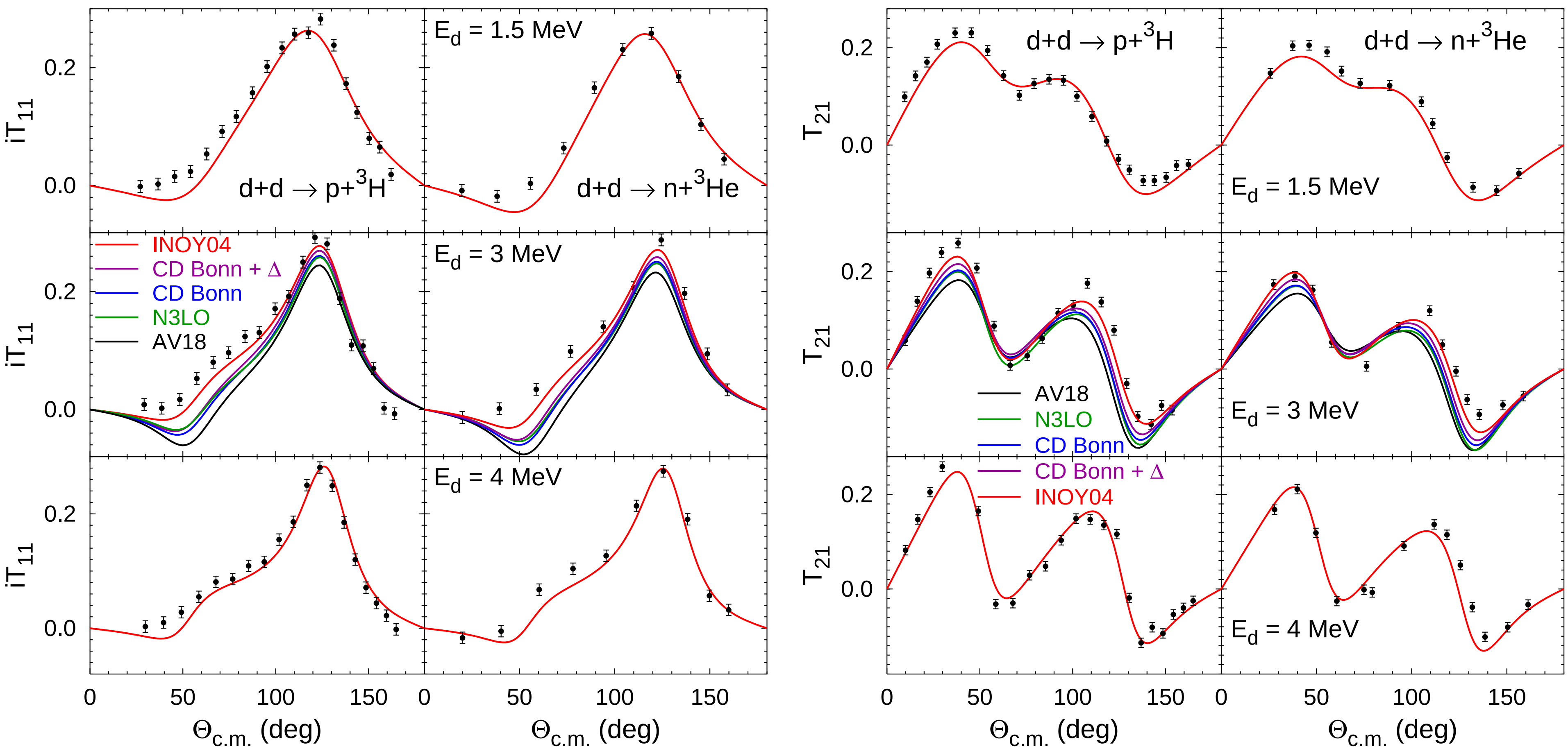


^{2}H(d,p)^{3}H and ^{2}H(d,n)^{3}He – cross section

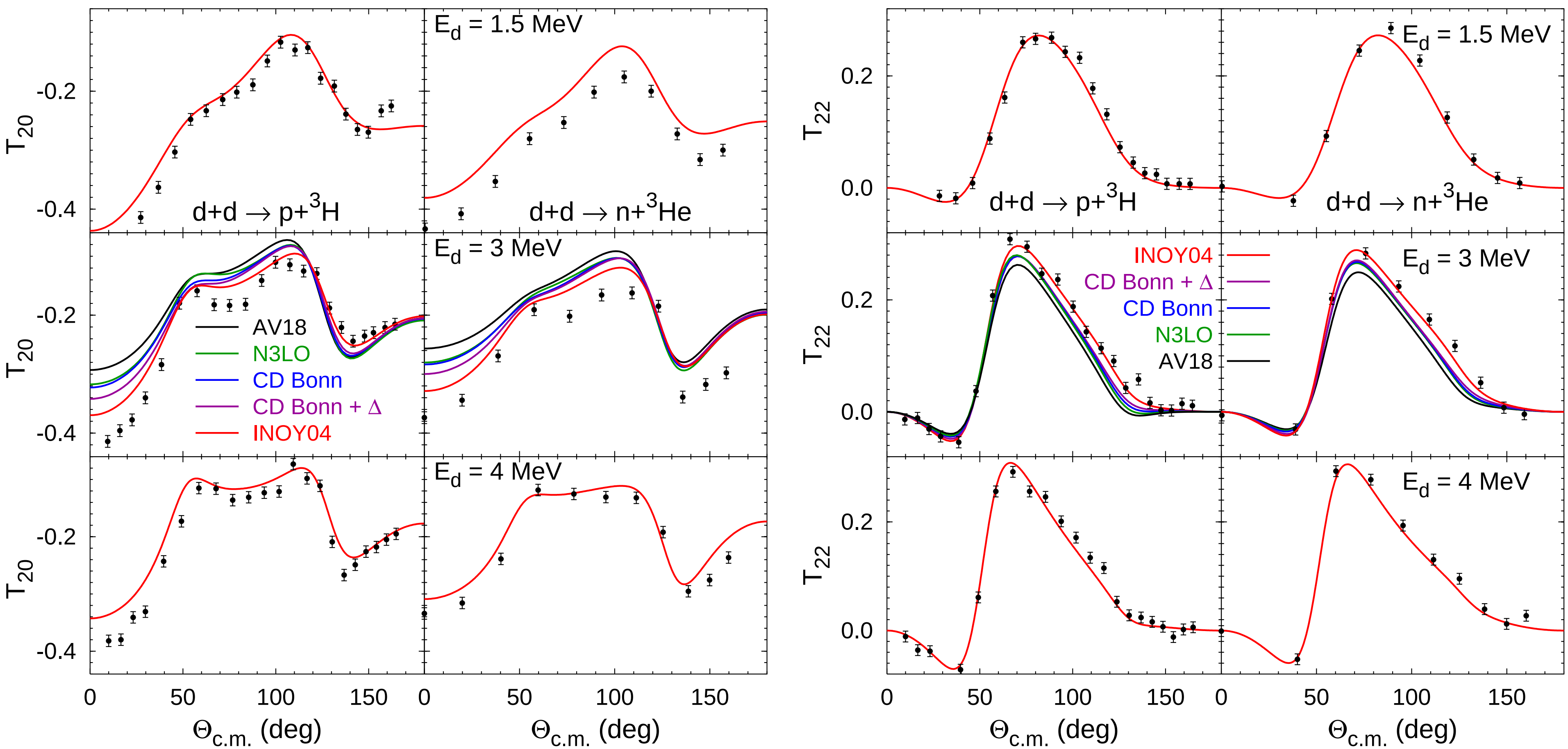


[PRC 81, 054002]

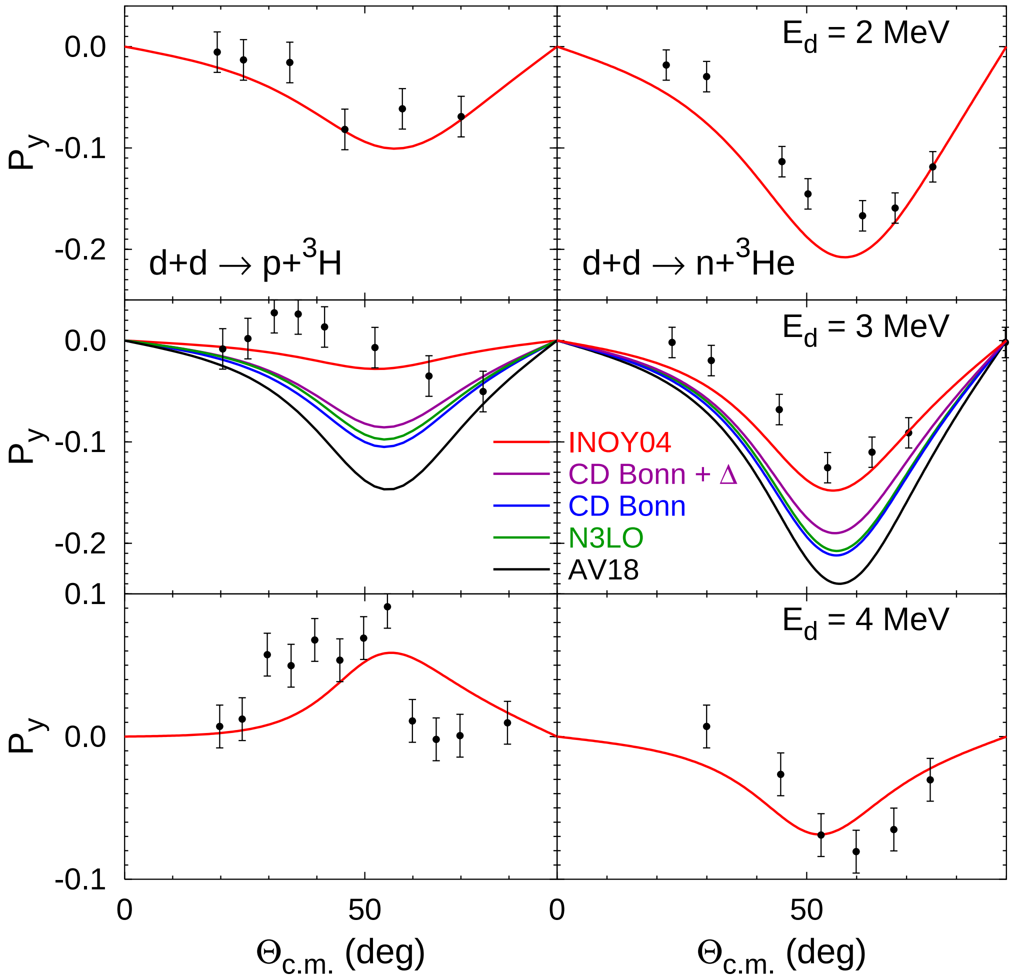
$^2H(d,p)^3H$ and $^2H(d,n)^3He$ – analyzing powers

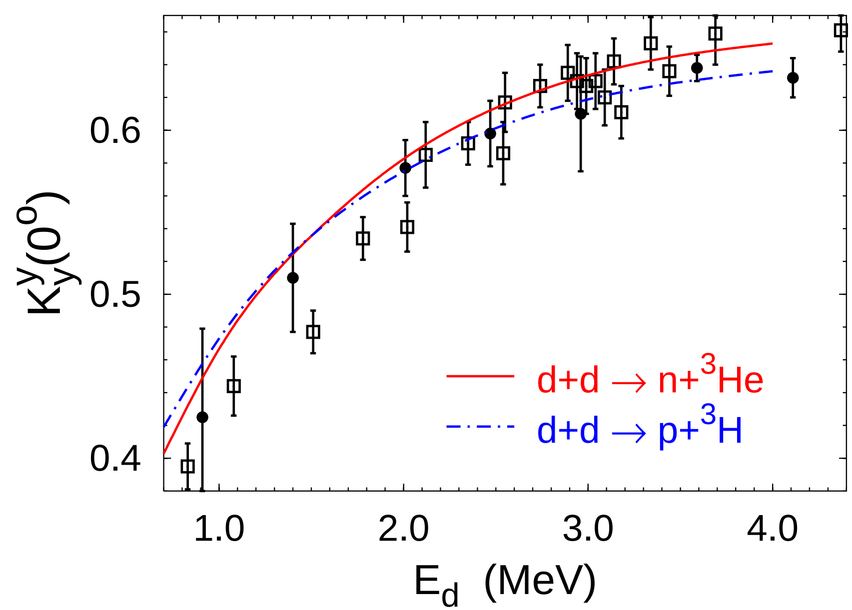


^{2}H(d,p)^{3}H and ^{2}H(d,n)^{3}He – analyzing powers

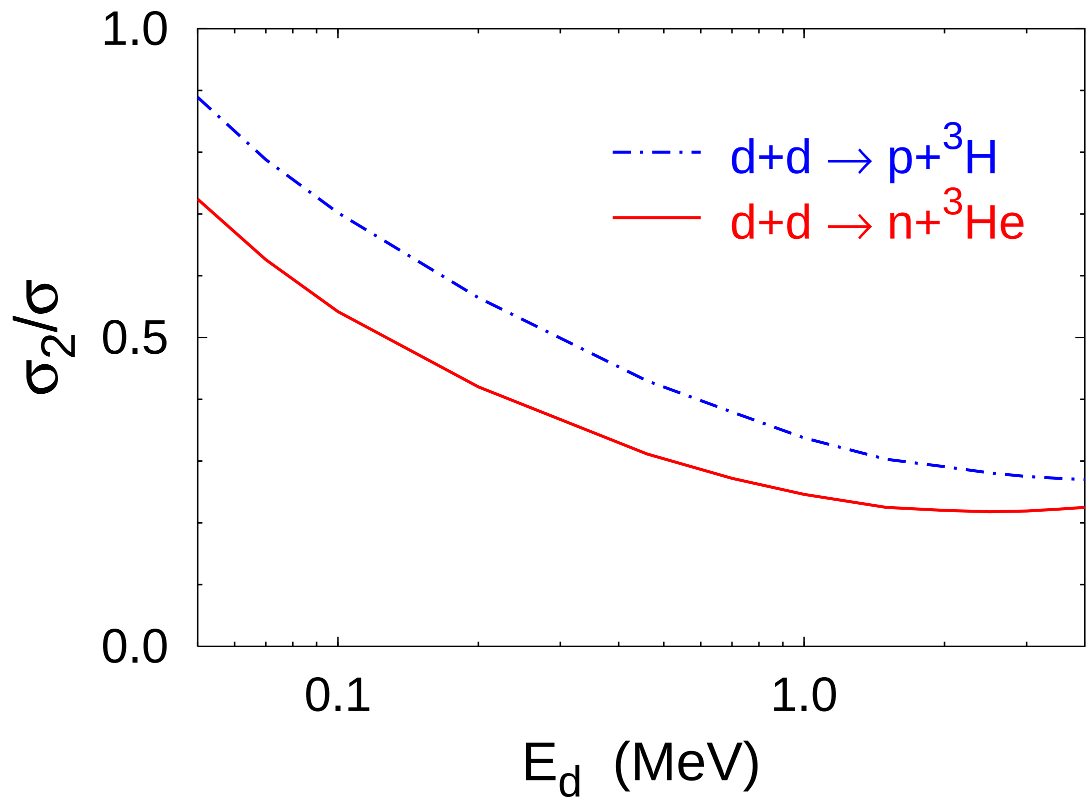


^{2}H(d,p)^{3}H and ^{2}H(d,n)^{3}He – final-state polarization



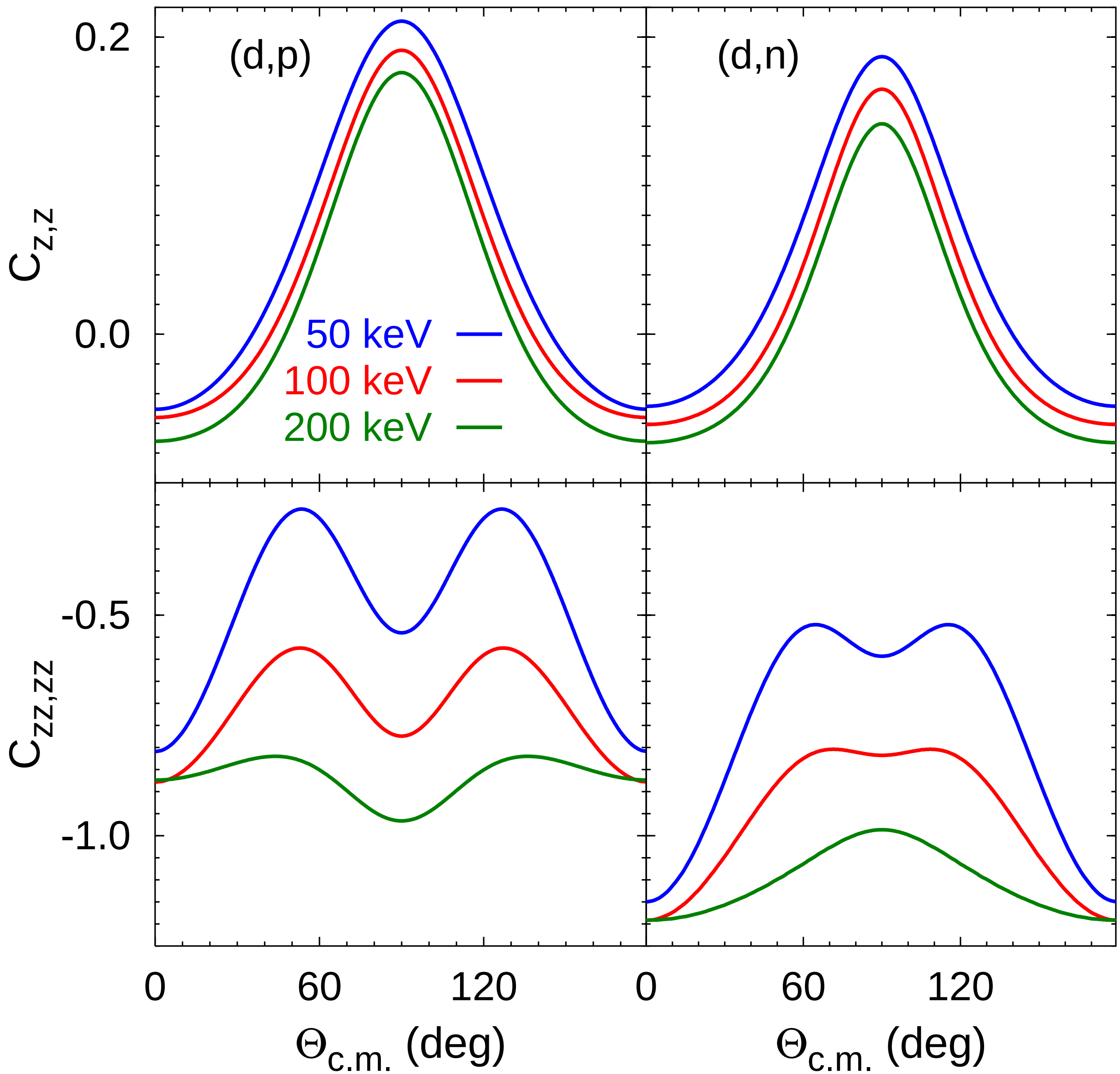


^{2}H(d,p)^{3}H and ^{2}H(d,n)^{3}He – QSF

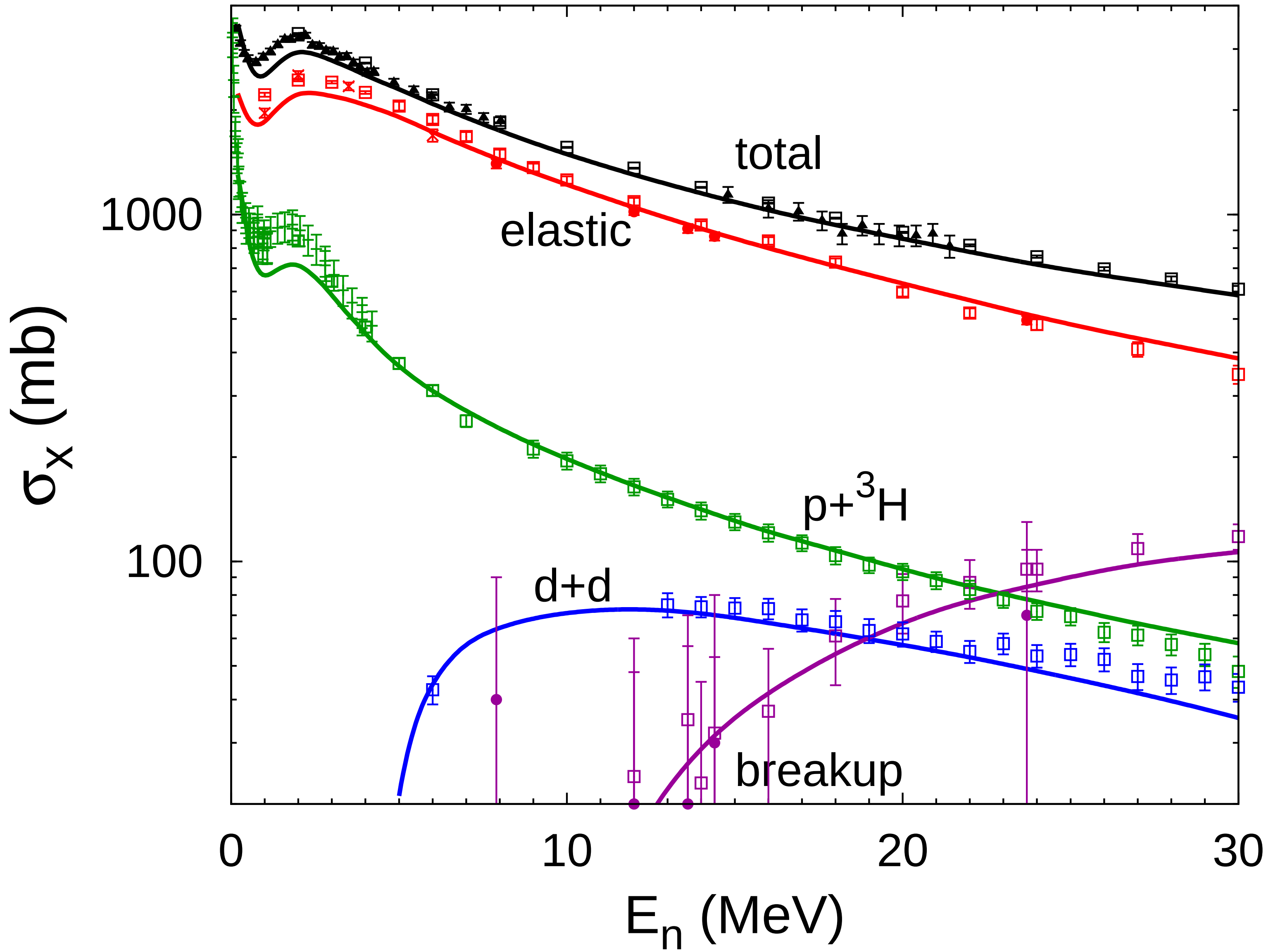


$$\sigma_{11}/\sigma - \sigma_2/\sigma < 0.025$$

^{2}H(d,p)^{3}H and ^{2}H(d,n)^{3}He – spin correlation

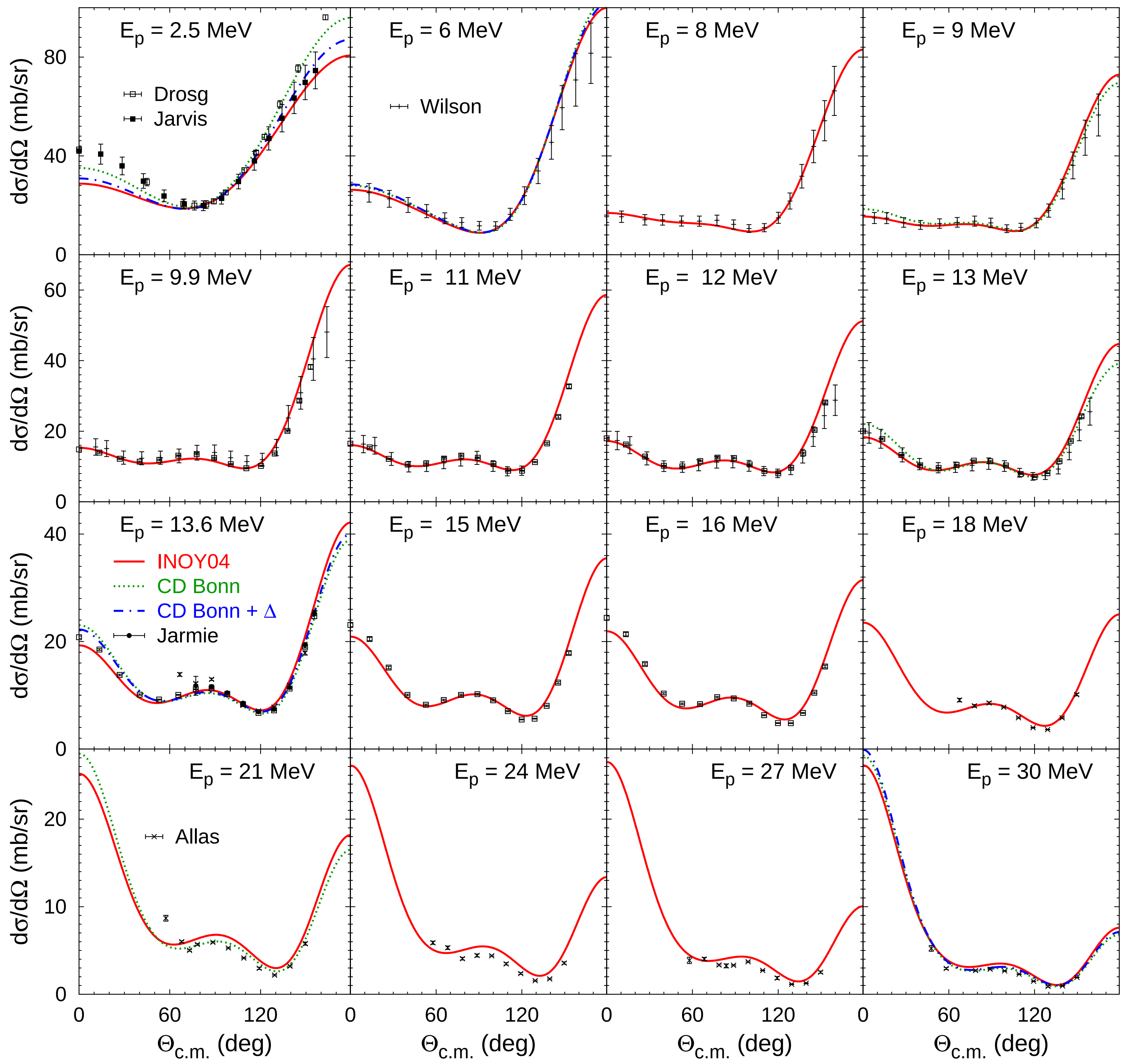


n+^{3}He total and partial cross sections

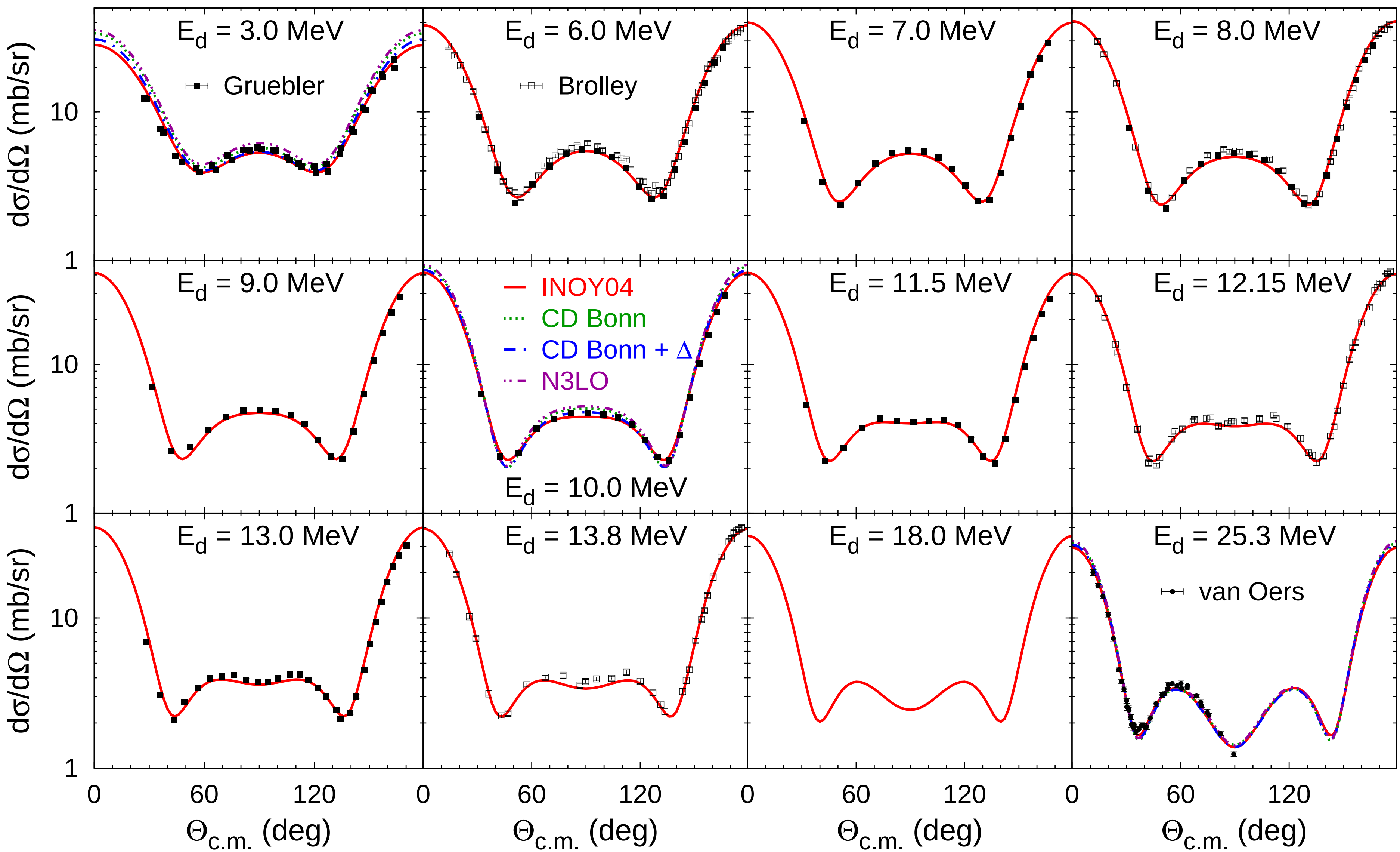


[PRL 113, 102502; PRC 90, 044002]

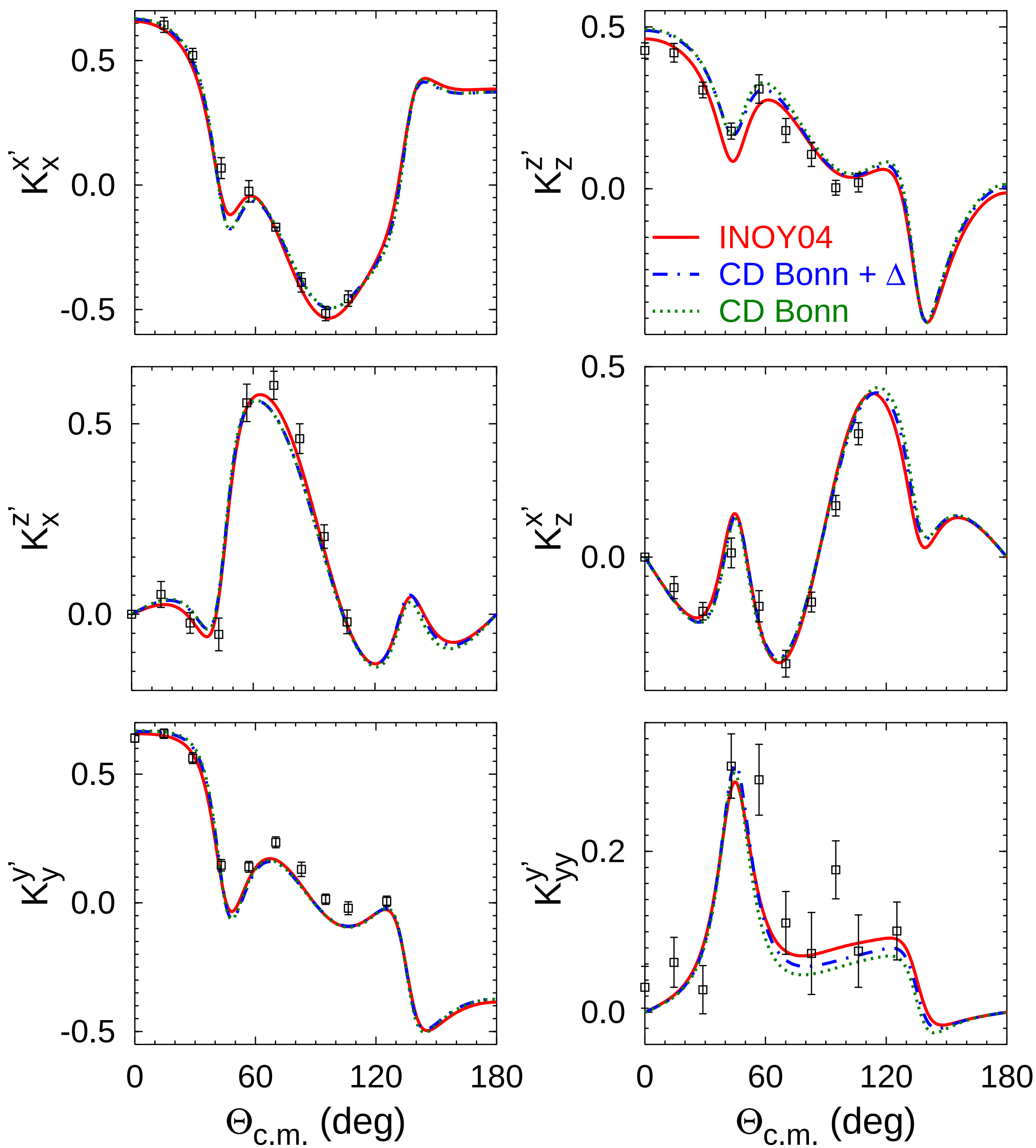
Charge exchange reaction $^3H(p,n)^3He$



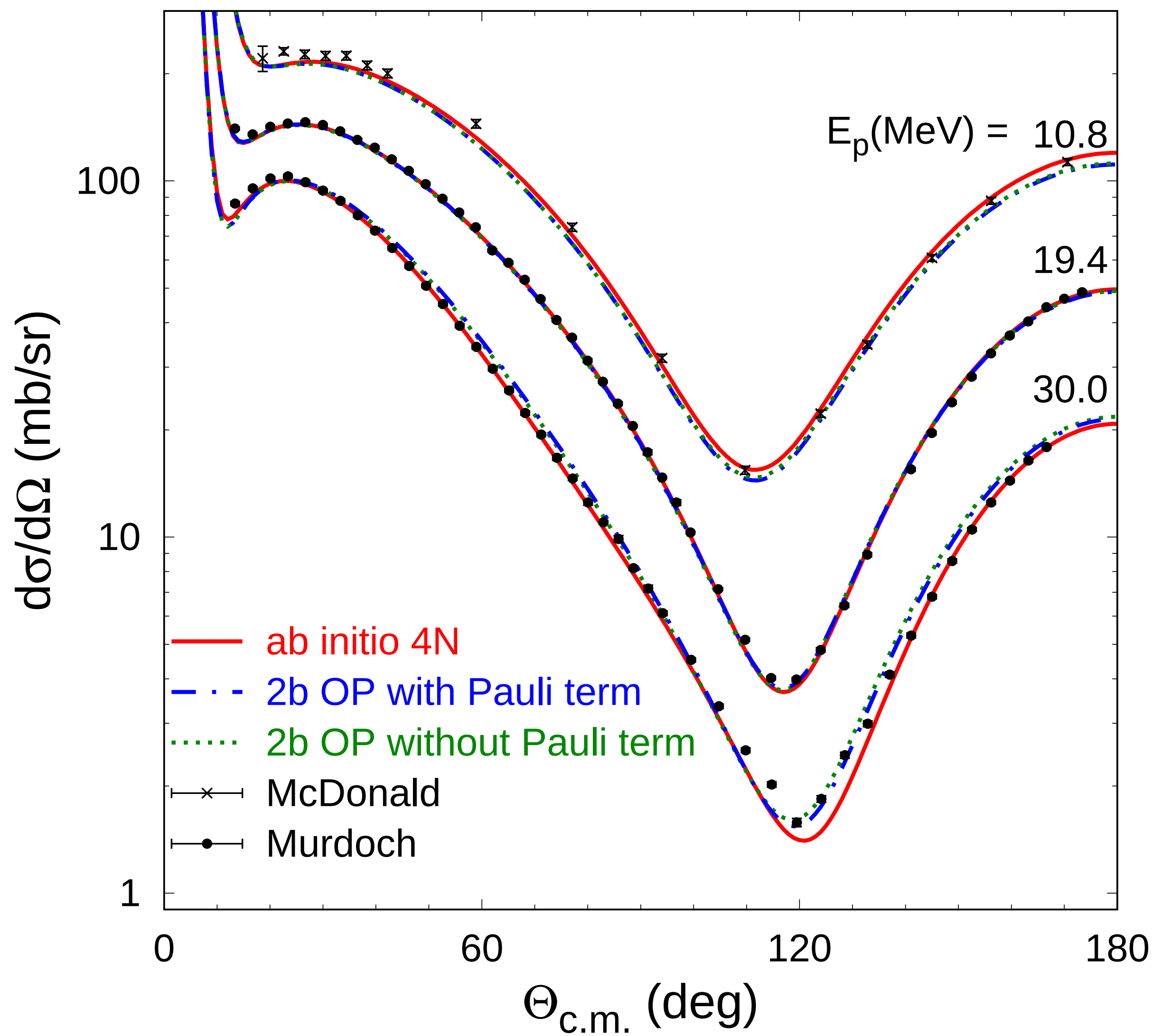
Transfer reaction $^{2}\mathrm{H}(d,p)^{3}\mathrm{H}$



Spin transfer in $^2\mathrm{H}(\vec{d},\vec{n})^3\mathrm{He}$ at 10 MeV

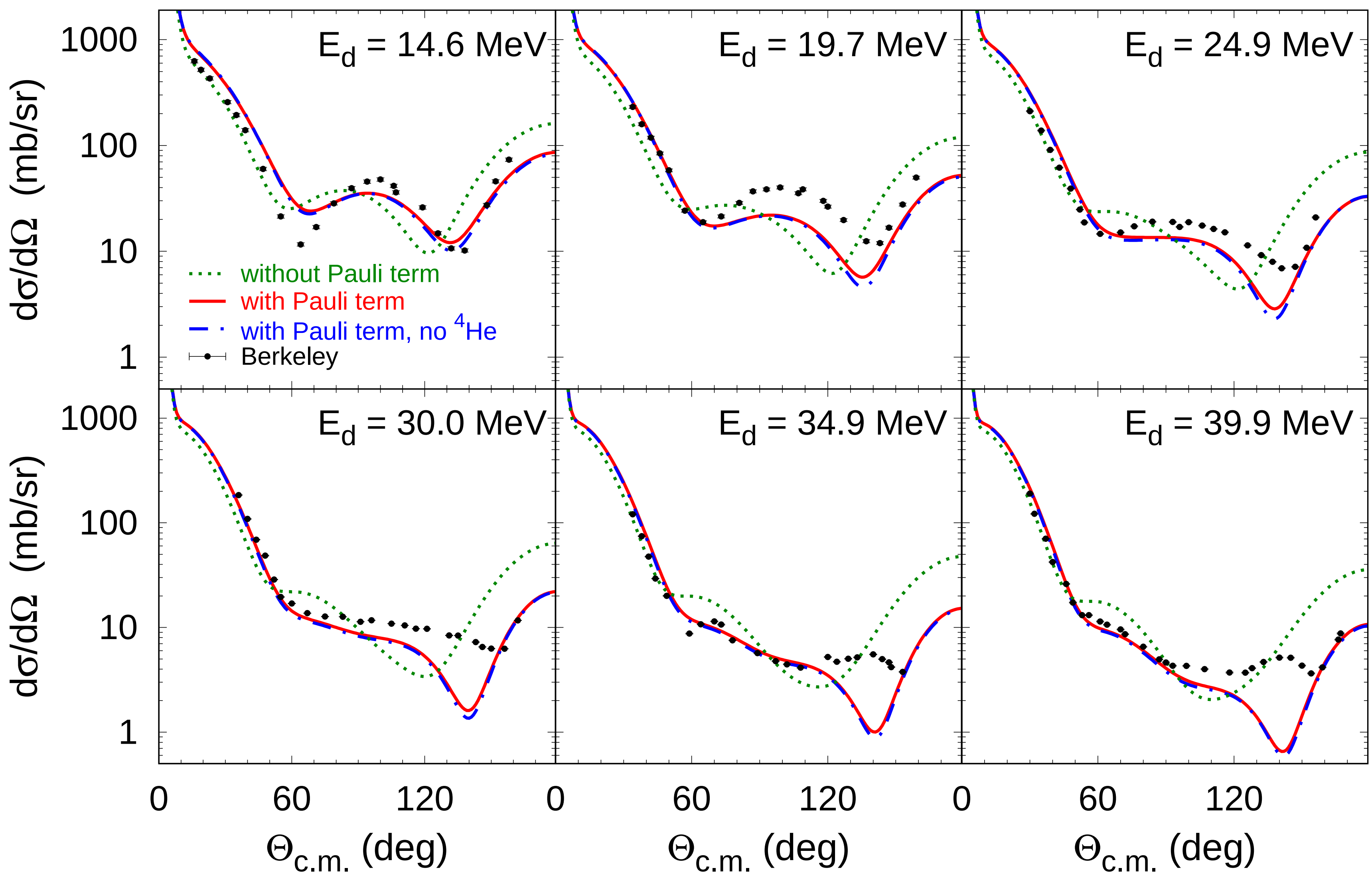


Optical potential for N-[3N] scattering



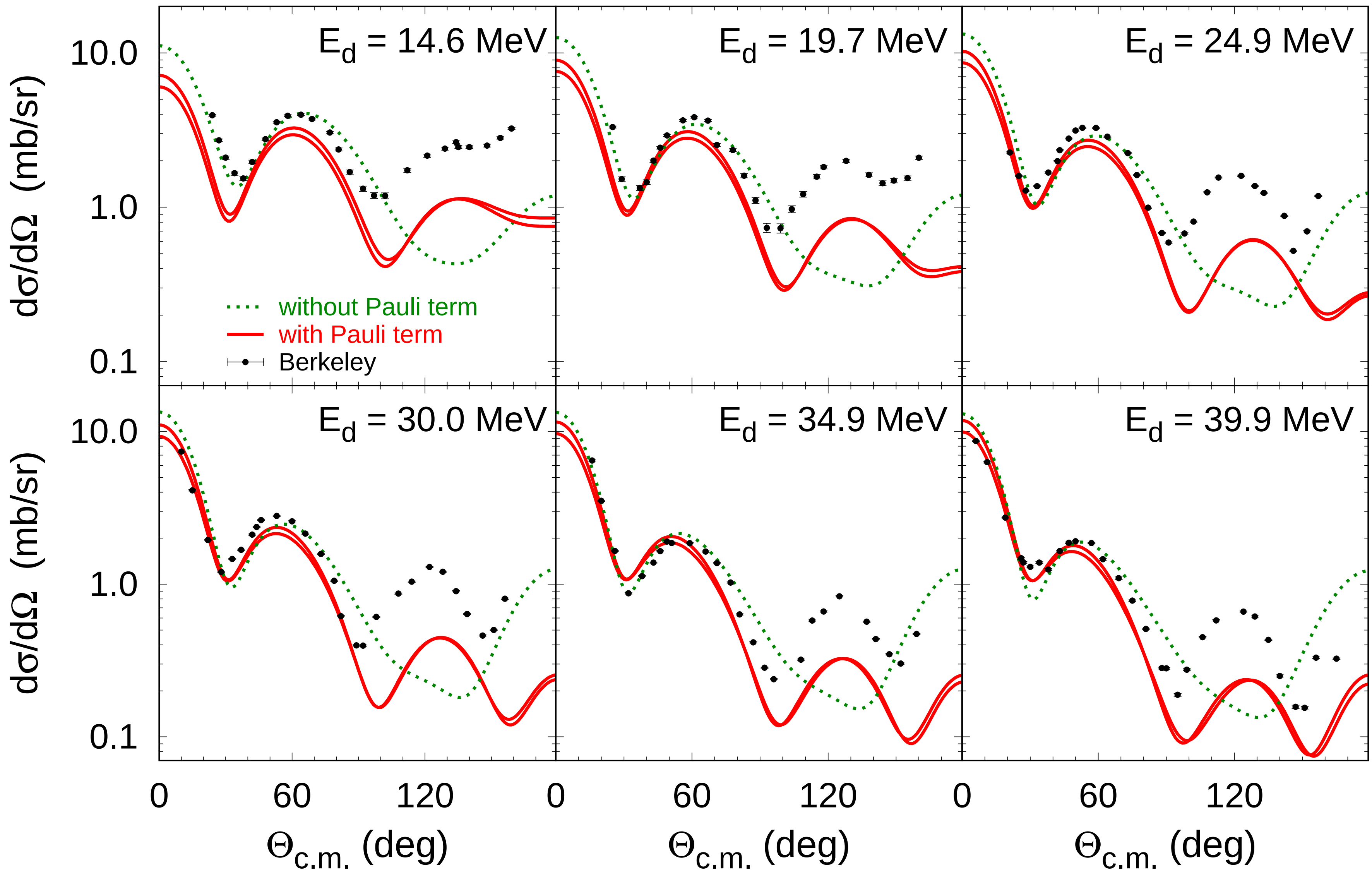
$$V + iW + |b\rangle S_P \langle b|$$

Elastic scattering d+^{3}He



[PLB 860, 139151]

Neutron transfer $^3He(d,p)^4He$



Nonlocal OP + core excitation + Faddeev equations

- 2009: Nonlocal OP in (d,p) reactions
  [AD, PRC 79, 021602]
- 2013: ADWA with nonlocal OP
  [Timofeyuk, Nunes, ...]
- 2018: ADWA with nonlocal OP is inaccurate,
  proven by Faddeev [AD, PRC 98, 021603]
  and CDCC [Gomez-Ramos et al, PRC 98, 011601]

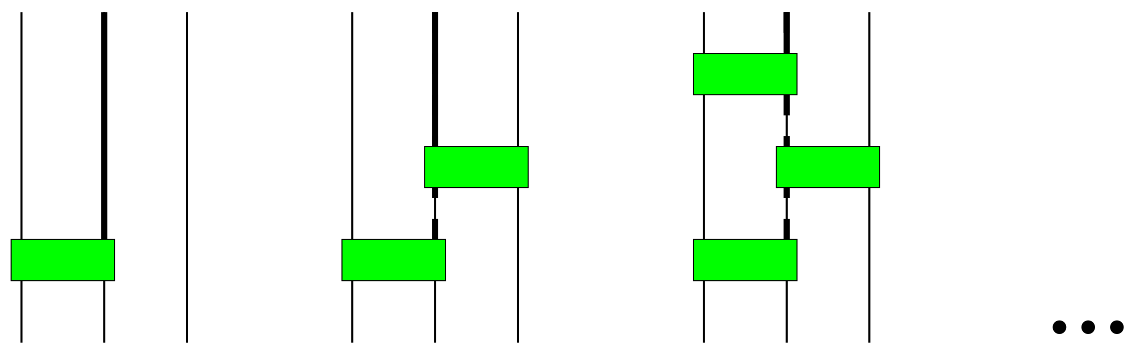
Faddeev/AGS equations with core excitation

$$U^{ba}_{\beta\alpha} = \bar{\delta}_{\beta\alpha}\delta_{ba}G_0^{-1} + \sum_{\sigma}\sum_{j}\bar{\delta}_{\beta\sigma}T^{bj}_{\sigma}G_0U^{ja}_{\sigma\alpha}$$

$$U^{ba}_{0\alpha} = \delta_{ba}G_0^{-1} + \sum_{\sigma}\sum_{j}T^{bj}_{\sigma}G_0U^{ja}_{\sigma\alpha}$$

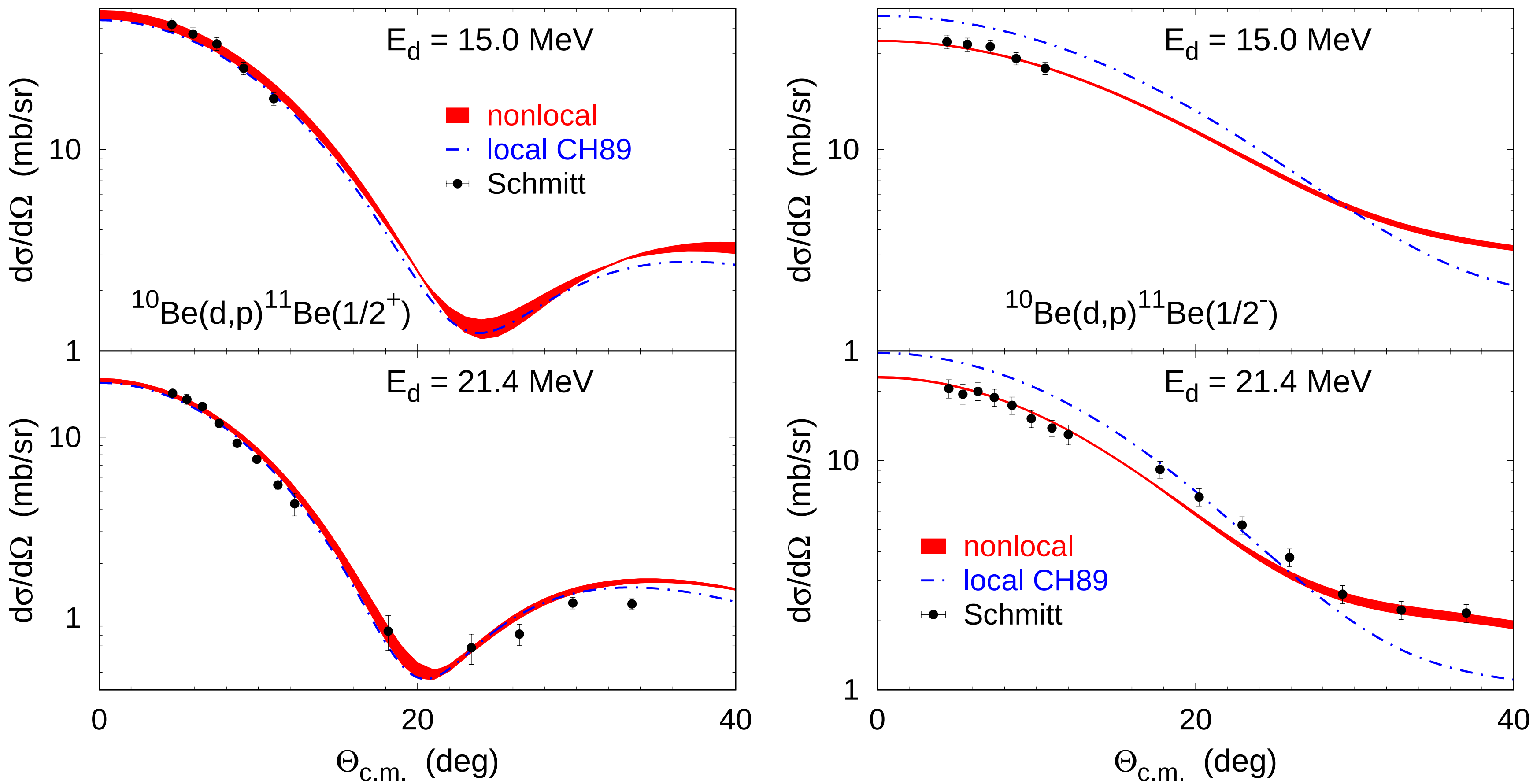
$$T^{ba}_{\sigma} = V^{ba}_{\sigma} + \sum_{j}V^{bj}_{\sigma}G_0T^{ja}_{\sigma}$$

$$G_0 = (E + i0 - H_0 - h_i)^{-1}$$



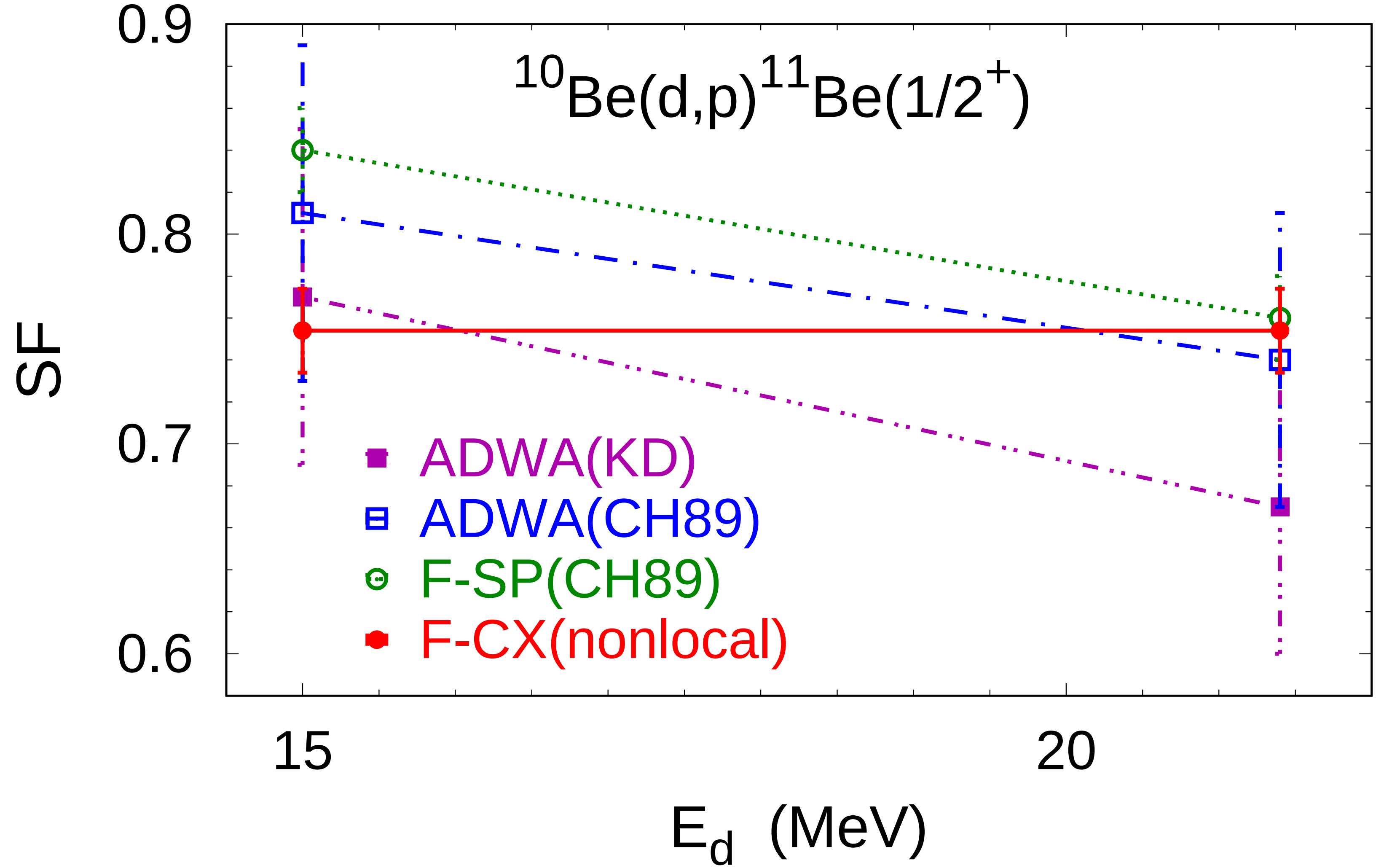
[PRC 88, 011601; PLB 769, 202; PRC 99, 024613]

Neutron transfer: ^{10}Be(d,p)^{11}Be



[PLB 840, 137867]

Spectroscopic factors: ^{10}Be(d,p)^{11}Be



[ADWA: Schmitt et al., PRL 108, 192701]

Inclusive breakup A(d,p)X

- Hussein & McVoy, Udagawa & Tamura, Ichimura et al.: approximate scattering wave functions (DWBA, Glauber, ...)
- Hussein, Frederico, Mastroleo, NPA 511, 269: Faddeev wave functions (no numerics)
- AD, PLB 868, 139825: Faddeev/AGS transition operators + numerical solution

Inclusive breakup A(d,p)X in AGS formalism

$$\frac{d^3\sigma}{d^3\mathbf{q}_\alpha} = (2\pi)^4 \frac{M_1}{f_s q_1^i} \sum_a \int d^3\mathbf{p}_\alpha \, \delta\left(E - \frac{p_\alpha^2}{2\mu_\alpha} - \frac{q_\alpha^2}{2M_\alpha} - E_a\right) \times |\langle \mathbf{p}_\alpha \mathbf{q}_\alpha a | U_{01} | \phi_1 \mathbf{q}_1^i \rangle|^2$$

$$\frac{d^3\sigma}{d^3\mathbf{q}_\alpha} = -(2\pi)^4 \frac{M_1}{\pi f_s q_1^i} \, \mathrm{Im}\left[\langle \phi_1 \mathbf{q}_1^i | U_{\alpha 1}^\dagger G_\alpha U_{\alpha 1} | \phi_1 \mathbf{q}_1^i \rangle_q\right]$$

Elastic and non-elastic breakup in AGS formalism

OP : $G_\alpha = (E + i0 - H_0 - v_\alpha - iw_\alpha)^{-1}$

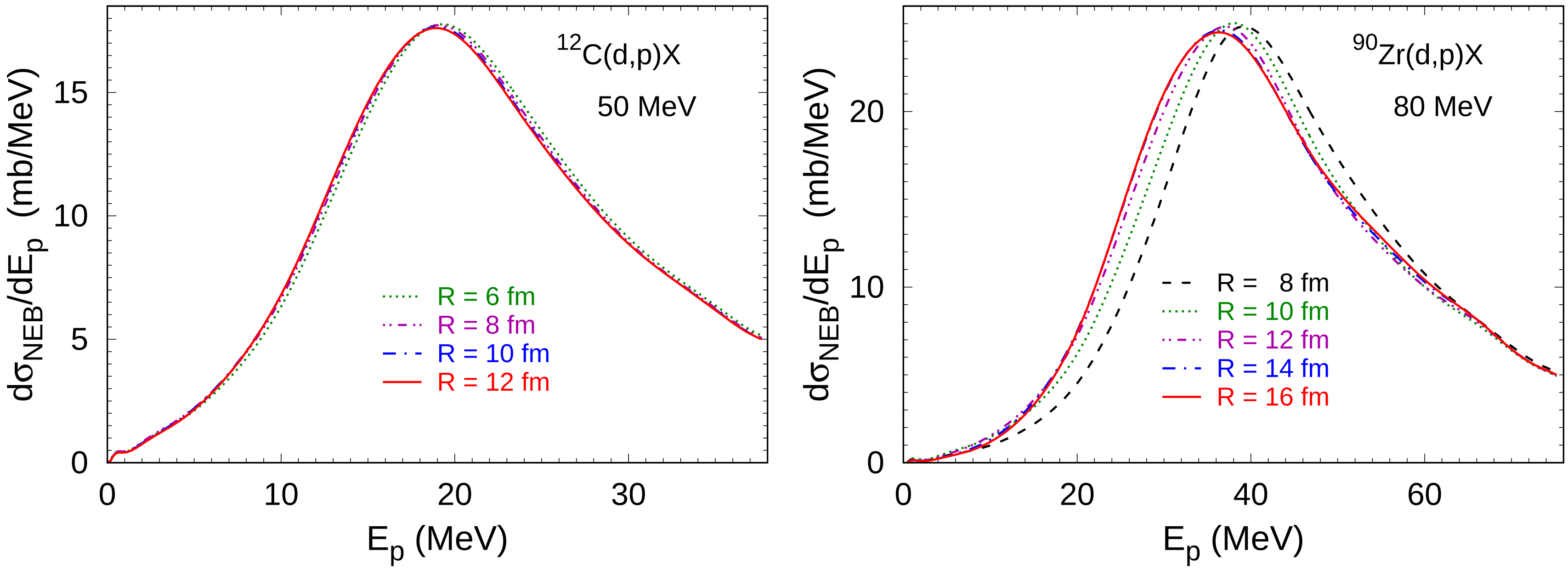
$$\mathrm{Im} G_\alpha = -\pi(1 + G_0^\dagger T_\alpha^\dagger)\delta(E - H_0)(1 + T_\alpha G_0) + G_\alpha w_\alpha G_\alpha$$

$$\frac{d^3\sigma_{\mathrm{EB}}}{d^3\mathbf{q}_\alpha} = (2\pi)^4 \frac{M_1}{f_s q_1^i} \int d^3\mathbf{p}_\alpha \, \delta\left(E - \frac{p_\alpha^2}{2\mu_\alpha} - \frac{q_\alpha^2}{2M_\alpha}\right) \times |\langle \mathbf{p}_\alpha \mathbf{q}_\alpha | U_{01} | \phi_1 \mathbf{q}_1^i \rangle|^2$$

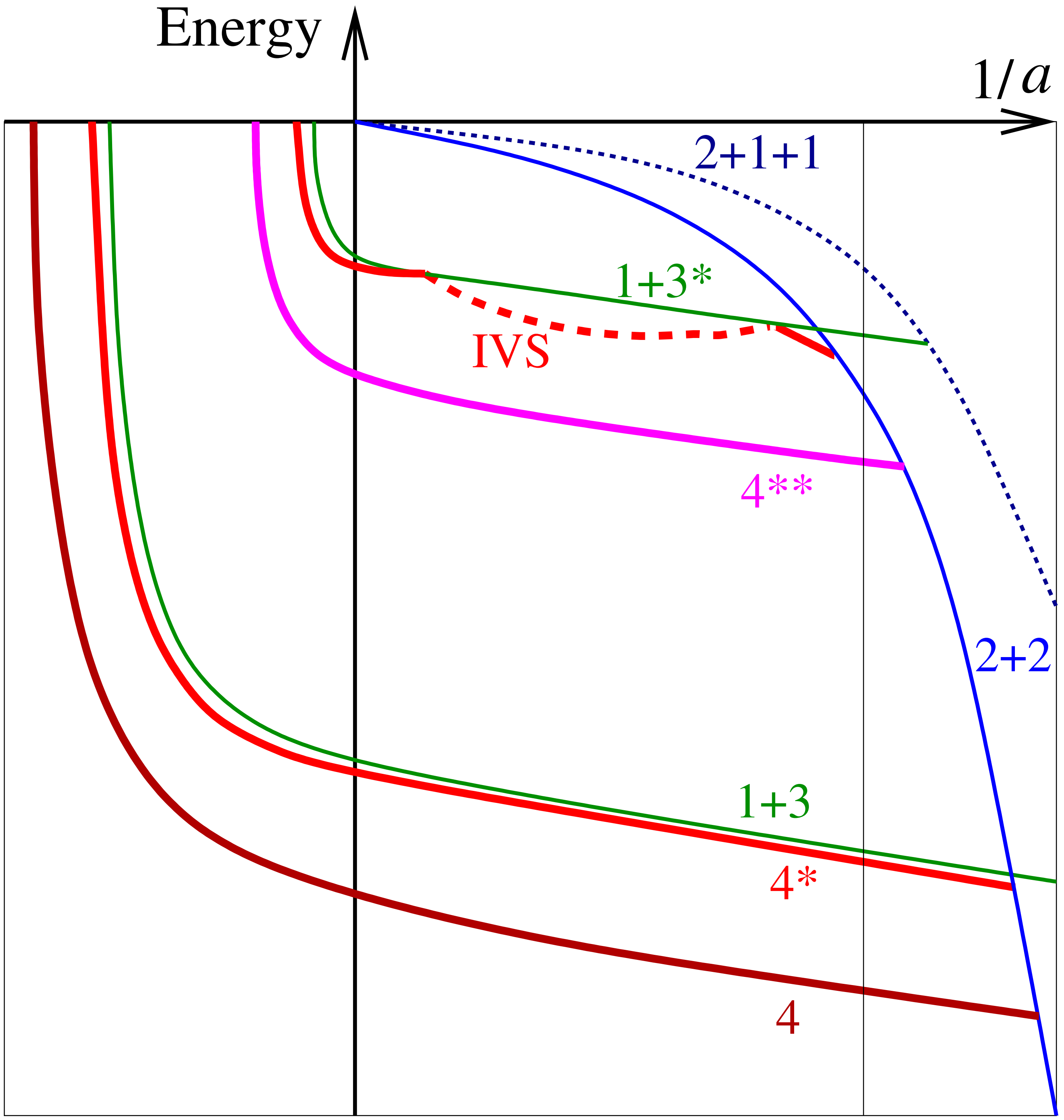
$$\frac{d^3\sigma_{\mathrm{NEB}}}{d^3\mathbf{q}_\alpha} = -(2\pi)^4 \frac{M_1}{\pi f_s q_1^i} \langle \phi_1 \mathbf{q}_1^i | U_{01}^\dagger G_0^\dagger w_\alpha G_0 U_{01} | \phi_1 \mathbf{q}_1^i \rangle_q$$

$$= -(2\pi)^4 \frac{M_1}{\pi f_s q_1^i} \int [d^3\mathbf{p}'_\alpha d^3\mathbf{p}_\alpha w_\alpha(\mathbf{p}'_\alpha, \mathbf{p}_\alpha) \times \langle \mathbf{p}'_\alpha \mathbf{q}_\alpha | G_0 U_{01} | \phi_1 \mathbf{q}_1^i \rangle^* \langle \mathbf{p}_\alpha \mathbf{q}_\alpha | G_0 U_{01} | \phi_1 \mathbf{q}_1^i \rangle]$$

NEB in A(d,p)X



$^{12}C(d,p)X$
50 MeV
R = 6 fm
R = 8 fm
R = 10 fm
R = 12 fm
$d\sigma_{NEB}/dE_p$ (mb/MeV)
E_p (MeV)
0
5
10
15
20
30
$^{90}Zr(d,p)X$
80 MeV
R = 8 fm
R = 10 fm
R = 12 fm
R = 14 fm
R = 16 fm
40
60


Energy
1/a
2+1+1
1+3*
IVS
4**
2+2
1+3
4*
4

Excited tetramer and atom-trimer scattering

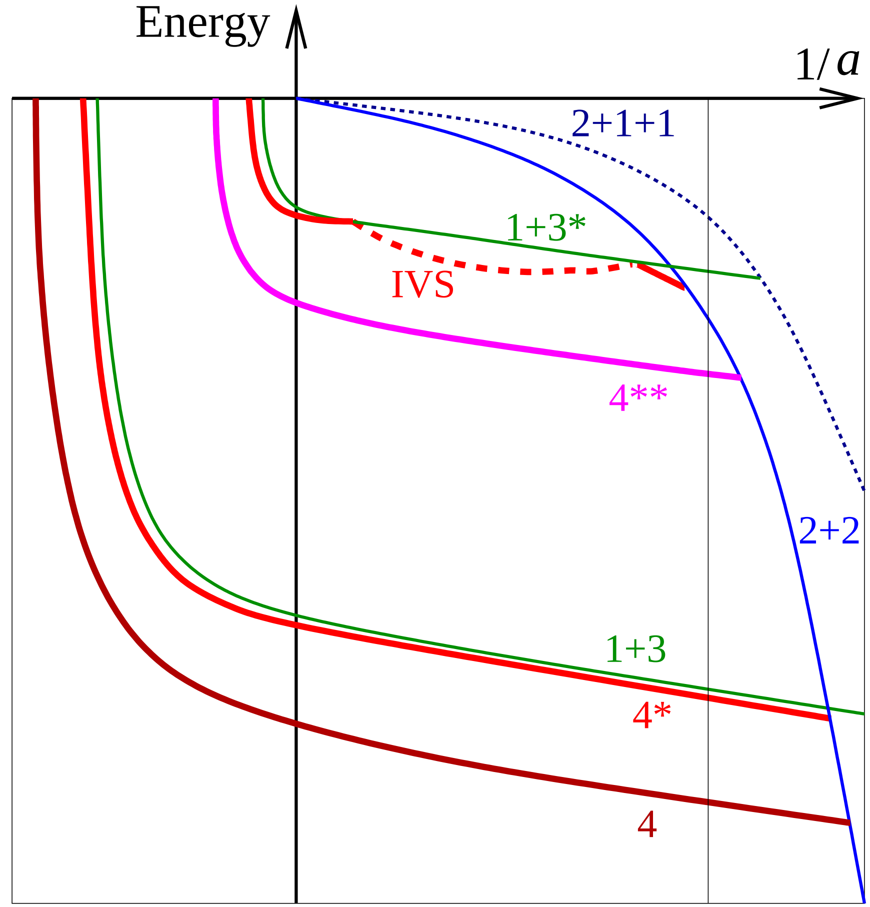
	$B_4 - B_3^*$ (mK)	A_{13} (Å)	R_{13} (Å)
Lazauskas et al.	1.09	103.7	29.1
Hiyama et al.	0.93		
AD	0.96	108.8	29.2

R. Lazauskas, J. Carbonell, PRA 73, 062717
E. Hiyama, M. Kamimura, PRA 85, 022502

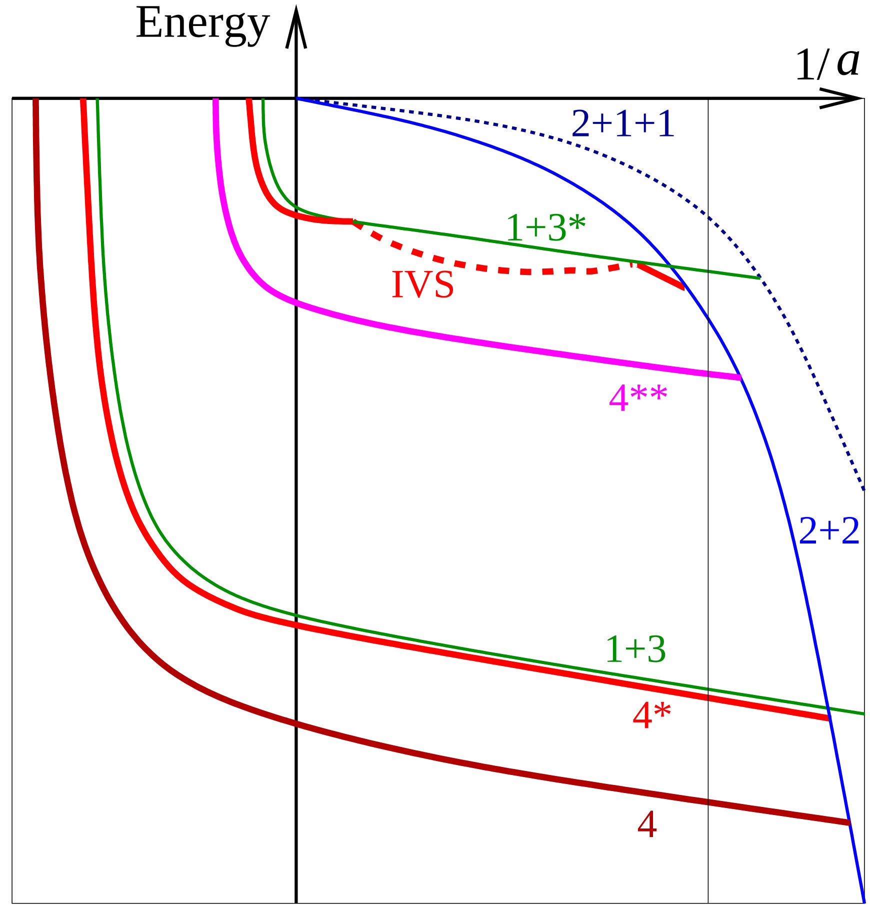
Scattering length and effective range

	A_{ab} (Å)	R_{ab} (Å)
atom-atom	100.0	7.33
atom-dimer	115.2	79.0
atom-trimer	108.8	29.2
dimer-dimer	$100.5 - i0.75$	$10.5 - i0.2$

Second excited tetramer state



Second excited tetramer state



$$E_4^{**}/B_3^* \approx -1.82 - i0.0045$$

Comparison with universal limit

	LM2M2	universal
A_{22}/a	$1.005 - i0.0075$	$0.945 - i0.004$
R_{22}/a	$0.105 - i0.002$	$-0.070 - i0.012$
E_4^{**}/B_3^*	$-1.82 - i0.0045$	$-1.888 - i0.0024$

Momentum-space FY/AGS

- 3-body reactions

$$\left.\begin{matrix} p+(nA) \\ d+A \end{matrix}\right\} \rightarrow \left\{\begin{matrix} n+(pA) \\ p+(nA) \\ d+A \\ p+n+A \end{matrix}\right.$$

- 4N reactions

$$\left.\begin{matrix} p+{}^3\mathrm{H} \\ n+{}^3\mathrm{He} \\ d+d \end{matrix}\right\} \rightarrow \left\{\begin{matrix} p+n+d \\ p+{}^3\mathrm{H} \\ n+{}^3\mathrm{He} \\ d+d \\ 2p+2n \end{matrix}\right.$$

Screening and renormalization (DFS version)

- **standard scattering equations**:
  transition operators, momentum space,
  partial waves, without separable approximation,
  straightforward extension to 4b scattering

- **limitations in practical applicability**:
  $>$ 2 charged clusters for breakup ?
  low energy, high charge ?
  large screening radius and angular momentum !

[Deltuva,Fonseca,Sauer:
PRC 71, 054005; PRC 72, 054004; PRC 73, 057001;
PRL 98, 162502; PRC 80, 064002; EPJ WoC 3, 01003]

Screening and renormalization: extending with ML

- 2b results for training:
  low energy, large screening

- extrapolating p+d results in R:
  talk by Darius Likandrovas

- low-energy (d,p) reactions at ISOLDE:
  ^{28}Na, ^{28}Mg, ^{31}Mg, ...