~~Nuclear~~ Superfluidity: from Cold Atoms to Neutron Stars

Michael Urban (IJCLab, Orsay, France)

Outline

1. Superfluidity in terrestrial laboratories
2. Superfluidity in neutron stars
3. Ultracold Fermi gases in the BCS-BEC crossover
4. Pairing in dilute neutron matter
5. Superfluidity in the inhomogeneous inner crust of neutron stars
6. Conclusions

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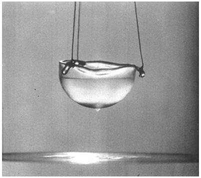
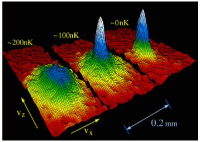
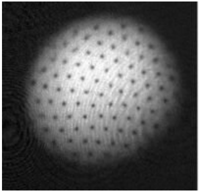
Superfluidity in terrestrial laboratories

- Superfluidity: absence of viscosity at low temperature (analogous to absence of resistance in superconductors)
- ^{4}He (bosons): $T_c \approx 2.2$ K
  [Kapitza (1938)]
- ^{3}He (fermions): $T_c \approx 2.6$ mK
  [Osheroff et al. (1972)]
- Bose-Einstein condensation in atom traps: $T_c \sim 100$ nK
  [Cornell, Wieman, Ketterle (1995)]
- Fermionic superfluid in atom traps: $T_c \sim 100$ nK
  [JILA, MIT, Innsbruck, ENS Paris (2004)]
- Since 1950s: indications for Cooper pairing and superfluidity in atomic nuclei:
  - odd-even mass staggering,
  - collective excitations,
  - reduced moments of inertia...





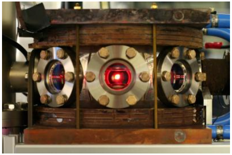
[JILA]



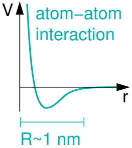
[MIT]

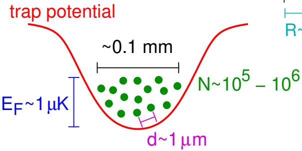
Ultracold atoms

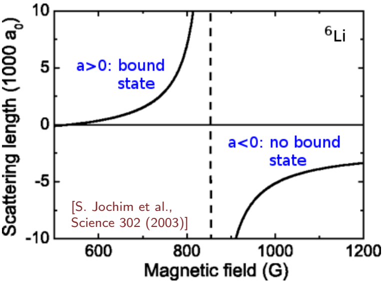
- 1995: first BEC of trapped bosonic atoms
- Fermions are more difficult to cool
  $\rightarrow$ Fermi superfluid realized only in 2004
- Interaction between the atoms:
  $R \sim 10^{-9}$ m $\ll d \sim 1/k_F \sim 10^{-6}$ m
  $\rightarrow$ contact interaction
- Pauli principle: interaction (s wave) only between atoms of opposite "spin" $(\uparrow, \downarrow)$
- Interaction strength is characterized by the scattering length a
- Feshbach resonance: scattering length a can be tuned experimentally by changing the magnetic field B
- The case $a \rightarrow \infty$ is called the unitary limit



[Heidelberg University]



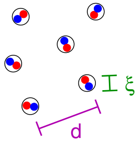




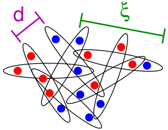
BCS-BEC crossover in ultracold atoms

- Superfluidity in bosonic system: Bose-Einstein condensation (BEC)

- **For $a > 0$**, fermionic atoms form dimers (molecules made from one $\uparrow$ and one $\downarrow$ atom)
- The bosonic dimers condense (BEC) at $T < T_c$
- Substructure of dimers negligible if $\xi \ll d \quad$ (i.e., $1/(k_F a) \gg 1$)



- **For $a < 0$**, two atoms in free space do not have a bound state
- But in the medium, they can form Cooper pairs
- BCS theory valid if $\xi \gg d \quad$ (i.e., $1/(k_F a) \ll -1$)

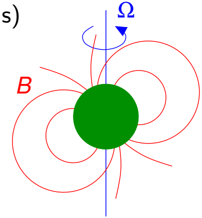


- **BCS-BEC crossover**: continuous transition between these two limits $\xi \sim d \quad (-1 \lesssim 1/(k_F a) \lesssim 1)$
- Particular case: unitary limit $1/(k_F a) = 0$

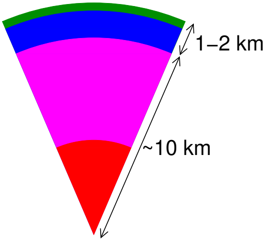
Outline

Basic properties of neutron stars

- Produced in core-collapse supernova explosions
- Very compact: $M \sim 1 - 2M_\odot$ $(2 - 4 \times 10^{30}$ kg) in a radius of $R \sim 10$ km
  $\rightarrow$ $\rho >$ nuclear saturation density
- Rapid rotation (periods range from seconds to milliseconds)
- Strong magnetic field B
  typically 10^{12} G, in "magnetars" up to 10^{14} G
- B not aligned with the rotation axis leads to periodic e.m. emission (pulsar) and slows down the rotation



- A neutron star has a complex inner structure:



outer crust: Coulomb lattice of neutron rich nuclei in a degenerate electron gas

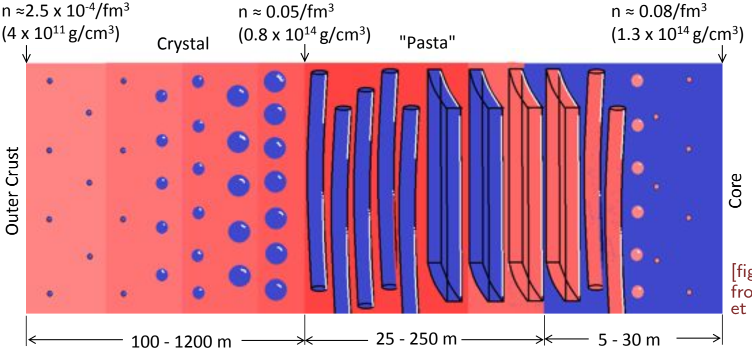
inner crust: unbound neutrons form a neutron gas between the nuclei (clusters)

outer core: homogeneous $n, p, e^-, (\mu^-)$ matter

inner core: densities up to a few times ρ_0,
new degrees of freedom: hyperons? quark matter?

Structure of the inner crust

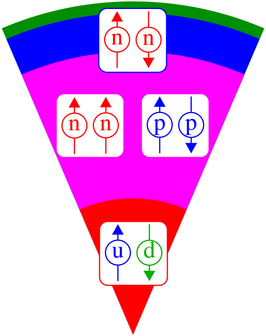
- Presence of a gas of unbound neutrons between the nuclei (clusters) + almost uniform degenerate electron gas to ensure global charge neutrality
- BCC crystal and "pasta phases": rods ("spaghetti"), slabs ("lasagne")



[figure adapted from W. Newton et al. (2011)]

Superfluidity in neutron stars

- Typical temperature of a neutron star: $T \sim 10^6 - 10^9$ K $\sim 0.1 - 100$ keV
- Compared to nuclear energy scales, this is very low!
- BCS gap equation: $$\Delta_p = -\sum_{p'} V_{p,p'} \frac{\Delta_{p'}}{2\sqrt{(\epsilon_{p'} - \mu)^2 + \Delta_{p'}^2}}$$
- Different types of superfluidity may exist in neutron stars:



inner crust:
neutron pairing in s wave (pairs with total spin $S = 0$),
$T_c \sim 1$ MeV $\longrightarrow$ main subject of this talk

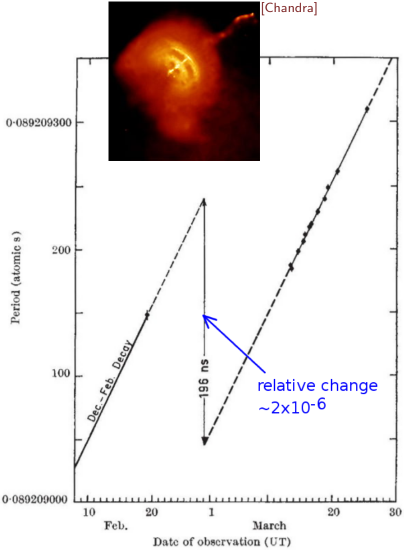
outer core:
neutron pairing in p wave (pairs with total spin $S = 1$)
proton pairing in s wave

quark core (speculative):
"color superconductivity", $T_c \sim 10$ MeV
[e.g. Alford et al. RMP (2008)]

Pulsar glitches

- Rotation of a neutron star: very regular, period increases slowly with time
- Glitch = sudden speed-up of the rotation, followed by a slow relaxation
- First glitch observed 1969 in the Vela pulsar, since then 520 glitches in 180 different pulsars [R.N. Manchester (2017)]
- Widely accepted explanation by Manchester and Itoh (1975): pinning of quantized vortices to the clusters in the inner crust
- While the normal part of the star is slowing down (Ω_n), the superfluid neutrons are spinning at constant frequency (Ω_s)
- When $\Omega_s - \Omega_n$ becomes too large, the vortices get unpinned and the superfluid transfers angular momentum to the normal part

[Chandra]



[Radhakrishnan and Manchester, Nature 222 (1969)]

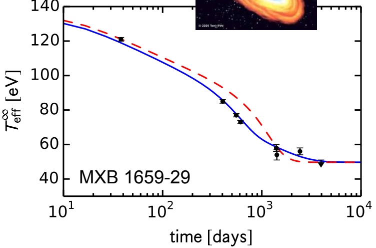
Cooling

- One day after the supernova, T has already dropped from $\sim 10^{11}$ to $\sim 10^{9}$ K
- For about 10^5 years, ν emission (from the core) is the dominant cooling mechanism
- For older stars, cooling is dominated by photon emission
- Cooper pairing affects cooling through:
  - $\nu\bar{\nu}$ emission via the PBF (pair breaking and formation) mechanism,
  - strongly reduced specific heat

Special case: accreting neutron stars

- Neutron star with a companion star
- Matter falling on the neutron star heats the surface
- Deep crustal heating: nuclear reactions in deeper layers of the crust
- X-ray outbursts take a few weeks or months (or even years), then cooling during a couple of years of quiescence
- Particularly sensitive to Cooper pairing in the neutron-star crust

[figure by T. Piro]



[Deibel et al., ApJ 839 (2017)]

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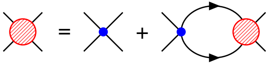
BCS mean field approach with contact interaction

- Determine gap Δ and chemical potential μ from gap and number equations $(\epsilon_k = \frac{k^2}{2m}, \quad E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2})$

$$\Delta = -g \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2E_k} \qquad\qquad n = 2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{2} - \frac{\epsilon_k - \mu}{2E_k} \right)$$

- Scattering length for coupling constant $g < 0$ and cutoff Λ

$$\frac{4\pi a}{m} = g + g \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{-2\epsilon_k} \frac{4\pi a}{m}$$

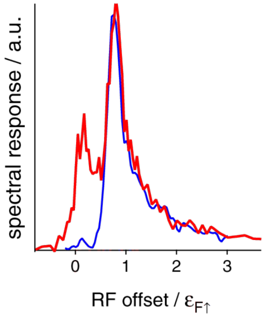


- Express in the gap equation g in terms of a: $\Delta = -\frac{4\pi a}{m} \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \left(\frac{\Delta}{2E_k} - \frac{\Delta}{2\epsilon_k} \right)$
- Coupling constant vanishes for $\Lambda \to \infty$: $\quad \frac{1}{g} = \frac{m}{4\pi a} - \frac{m\Lambda}{2\pi^2}$
- Hartree field vanishes in this limit: $U_\sigma = g n_{-\sigma} \overset{\Lambda\to\infty}{\longrightarrow} 0$
- In order to get the simplest weak-coupling correction $\frac{4\pi a}{m} n_\uparrow \, n_\downarrow$ to the GS energy, resummation of ladder diagrams is necessary

Gap and T_c at unitarity ($a \to \infty$): experiments

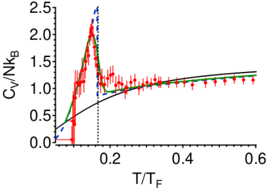
- Advantage of unitarity: all quantities scale with $\epsilon_F = \frac{k_F^2}{2m}$
- Radio frequency (RF) spectroscopy:
  measure energy needed to transfer atoms of state
  1 = "$\uparrow$" or 2 = "$\downarrow$" into a third hyperfine state 3.
- Schirotzek et al. PRL 101, 140403 (2008):
  two-peak structure if n_1 slightly larger than n_2
  (excess 1 particles have already energy $\sim \Delta$ while
  paired 1 and 2 particles require energy to dissociate the pair)

  Gap: $\Delta/\epsilon_F = 0.44(3)$
  Hartree shift: $U/\epsilon_F = -0.43(3)$



- Harmonic trap + local-density-approximation (LDA):
  range of densities and hence of T/ϵ_F in one system
- Ku et al., Science 335, 563 (2012):
  all thermodynamic quantities can be obtained from
  high-precision measurements of the density profile

  Superfluid transition: $T_c/\epsilon_F = 0.167(13)$
  Bertsch parameter: $\xi = \mu_{T=0}/\epsilon_F = 0.376(4)$



Effects beyond BCS theory in the BCS-BEC crossover

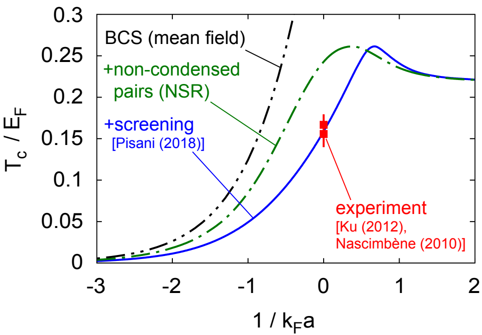
(a) Non-condensed pairs

- BEC limit: dimers exist at $T > T_c$ but are not condensed
- BCS limit: pair formation and condensation take place at the same temperature
- Crossover: necessary to include non condensed pairs at $T > T_c$
  [Nozières and Schmitt-Rink (NSR), JLTP 59 (1985)]
- BCS theory gives the pair dissociation temperature $T^* > T_c$

(b) Screening of the interaction

- Interaction modified by medium polarisation
  (similar to Debye screening)
- In the BCS limit, this effect reduces T_c by more than 50%
  [Gor'kov and Melik-Barkhudarov (1961)]

To explain the experimental T_c in the unitary limit, one has to include both effects [Pisani et al., PRB 14528 (2018)]



Testing nuclear-physics techniques with cold atoms

- Quantum Monte-Carlo (QMC):
  used in cold atoms and neutron matter
  reproduces ξ, Δ, U, ... in the unitary limit

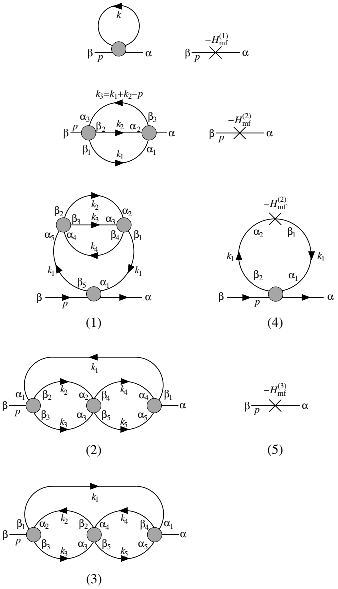
- Let's try Bogoliubov Many-Body Perturbation Theory (BMBPT)

- Soften the interaction $\Leftrightarrow$ finite cutoff Λ

- $V_{\text{low-}k}$-like s-wave interaction $V(q, q')$ that reproduces the phase shifts of the contact interaction for $q < \Lambda$
  [MU & S. Ramanan, PRA 103, 063306 (2021)]

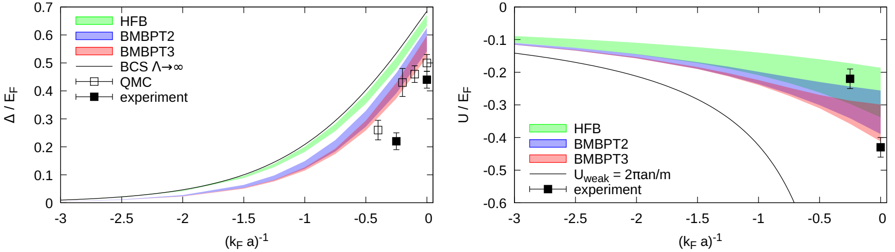
- Nambu-Gor'kov formalism:
  2×2 self-energy $\Sigma = \begin{pmatrix} U & \Delta \\ \Delta & -U \end{pmatrix}$

- Better don't start from the HFB (Hartree-Fock-Bogoliubov) ground state but from a reference state with corrected gap (counterterms shown as $\times$)



BMBPT3 results for Δ and U [S. Ramanan & MU, in preparation]

- Vary cutoff in the range $1.5k_F \leq \Lambda \leq 2.5k_F$: cutoff dependence as indicator for missing contributions (induced 3-body force, higher orders of BMBPT)



QMC: [Carlson& Reddy PRL (2005), Gezerlis& Carlson PRC (2008)];
exp: [Schirotzek et al. PRL (2008)]; GMB: [Gor'kov & Melik-Barkhudarov JETP (1961)]

- Weak coupling: $\Delta \to (4e)^{-1/3}\Delta_{\text{BCS}} \approx 0.45\Delta_{\text{BCS}}$, $\qquad U \to \frac{4\pi a}{m} n_\sigma$

- At 3rd order, the gap has corrections from many effects: effective mass, Z factor, quasiparticle interaction in the screening, vertex correction, ...

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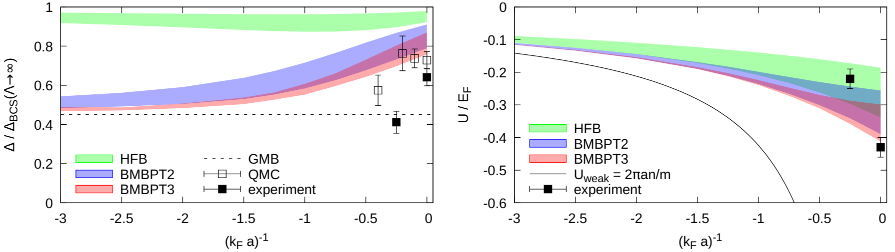

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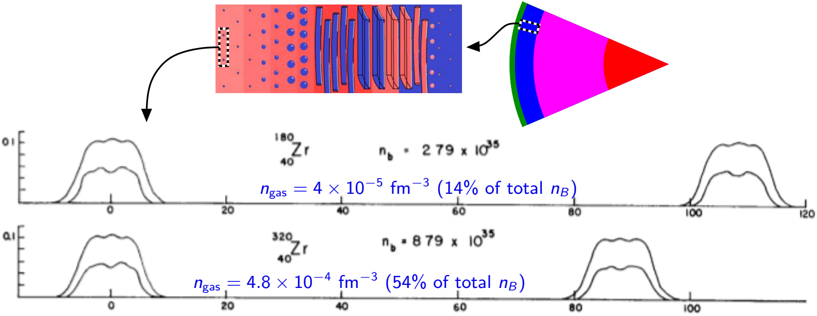
Outline

What is “dilute” neutron matter?

- Upper layers of the inner crust (close to neutron-drip density $\sim 2.5 \times 10^{-4}$ fm^{-3})



[Negele and Vautherin, NPA 207 (1973); similar results by Baldo et al., PRC 76 (2007)]

- In spite of its “low” density (still $\rho \gtrsim 10^{11}$ g/cm^3), the neutron gas is relevant because it occupies a much larger volume than the clusters

- Deeper in the crust: $n_{\rm gas}$ increases up to $\sim n_0/2 = 0.08$ fm^{-3}

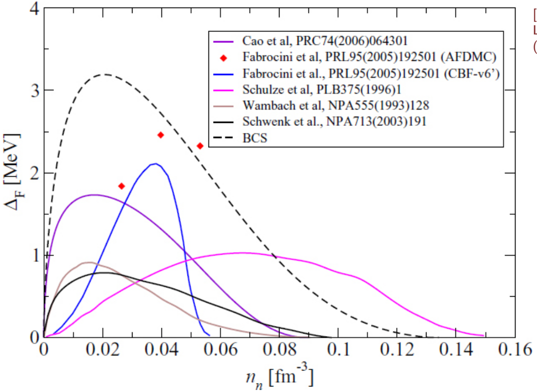
Comparison with ultracold trapped Fermi gases

	neutron gas	trapped Fermi gas (e.g. ^{6}Li)
n	$4 \times 10^{-5} \dots 0.08\ \mathrm{fm}^{-3}$	$\sim 1\ \mu\mathrm{m}^{-3}$
$k_F = (3\pi^2 n)^{1/3}$	$0.1 \dots 1.3\ \mathrm{fm}^{-1}$	$\sim 1\ \mu\mathrm{m}^{-1}$
$E_F = k_F^2/2m$	$0.2 \dots 35$ MeV	$\sim 1\ \mu\mathrm{K} \sim 10^{-10}$ eV
scattering length a	-18 fm	adjustable (Feshbach resonance)
effective range r_{eff}	2.5 fm	~ 1 nm
$1/(k_F a)$	$-0.5 \cdots - 0.07$	unitary limit: 0 BCS-BEC crossover: $-1 \dots 1$
$k_F r_{\mathrm{eff}}$	$0.25 \dots 3$	10^{-3}

- r_{eff} can be neglected in cold atoms but not in neutron matter
- the neutron gas is close to the crossover regime but not in the unitary limit

Pairing in neutron matter: results in the literature

- Concentrate on *s*-wave pairing (*p*-wave pairing expected at higher densities)

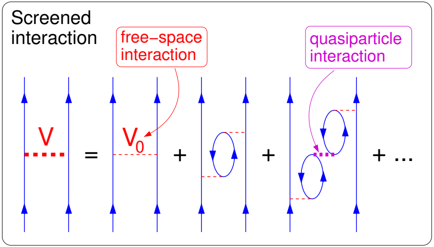


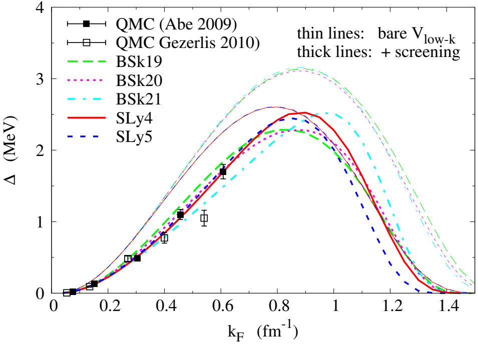
[Chamel and Haensel, Liv. Rev. Relativity (2008)]

- Gap first increases with density (because of density of states, as in cold atoms) but then it decreases (because of the finite range of the interaction)
- Large corrections beyond BCS, but no consensus (status 2008)

Recent progress at low densities

- Screening calculation with low-momentum interaction $V_{\text{low-}k}$ for the pairing and Skyrme functionals for m^* and the RPA [M.U. and S.Ramanan, PRC (2020), EPJ ST (2021)]
- Zoom on low density: $k_F \propto n^{1/3}$
- Necessary to scale the cutoff with k_F ($\Lambda = 2.5k_F$, as in cold atoms) to recover $\Delta/\Delta_{\text{BCS}} \to 0.45$ for $k_F a \to 0$
- $\Delta/\Delta_{\text{BCS}} \approx 0.6$ at relevant low densities, in good agreement with QMC calculations
- But inner crust involves densities up to $n \simeq 0.08\,\text{fm}^{-3}$ ($k_F \simeq 1.3\,\text{fm}^{-1}$) where large uncertainties persist: m^*, quasiparticle interaction (Landau parameters), 3-body force, ...





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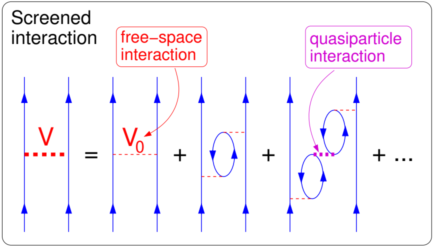
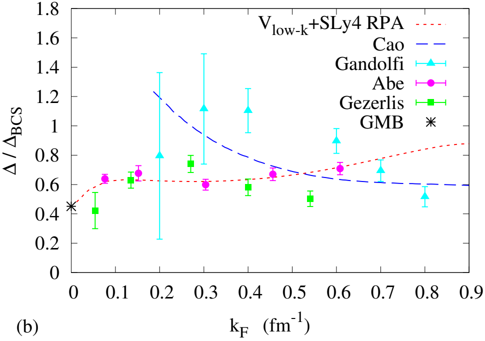




(b)

Critical temperature including screening and non-condensed pairs

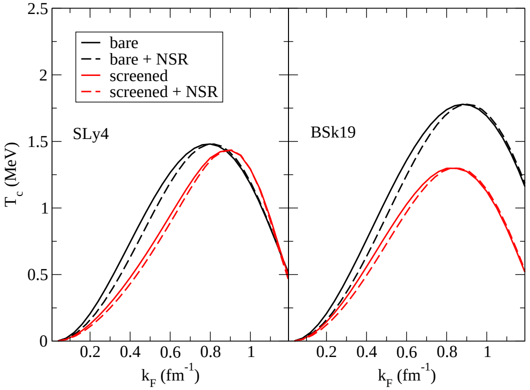
- In the BCS-BEC crossover:

$$T_c < T^*$$

  T_c = pair condensation temp.
  T^* = pair dissociation temp.

- Nozières-Schmitt-Rink (NSR) theory [JLTP 59 (1985)]: compute density including non-condensed pairs

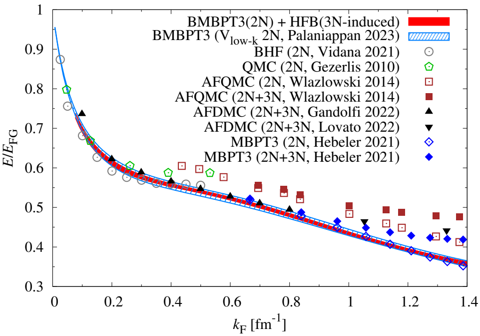
- NSR approach for neutron matter [S. Ramanan and MU, PRC 88 (2013); PRC 101 (2020)]



- Unlike the unitary Fermi gas, in neutron matter, the screening effect is much stronger than the NSR effect
- The BCS relation $T_c = 0.57\Delta(T = 0)$ remains a good approximation

Description of dilute neutron matter with BMBPT

- Goal: eliminate uncertainties due to different Skyrme functionals
- E/N in units of $E_{\mathrm{FG}}/N = \frac{3}{5}E_F$
- Notice: E/E_{FG} is far from $\xi = 0.376$ of the unitary Fermi gas
- Our most recent calculation:
  3rd order BMBPT with chiral N4LO 2-body force (2BF), softened with the similarity renormalization group (SRG)
  [Palaniappan et al. PRC 111 (2025)]



- To get right asymptotics at low density, it is again necessary to scale the SRG cutoff λ with k_F (error band: residual cutoff dependence for $1.3 \leq \lambda/k_F \leq 2.5$)
- Even if the bare 3BF is negligible at low density, the SRG induced 3BF is necessary at $\lambda \lesssim 2.5k_F$ to reduce cutoff dependence
- To be done: BMBPT corrections to the pairing gap (as in cold atoms)

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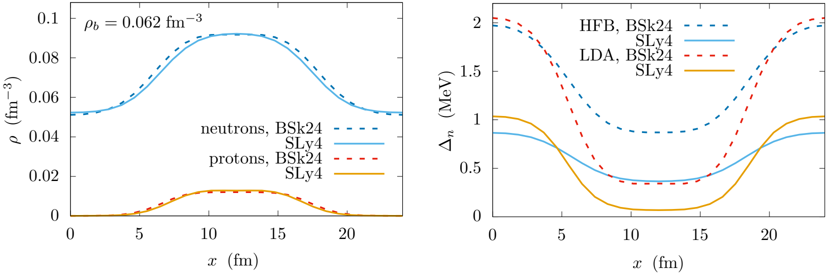
1. Superfluidity in terrestrial laboratories
2. Superfluidity in neutron stars
3. Ultracold Fermi gases in the BCS-BEC crossover
4. Pairing in dilute neutron matter
5. Superfluidity in the inhomogeneous inner crust of neutron stars
6. Conclusions

Inhomogeneous crust vs. infinite matter calculations

- Local-density approximation: $\Delta_{\text{LDA}}(r) = \Delta_{\text{inf.mat.}}(\rho(r))$
- Compare with full HFB calculation for inhomogeneous crust
  example: "spaghetti phase" [G. Almirante and MU, PRC 110, 065802 (2024)]



- HFB gap of the neutron gas extends into the cluster ("proximity effect")
- HFB gap shows much less variations than the LDA one
- LDA reproduces quite well the HFB gap in the gas

Superfluid fraction (entrainment)

- Current in a uniform superfluid ($T = 0$):

$$j = n\frac{\hbar}{2m}\nabla\phi \qquad \text{where} \qquad \Delta = |\Delta|e^{i\phi}$$

assuming that ϕ varies only on large enough length scales

- In a non-uniform system, define superfluid and normal densities n_S and n_N in terms of coarse grained quantities $\bar{j}$, $\bar{\phi}$, $\bar{n}$ such that:

$$\bar{j} = n_S\frac{\hbar}{2m}\nabla\bar{\phi} + n_N v_N \qquad \text{with} \qquad n_S + n_N = \bar{n}$$

(v_N = velocity of the inhomogeneities)

- If the system is non-uniform, then $n_S < n$ even at $T = 0$
[A. Leggett, J. Stat. Phys. 93, 927 (1998)]

- Some of the particles are "entrained" by the motion of the inhomogeneities

- Superfluid fraction n_S/n is crucial for glitches (also relevant for cooling): large Vela glitches require substantial superfluid fraction in the inner crust

Band theory vs. hydrodynamics

- Normal band theory

  [N. Chamel & P. Haensel, Liv. Rev. Relativity 11 (2008)]

  analogous to band theory in solids

  valid for weak coupling ($\Delta \to 0$)

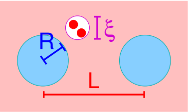
- Superfluid hydrodynamics

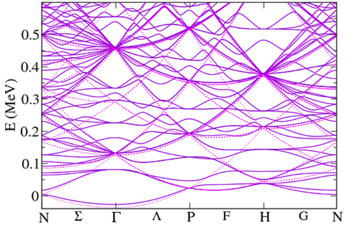
  [N. Martin & MU, PRC 94 (2016)]

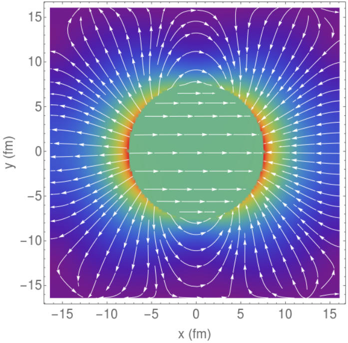
  assume also microscopic current j and microscopic phase ϕ fulfil $j = n\frac{\hbar}{2m}\nabla\phi$

  valid for strong coupling

$$\xi \propto \frac{k_F}{\pi m \Delta} \ll L$$

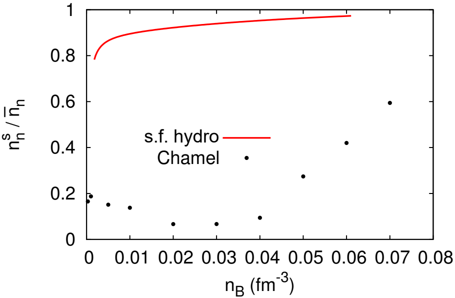


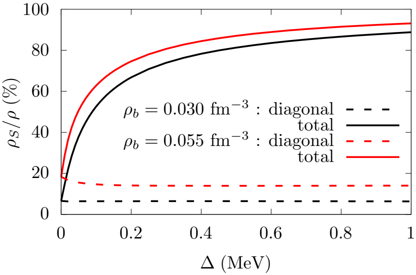




Vela glitch puzzle and its solution

- Normal band theory predicts much stronger suppression of superfluid fraction than superfluid hydrodynamics
- With the band theory result, one would have to include also the core to explain observed Vela glitches
- Full HFB calculation (including bands) interpolates between these two extremes [G. Almirante & MU, PRC 110 (2024)]
- Reason for failure of normal band theory: neglect of non-diagonal terms in the linear response formula [G. Almirante & MU, arxiv:2503.21635] ("geometric contribution" in multiband superconductors)
- Superfluid fraction depends on the value of the gap!





Outline

Conclusions

- Superfluidity in cold atoms is directly observable, Δ and T_c can be measured
- Superfluidity has important observational consequences in neutron stars, but difficult to pin down the values of Δ or T_c from observations
- Common features of ultracold atoms and the inner crust of neutron stars:
  - Large s-wave scattering length
  - Strongly correlated (Δ can be comparable with E_F)
  - Corrections beyond BCS are important (even at weak coupling)
- But: neutron matter is **not** close to a unitary Fermi gas at any density (finite range, higher partial waves, 3-body force ...)
- Ultracold atoms can serve as a test case for methods to be applied to neutron-star matter
- Inner crust of neutron stars $\neq$ infinite uniform matter