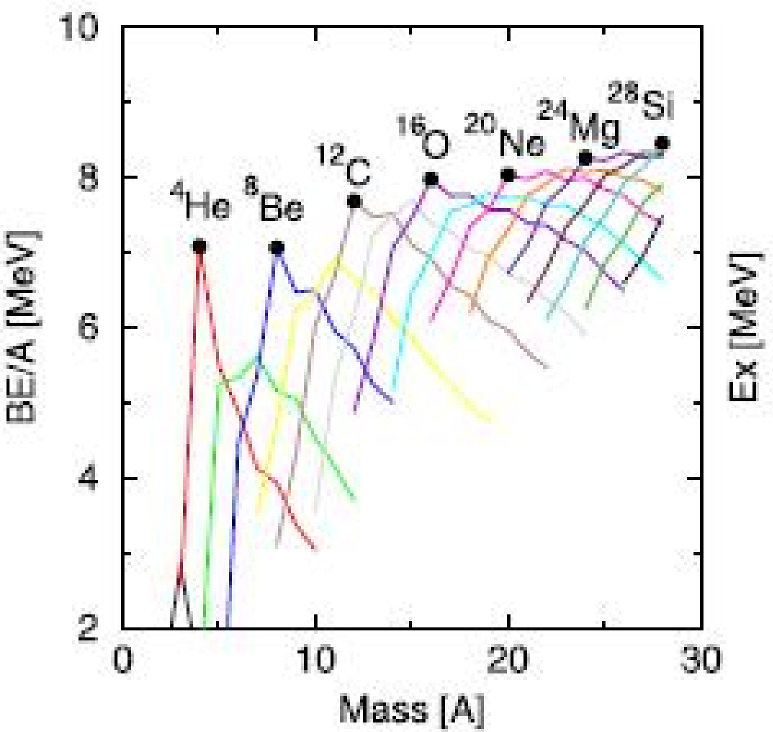
Discrete Symmetries in the Cluster Shell Model

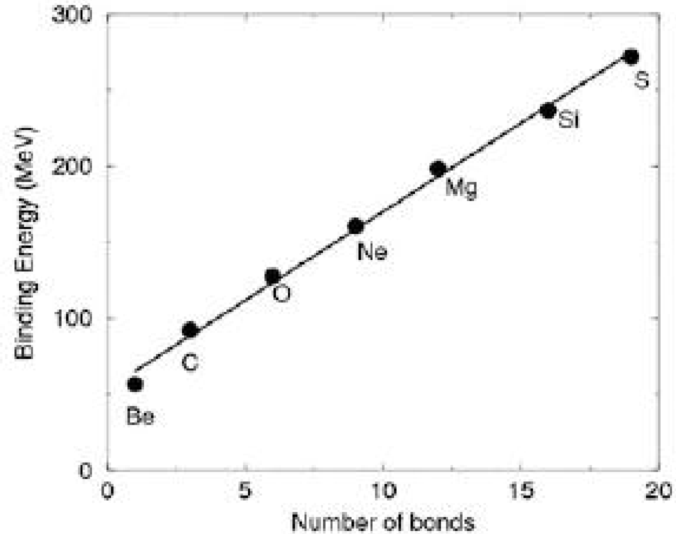
- Introduction
- Algebraic Cluster Model (ACM)
- Point group symmetry
- Applications: ^{12}C and ^{16}O
- Cluster Shell Model (CSM)
- Double point group symmetry
- Applications: ^{13}C and ^{17}O
- Summary and conclusions

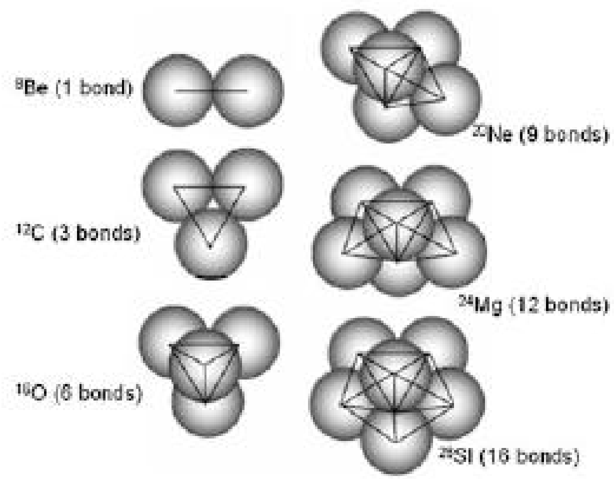
Instituto de Ciencias Nucleares UNAM

Alpha-Cluster Nuclei



3. (Left panel) Binding energy per nucleon of lig





Recent Theoretical Work

- Alpha-cluster model (Wheeler 1937, Brink 1966, Robson 1978)

- AMD (Kanada-Enyo, PTP, 2007)
- FMD model (Chernykh et al, PRL, 2007)
- BEC-like cluster model (Funaki et al, PRC, 2009)
- Ab initio no-core shell model (Roth et al, PRL, 2011)
- Lattice EFT (Epelbaum et al, PRL, 2011, 2012)
- No-core symplectic model (Dreyfuss et al, PLB, 2013)
- Algebraic Cluster Model (2000, 2002, 2014, 2017)
- and many others

- Recent reviews: Freer & Fynbo, PPNP 78, 1 (2014), Freer, Horiuchi, Kanada-En'yo, Lee & Meissner, RMP 90, 035004 (2018)

Few-Body Systems

- Integro-differential methods
- Algebraic Cluster Model (ACM)
- For k dof introduce a SGA of U(k+1)
- Two-body systems: U(4) vibron model (diatomic molecules, mesons)
- Three-body systems: U(7) model (baryons, nuclear clusters, molecules)
- Four-body systems: U(10) model
- n-body systems: U(3n-2) model

ACM for n-Body Systems

3n-3 relative degrees of freedom: Jacobi coordinates

$$\vec{\rho}_k = \frac{1}{\sqrt{k(k+1)}} \left(\sum_{i=1}^{k} \vec{r}_i - k\vec{r}_{k+1} \right) , \qquad k = 1, 2, \ldots, n-1$$

Introduce n-1 dipole bosons and a scalar boson

$$b^{\dagger}_{k,m} , \; s^{\dagger} , \qquad k = 1, 2, \ldots, n-1$$

Number conserving one- and two-body Hamiltonian

$$\begin{aligned} H &= \epsilon_0 \, s^{\dagger}\tilde{s} - \epsilon_1 \sum_k b^{\dagger}_k \cdot \tilde{b}_k + u_0 \, s^{\dagger}s^{\dagger} \, \tilde{s}\tilde{s} \\ & \quad - u_1 \sum_k s^{\dagger} b^{\dagger}_k \cdot \tilde{b}_k \tilde{s} + v_0 \sum_k \left[b^{\dagger}_k \cdot b^{\dagger}_k \, \tilde{s}\tilde{s} + \text{h.c.} \right] \\ & \quad + \sum_L \sum_{ijkl} v^{(L)}_{ijkl} \, [b^{\dagger}_i \times b^{\dagger}_j]^{(L)} \cdot [\tilde{b}_k \times \tilde{b}_l]^{(L)} , \end{aligned}$$

Mixing between oscillator shells

Identical Clusters

Explicit construction of harmonic oscillator states with good permutation symmetry

Kramer & Moshinsky, NP 82, 241 (1966)

$$U(3n-2) \supset U(3n-3) \supset \mathcal{U}(3) \otimes U(n-1)$$

$\mathcal{U}(3) \downarrow$ 3-dim ho $\qquad U(n-1) \downarrow$ index space

ho space : $\mathcal{U}(3) \supset \mathcal{SO}(3)$

index space : $U(n-1) \supset O(n-1) \supset S_n$

Doable for a small number of quanta

However, ACM includes large number of quanta and mixing between different oscillator shells

Permutation Symmetry

Solution: generate wave functions of good permutation symmetry numerically by diagonalizing a S_n invariant H

Permutation symmetry determined by the interchange P(12) and the cyclic permutation P(12…n)

$$\langle\psi|\,P(12)\,|\psi\rangle = \langle\psi|\,\mathrm{e}^{i\pi b_1^\dagger b_1}\,|\psi\rangle$$

$$\langle\psi|\,P(123)\,|\psi\rangle = \langle\psi|\,\mathrm{e}^{i\pi(b_1^\dagger b_1+b_2^\dagger b_2)}\,\mathrm{e}^{\theta_1(b_1^\dagger b_2-b_2^\dagger b_1)}\,|\psi\rangle$$

for n=3 with $\theta_1 = \arctan\sqrt{3}$

Cyclic permutation related to a change of oscillator coordinates: Talmi-Brody-Moshinsky brackets

Wave Functions

- Labeled by $\left|[N],\alpha,L_t^P\right\rangle$
- Total number of bosons: N
- Angular momentum: L
- Parity: P
- Permutation symmetry: t

Three-Body Clusters

Permutation symmetry

$$
\begin{aligned}
H &= \xi_1\,(R^2\,s^\dagger s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)\,(\text{h.c.}) \\
&\quad + \xi_2\left[(b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)\,(\text{h.c.}) + 4(b_1^\dagger \cdot b_2^\dagger)(\text{h.c.})\right] \\
&\quad + \kappa\,\vec{L}\cdot\vec{L}
\end{aligned}
$$

Classical limit: potential energy surface

$$
\begin{aligned}
E_N(r,\chi,\theta) &= \langle N;r,\chi,\theta\,|:H:|\,N;r,\chi,\theta\rangle \\
|N;r,\chi,\theta\rangle &= \frac{1}{\sqrt{N!}}\,(b_c^\dagger)^N\,|0\rangle \\
b_c^\dagger &= \frac{s^\dagger + r\cos\chi\,b_{2,y}^\dagger + r\sin\chi\,(\cos\theta\,b_{1,y}^\dagger + \sin\theta\,b_{1,x}^\dagger)}{\sqrt{1+r^2}}
\end{aligned}
$$

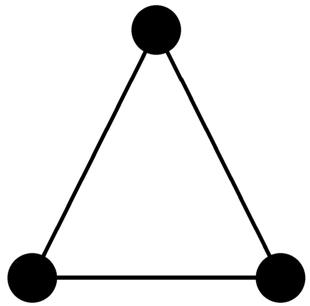
Equilibrium Shape

Energy surface

$$E_N(r,\chi,\theta) = \frac{N(N-1)}{(1+r^2)^2}\Big[\xi_1(R^2-r^2)^2 + \xi_2 r^4\left\{4(\cos\chi\sin\chi\cos\theta)^2+(\sin^2\chi-\cos^2\chi)^2\right\}\Big]$$

Equilibrium shape: two Jacobi coordinates have equal length ($r_0=R$ and $\chi_0=\pi/4$) and are perpendicular ($\theta=\pi/2$)

Equilateral triangle:
Oblate top



Rotation-Vibration Spectrum

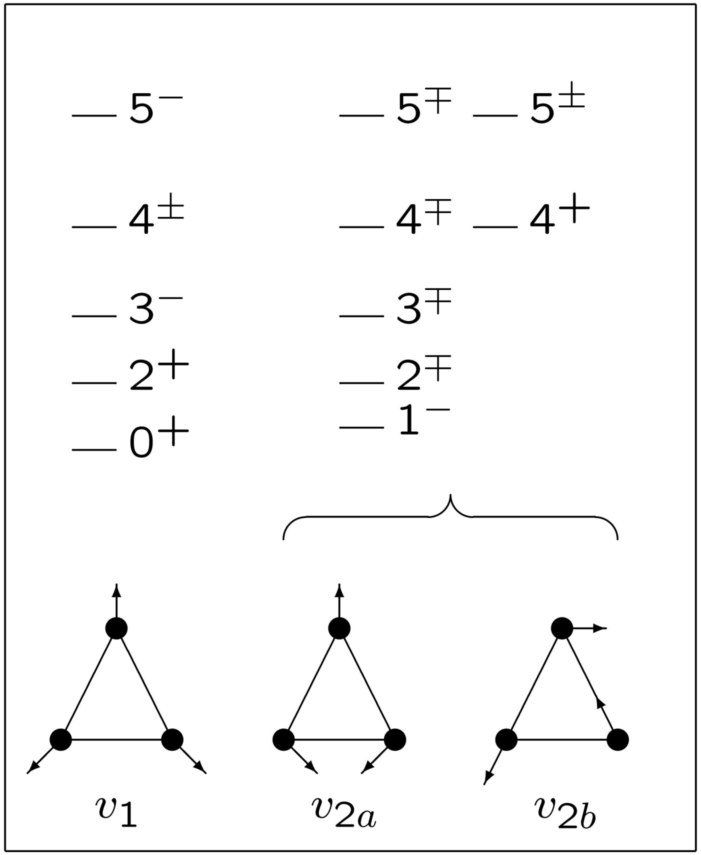
$$E = E_{\text{vib}} + E_{\text{rot}}$$

$$E_{\text{vib}} \approx \omega_1(\nu_1 + \frac{1}{2}) + \omega_2(\nu_2 + 1)$$

$$\omega_1 = 4NR^2\xi_1$$

$$\omega_2 = \frac{4NR^2}{1+R^2}\xi_2$$

$$E_{\text{rot}} = \kappa L(L+1)$$

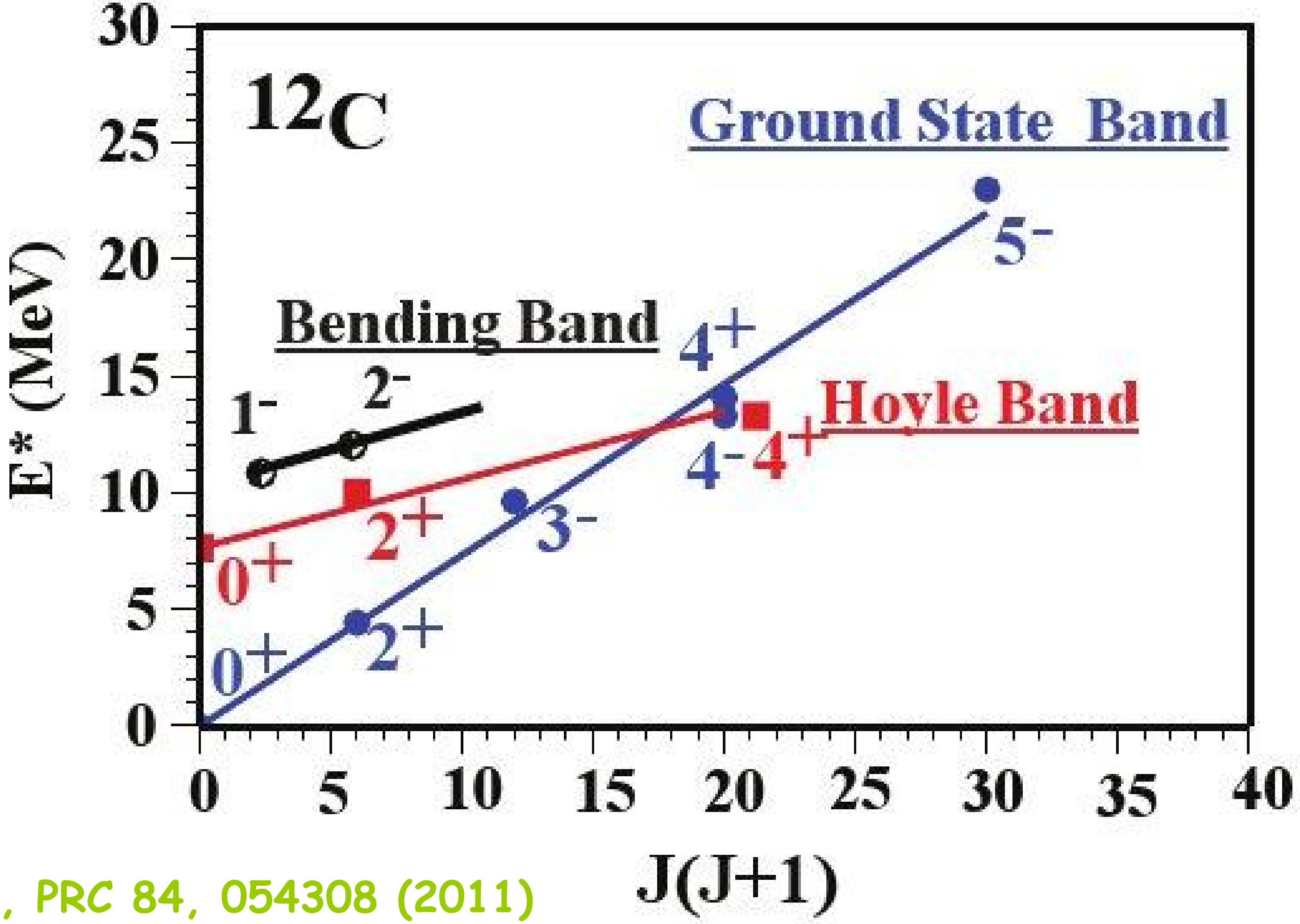


Rotational sequence

$$L^P = 0^+, 2^+, 3^-, 4^\pm, 5^-, \ldots$$

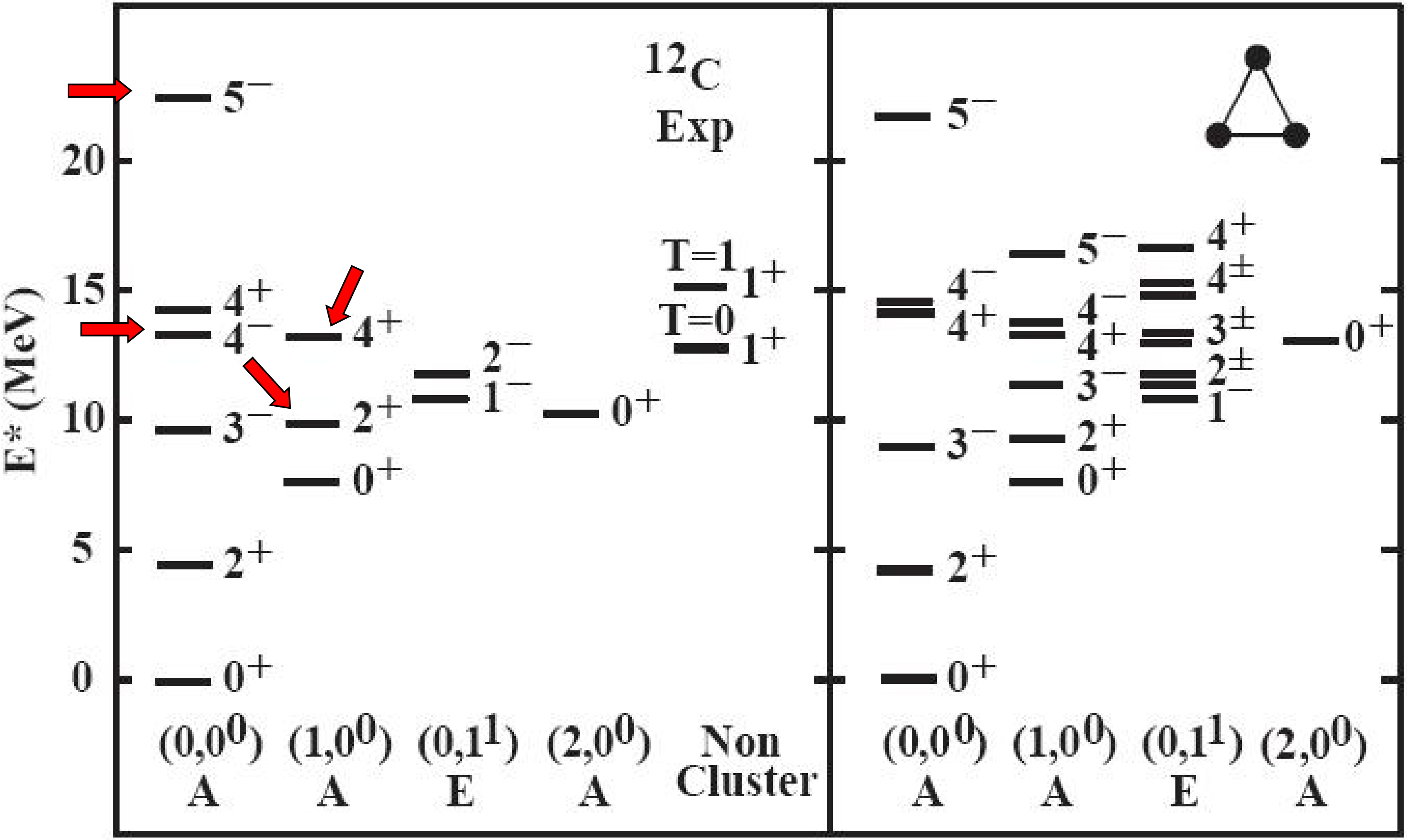
Fingerprint of triangular symmetry (D_{3h})

Bijker & Iachello, AP 298, 334 (2002)



Itoh et al, PRC 84, 054308 (2011)
Freer et al, PRC 86, 034320 (2012)
Zimmerman et al, PRL 110, 152502 (2013)

Marín-Lámbarri et al, PRL 113, 012502 (2014)



Marín-Lámbarri, Bijker et al, PRL 113, 012502 (2014)

Electric Transitions

$$\rho(\vec{r}) = \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2}\sum_{i=1}^{k} e^{-\alpha(\vec{r}-\vec{r}_i)^2}$$

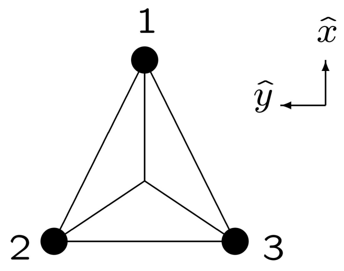
$$\vec{r}_1 = (\beta, \frac{\pi}{2}, 0)$$

$$\vec{r}_2 = (\beta, \frac{\pi}{2}, \frac{2\pi}{3})$$

$$\vec{r}_2 = (\beta, \frac{\pi}{2}, \frac{4\pi}{3})$$

$$\alpha = 0.56 \ \ 1/\text{fm}^2$$

$$\beta = 1.82 \ \ \text{fm}$$



$$\mathcal{F}_L(q) = c_L\, j_L(q\beta)\, e^{-q^2/4\alpha}$$

$$\langle r^2\rangle^{1/2} = \sqrt{\beta^2 + 3/2\alpha}$$

$$B(EL; 0^+ \rightarrow L^P) = \frac{(Ze)^2\, c_L^2\, \beta^{2L}}{4\pi}$$

$$c_L^2 = \frac{2L+1}{3}\left[1 + 2P_L(-\tfrac{1}{2})\right]$$

$$Q_{2^+} = +\frac{2}{7} Ze\, \beta^2$$

		ACM	Exp.	
^{12}C	$B(E2; 2_1^+ \rightarrow 0_1^+)$	7.8	7.63 ± 0.19	$e^2\text{fm}^4$
	$B(E3; 3_1^- \rightarrow 0_1^+)$	65	104 ± 14	$e^2\text{fm}^6$
	$B(E4; 4_1^+ \rightarrow 0_1^+)$	48		$e^2\text{fm}^8$
	$Q_{2_1^+}$	5.7	5.3 ± 4.4	$e\text{fm}^2$
	$\langle r^2\rangle^{1/2}$	2.448	2.468 ± 0.012	fm

Bijker & Iachello,
AP 298, 334 (2002)

ACM for ^{16}O

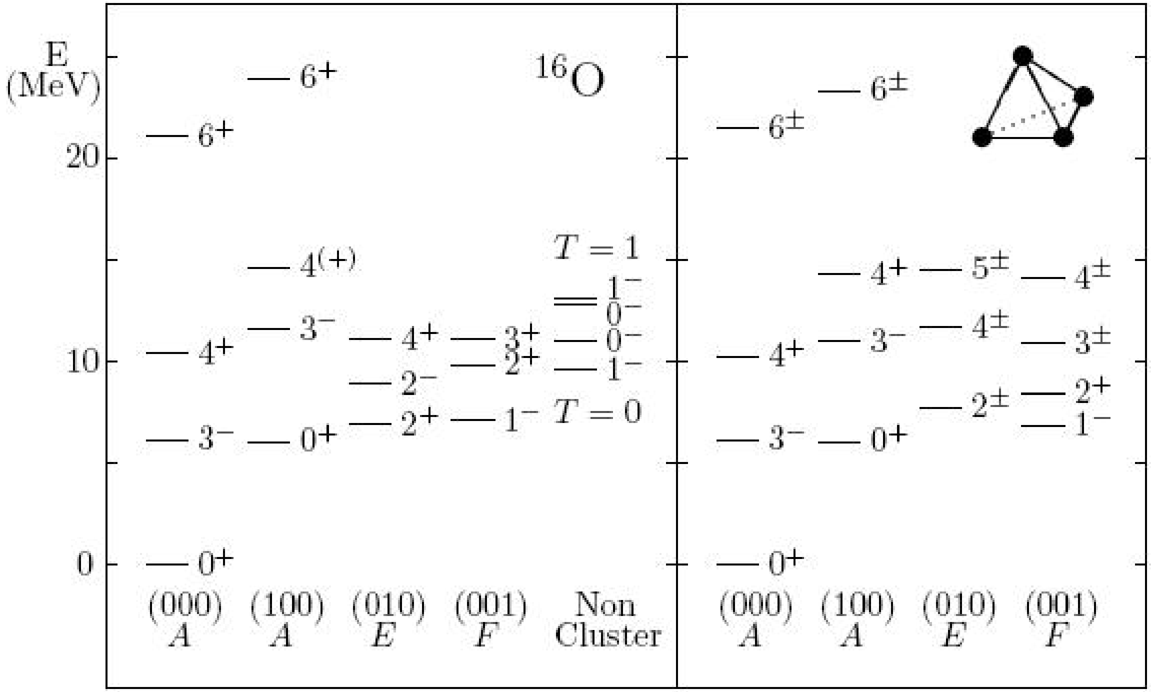


Figure 5: Comparison between the observed spectrum of ^{16}O (left) and the theoretical spectrum (right). The levels are organized in columns corresponding to the ground state band and the three vibrational bands with A, E and F symmetry of a spherical top with tetrahedral symmetry. The last column shows the lowest non-cluste r levels.

Electric Transitions: ^{8}Be, ^{12}C, ^{16}O

$$\rho(\vec{r}) = \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2} \sum_{i=1}^{k} e^{-\alpha(\vec{r}-\vec{r}_i)^2}$$

$$\mathcal{F}_L(q) = c_L\, j_L(q\beta)\, e^{-q^2/4\alpha}$$

$$\langle r^2\rangle^{1/2} = \sqrt{\beta^2 + 3/2\alpha}$$

$$B(EL; 0^+ \to L^P) = \frac{(Ze)^2\, c_L^2\, \beta^{2L}}{4\pi}$$

$$c_L^2 = \begin{cases} \frac{2L+1}{2}\,[1 + P_L(-1)] & 2\alpha\text{-cluster} \\ \frac{2L+1}{3}\left[1 + 2P_L(-\frac{1}{2})\right] & 3\alpha\text{-cluster} \\ \frac{2L+1}{4}\left[1 + 3P_L(-\frac{1}{3})\right] & 4\alpha\text{-cluster} \end{cases}$$

$$Q_{2^+} = \begin{cases} -\frac{4}{7} Ze\beta^2 & 2\alpha\text{-cluster} \\ +\frac{2}{7} Ze\beta^2 & 3\alpha\text{-cluster} \end{cases}$$

	β (fm)	α ($1/\text{fm}^2$)
^{4}He	0.00	0.56
^{8}Be	1.82	0.56
^{12}C	1.74	0.56
^{16}O	2.07	0.56

$\langle r^2\rangle^{1/2}$	ACM (fm)	Exp. (fm)
^{4}He	1.674	1.674 ± 0.012
^{8}Be	2.448	
^{12}C	2.389	2.468 ± 0.012
^{16}O	2.639	2.710 ± 0.015

Electric Transitions

		ACM	Exp.	GFMC	
^{8}Be	$B(E2;2_1^+ \to 0_1^+)$	14		20.0 ± 0.8	e^2fm^4
	$B(E2;4_1^+ \to 2_1^+)$	20	$21 \pm 2.3^*$	27.2 ± 1.5	e^2fm^4
	$B(E4;4_1^+ \to 0_1^+)$	153			e^2fm^8
^{12}C	$B(E2;2_1^+ \to 0_1^+)$	7.8	7.63 ± 0.19		e^2fm^4
	$B(E3;3_1^- \to 0_1^+)$	65	104 ± 14		e^2fm^6
	$B(E4;4_1^+ \to 0_1^+)$	48			e^2fm^8
^{16}O	$B(E3;3_1^- \to 0_1^+)$	215	205 ± 10		e^2fm^6
	$B(E4;4_1^+ \to 0_1^+)$	425	378 ± 133		e^2fm^8
	$B(E6;6_1^+ \to 0_1^+)$	9626			e^2fm^{12}

$Q_{2_1^+}$	ACM (efm^2)	Exp. (efm^2)	GFMC (efm^2)
^{8}Be	-7.6		-9.1 ± 0.2
^{12}C	$+5.7$	$+5.3 \pm 4.4$	

Parameter Free:

Consequence of Symmetry

* Estimated value GFMC

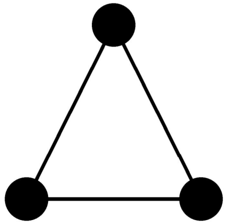
Datar et al, PRL 111, 062502 (2013)

Algebraic Cluster Model

	2α	3α	4α
ACM	$U(4)$	$U(7)$	$U(10)$
Point group	$\mathcal{Z}_2$	$\mathcal{D}_{3h}$	$\mathcal{T}_d$
Geom. conf.	Linear	Triangle	Tetrahedron
Model	Rotor	Oblate top	Spherical top
Vibrations	1	3	6
Rotations	2	3	3
G.s. band	0^+	0^+	0^+
	2^+	2^+	
		3^-	3^-
	4^+	$4^\pm$	4^+
		5^-	
	6^+	$6^{\pm +}$	$6^\pm$
Large $E\lambda$ electric transitions!			

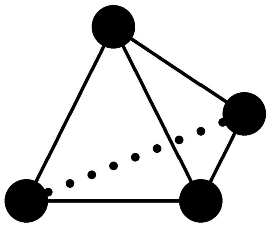
Z_2 Dumbbell ^{8}Be

NPA 973, 1 (2018)



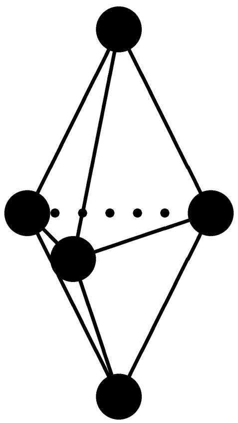
D_{3h} Triangle ^{12}C

PRC 61, 067305 (2000)
Ann Phys 298, 334 (2002)
PRL 113, 012502 (2014)



T_d Tetrahedron ^{16}O

PRL 112, 152501 (2014)
NPA 957, 154 (2017)

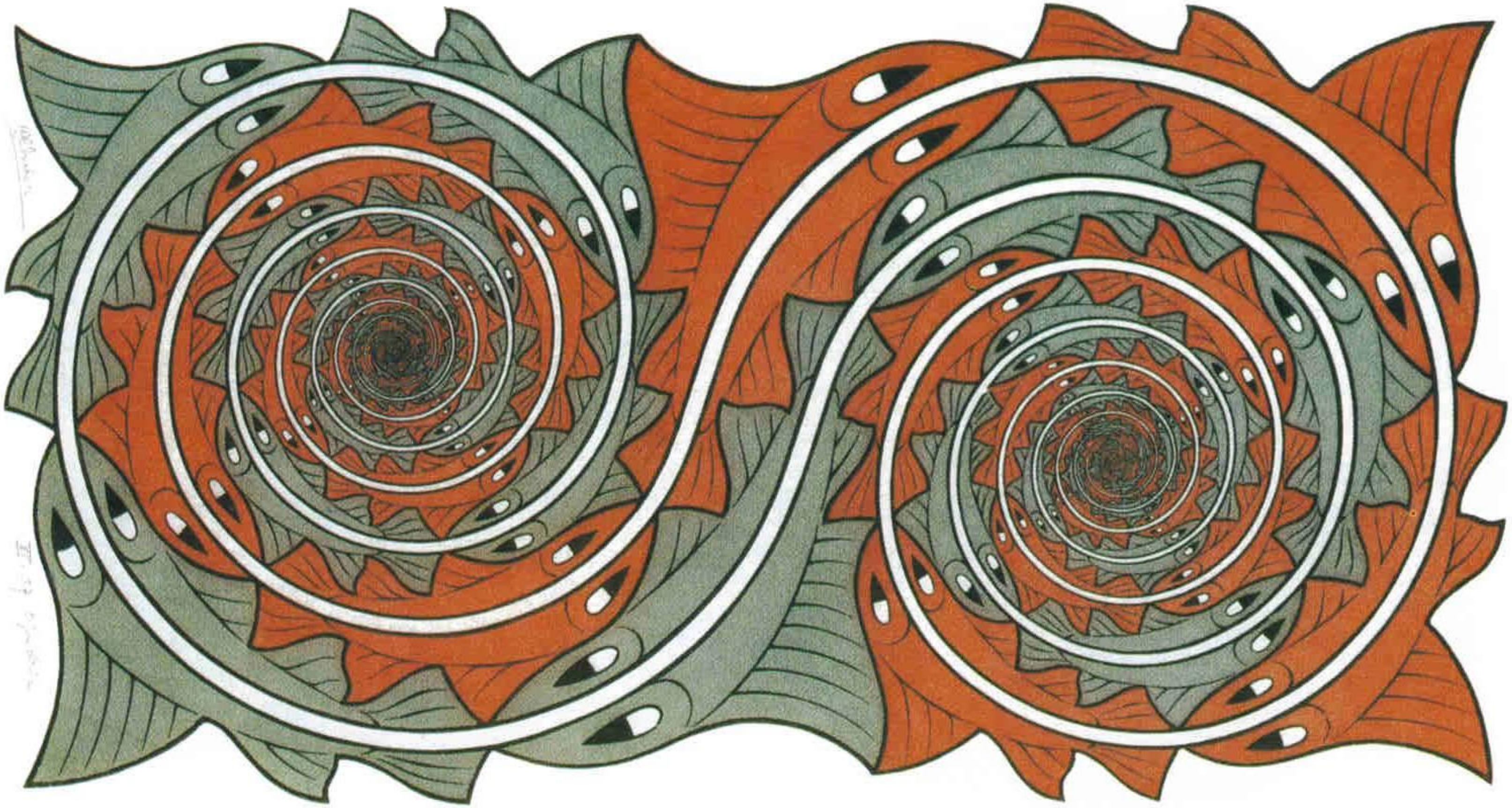


D_{3h} Bi-pyramid ^{20}Ne

NPA 1006, 122077 (2021)

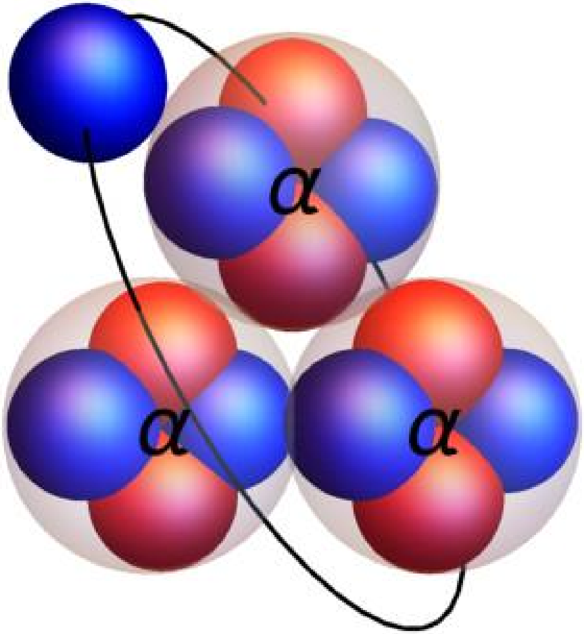
Summary and Conclusions

- Algebraic Cluster Model
- SGA of U(3n-2) for n-body systems
- Discrete and continuous symmetries
- Rotational bands fingerprints of point group symmetries
- Hoyle band: linear, bent or triangular?
- Applications in molecular, nuclear, hadronic physics



Odd Cluster Nuclei

- What are the signatures of α-clustering in odd-mass nuclei?
- Cluster Shell Model (CSM)
- Splitting of sp levels in cluster potentials (analogous to Nilsson model)
- Double point groups



Cluster Shell Model

Cluster density

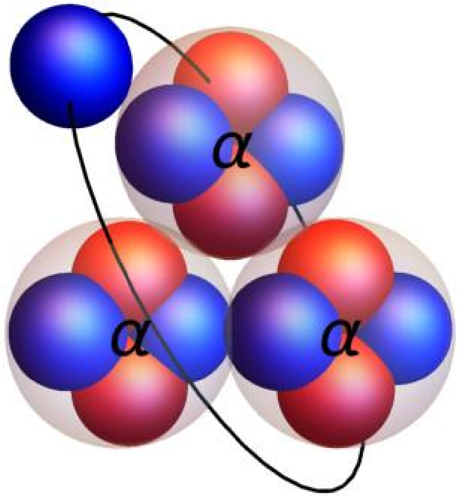
$$\rho(\vec{r}) = \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2}\sum_{i=1}^{k}\exp\left[-\alpha\left(\vec{r}-\vec{r}_i\right)^2\right]$$

$$= \frac{Ze}{k}\left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha(r^2+\beta^2)} 4\pi \sum_{\lambda\mu} i_\lambda(2\alpha\beta r) Y_{\lambda\mu}(\theta,\phi) \sum_{i=1}^{k} Y^*_{\lambda\mu}(\theta_i,\phi_i)$$

$$\vec{r_i} = (\beta, \theta_i, \phi_i)$$

Cluster potential

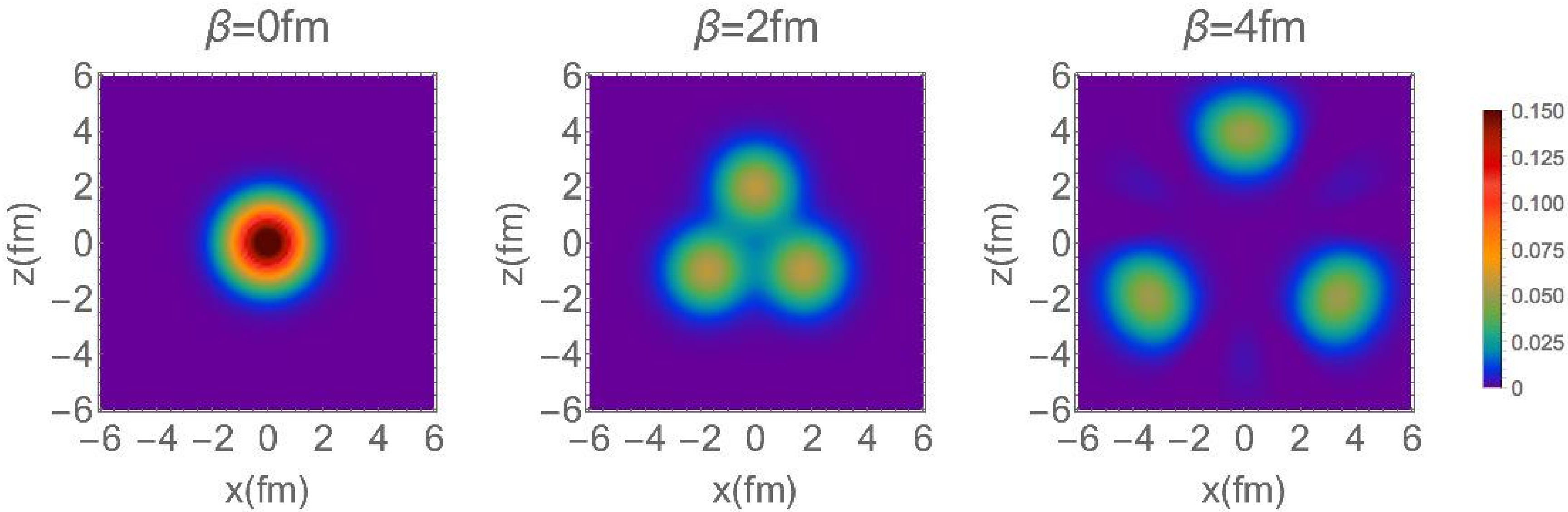
$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + V(\vec{r}) + V_{\mathrm{SO}}(\vec{r}) + V_{\mathrm{C}}(\vec{r})$$



Della Rocca, Bijker & Iachello
NPA 966, 158 (2017)

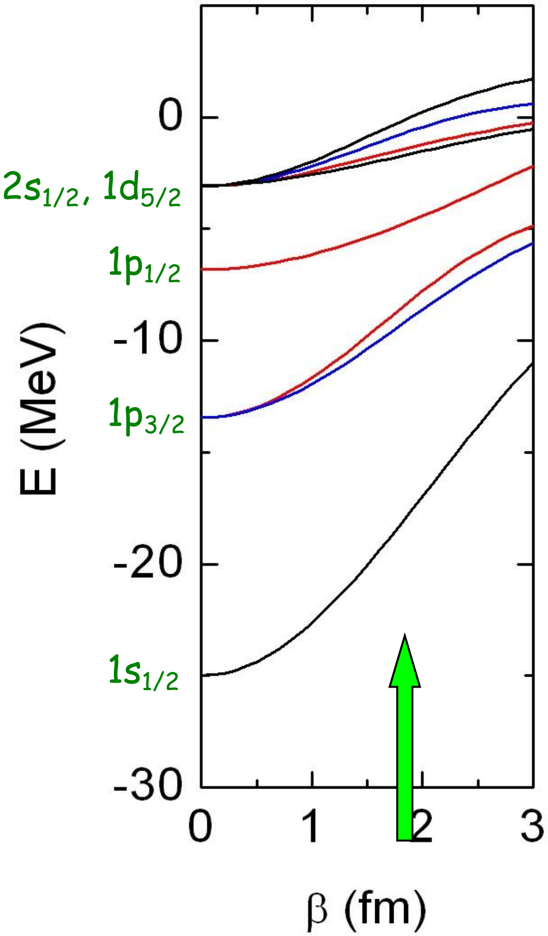
Adrian Horacio Santana Valdés
M.Sc. Thesis, UNAM (2018)

Densitities of 3-α cluster



Della Rocca, Bijker & Iachello, NPA 966, 158 (2017)

Triangular Symmetry



Symmetry: $\mathcal{D}'_{3h}$

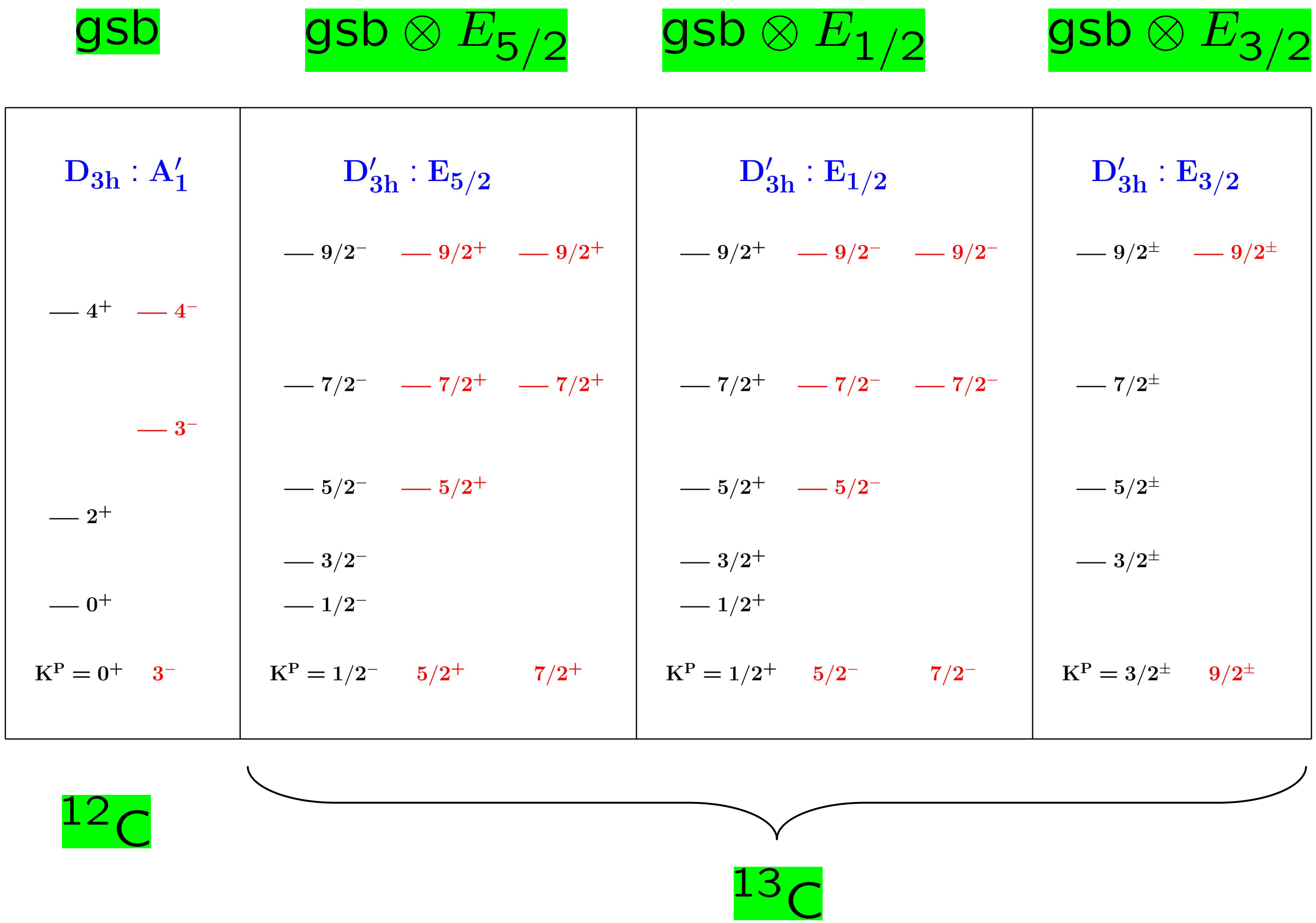
Ω	$E_{1/2}$	$E_{5/2}$	$E_{3/2}$
Deg	2	2	2
$s_{1/2}$	1		
$p_{1/2}$		1	
$p_{3/2}$		1	1
$d_{3/2}$	1		1
$d_{5/2}$	1	1	1

Bijker & Iachello,
PRL 122, 162501 (2019)

Vibrations

	$^{12}\mathrm{C}$	$^{12}\mathrm{C} \otimes \mathrm{E}_{5/2}$	$^{12}\mathrm{C} \otimes \mathrm{E}_{1/2}$	$^{12}\mathrm{C} \otimes \mathrm{E}_{3/2}$
Bend	$\surd$ E'	— $\mathrm{E}_{3/2}$ — $\mathrm{E}_{1/2}$	— $\mathrm{E}_{3/2}$ — $\mathrm{E}_{5/2}$	— $\mathrm{E}_{5/2}$ — $\mathrm{E}_{1/2}$
Hoyle	$\surd$ A'_1	$\surd$ $\mathrm{E}_{5/2}$	— $\mathrm{E}_{1/2}$	— $\mathrm{E}_{3/2}$
Gsb	$\surd$ A'_1	$\surd$ $\mathrm{E}_{5/2}$	$\surd$ $\mathrm{E}_{1/2}$	— $\mathrm{E}_{3/2}$

Rotations



gsb
gsb ⊗ $E_{5/2}$
gsb ⊗ $E_{1/2}$
gsb ⊗ $E_{3/2}$
$D_{3h} : A'_1$
4^+ 4^-
3^-
2^+
0^+
$K^P = 0^+$ 3^-
$D'_{3h} : E_{5/2}$
$9/2^-$ $9/2^+$ $9/2^+$
$7/2^-$ $7/2^+$ $7/2^+$
$5/2^-$ $5/2^+$
$3/2^-$
$1/2^-$
$K^P = 1/2^-$ $5/2^+$ $7/2^+$
$D'_{3h} : E_{1/2}$
$9/2^+$ $9/2^-$ $9/2^-$
$7/2^+$ $7/2^-$ $7/2^-$
$5/2^+$ $5/2^-$
$3/2^+$
$1/2^+$
$K^P = 1/2^+$ $5/2^-$ $7/2^-$
$D'_{3h} : E_{3/2}$
$9/2^\pm$ $9/2^\pm$
$7/2^\pm$
$5/2^\pm$
$3/2^\pm$
$K^P = 3/2^\pm$ $9/2^\pm$
^{12}C
^{13}C

Rotational Energy

$$H_{\rm rot} = \frac{L_1^2 + L_2^2}{2\mathcal{I}} + \frac{L_3^2}{2\mathcal{I}_3} = \sum_{i=1}^{2} \frac{(J_i - j_i)^2}{2\mathcal{I}} + \frac{(J_3 - j_3)^2}{2\mathcal{I}_3}$$

Wave function

$$|\Omega, \mu; J^P K M\rangle = \frac{1}{\sqrt{2}} \left(1 + \hat{P}\, \mathrm{e}^{i\pi J_2}\, \hat{p}\, \mathrm{e}^{-i\pi j_2} \right) |J^P K M\rangle\, |\Omega, \mu\rangle$$

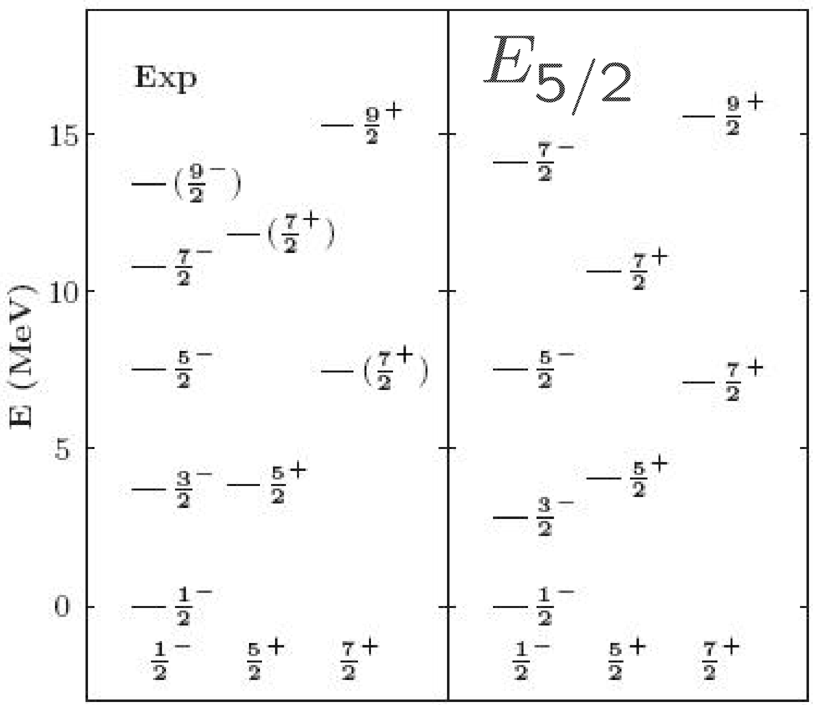
Energies

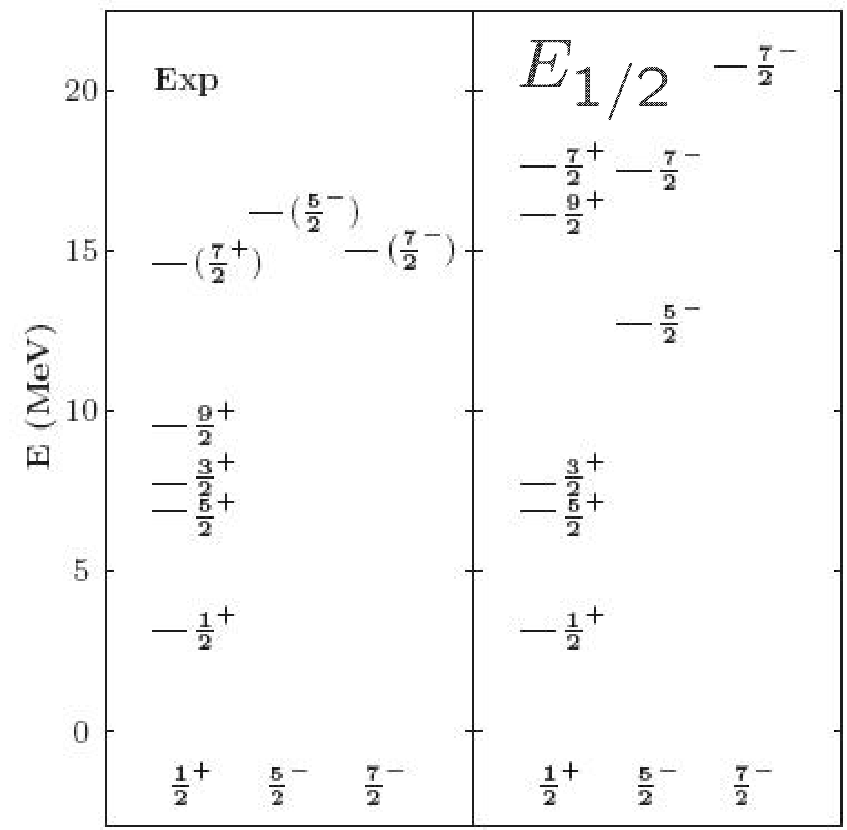
$$E_\Omega(J) \approx \frac{1}{2\mathcal{I}} \left[J(J+1) - 2K^2 + \delta_{K,1/2}\, a_\Omega (-1)^{J+1/2} \left(J + \frac{1}{2} \right) \right]$$

a_Ω Decoupling parameter

Generalized formula for triangular symmetry

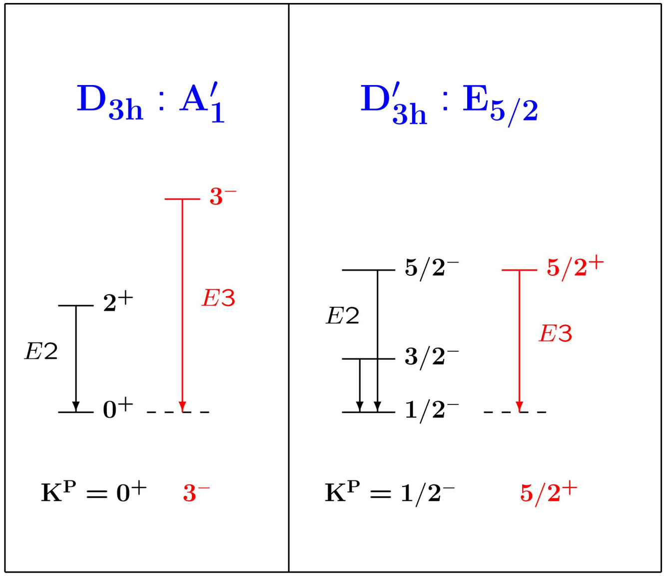
Bandas Rotacionales en ^{13}C





Electric Transitions

$$\frac{B(E3;3_1^- \to 0_1^+)}{B(E2;2_1^+ \to 0_1^+)} = \frac{5}{2}\beta^2$$



Dominated by collective part

$$\begin{aligned} B(E2;3/2_1^- \to 1/2_1^-) &= B(E2;5/2_1^- \to 1/2_1^-) \\ &= B(E2;2_1^+ \to 0_1^+) \\ B(E3;5/2_1^+ \to 1/2_1^-) &= B(E3;3_1^- \to 0_1^+) \\ Q_{5/2_1^-} = \frac{10}{7} Q_{3/2_1^-} &= Q_{2_1^+} \end{aligned}$$

	$B(EL)$	Th	Exp	
^{12}C	$B(E2;2_1^+ \to 0_1^+)$	7.8	7.63 ± 0.19	e^2fm^4
	$B(E3;3_1^- \to 0_1^+)$	65.0	104 ± 14	e^2fm^6
	$Q_{2_1^+}$	5.7	5.3 ± 4.4	$e$fm^2
^{13}C	$B(E2;3/2_1^- \to 1/2_1^-)$	7.8	6.4 ± 1.5	e^2fm^4
	$B(E2;5/2_1^- \to 1/2_1^-)$	7.8	5.6 ± 0.4	e^2fm^4
	$B(E3;5/2_1^+ \to 1/2_1^-)$	65.0	100 ± 40	e^2fm^6
	$Q_{5/2_1^-}$	5.7		$e$fm^2
	$Q_{3/2_1^-}$	4.0		$e$fm^2

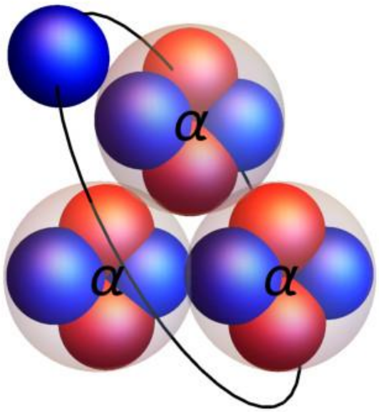
Form Factors

Charge distribution

$$\rho_{\text{ch}}(\vec{r}) = \rho^{\text{c}}_{\text{ch}}(\vec{r}) + \rho^{\text{sp}}_{\text{ch}}(\vec{r})$$

$$\rho^{\text{c}}_{\text{ch}}(\vec{r}) = \frac{(Ze)_{\text{c}}}{3}\left(\frac{\alpha}{\pi}\right)^{3/2}\sum_{i=1}^{3} \mathrm{e}^{-\alpha(\vec{r}-\vec{r}_i)^2}$$

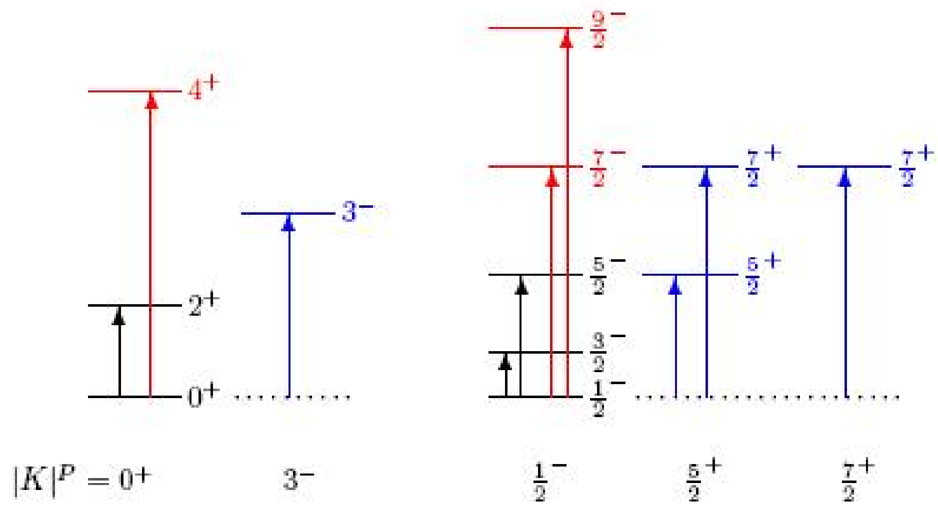
$$\rho^{\text{sp}}_{\text{ch}}(\vec{r}) = \tilde{e}\,\delta(\vec{r}-\vec{r}_{\text{sp}})$$



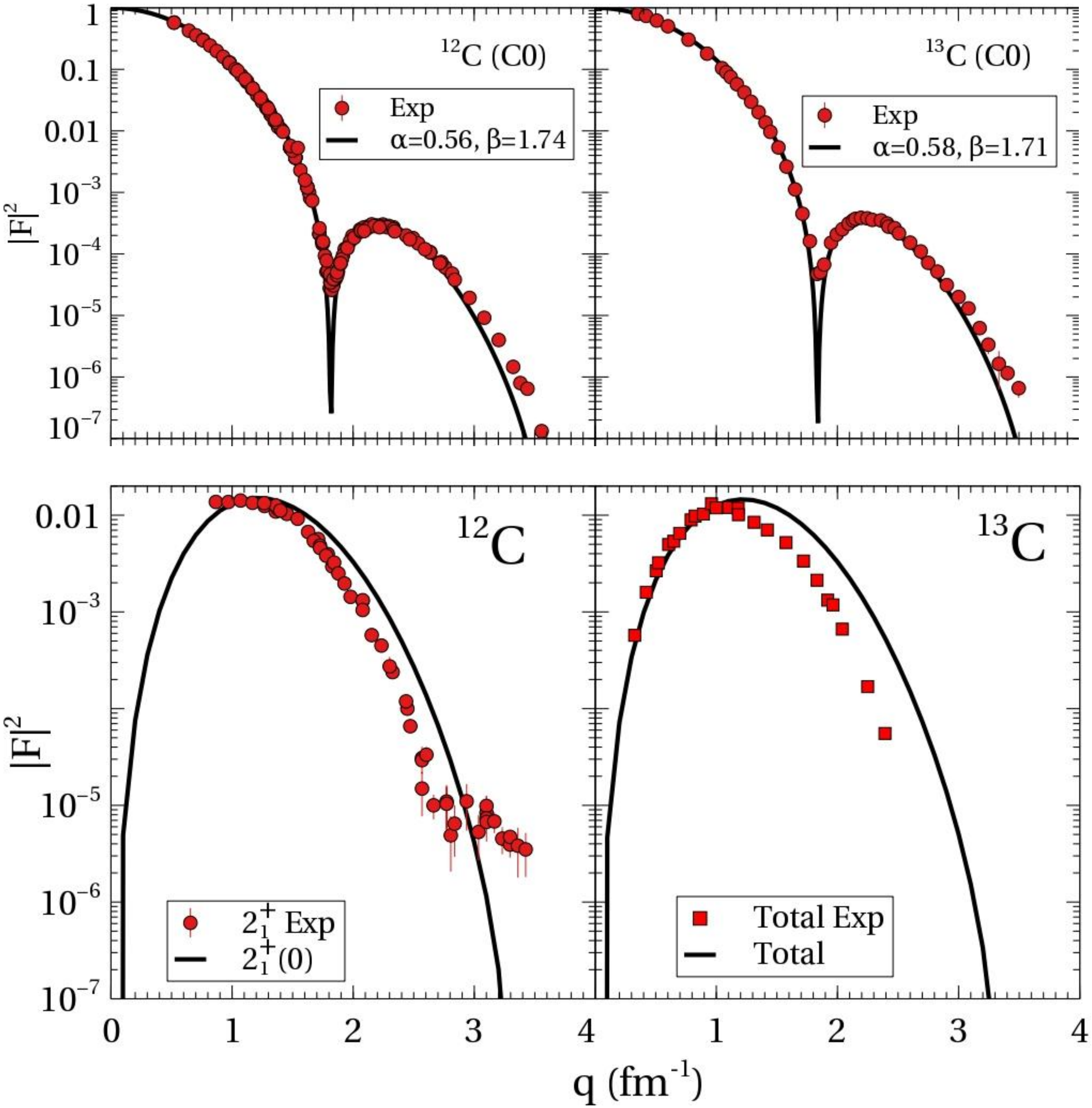
Form factors

$$\left\langle \psi_f \left| \int \rho_{\text{ch}}(\vec{r})\, \mathrm{e}^{i\vec{q}\cdot\vec{r}} d^3r \right| \psi_i \right\rangle$$

Correspondence ^{12}C y ^{13}C



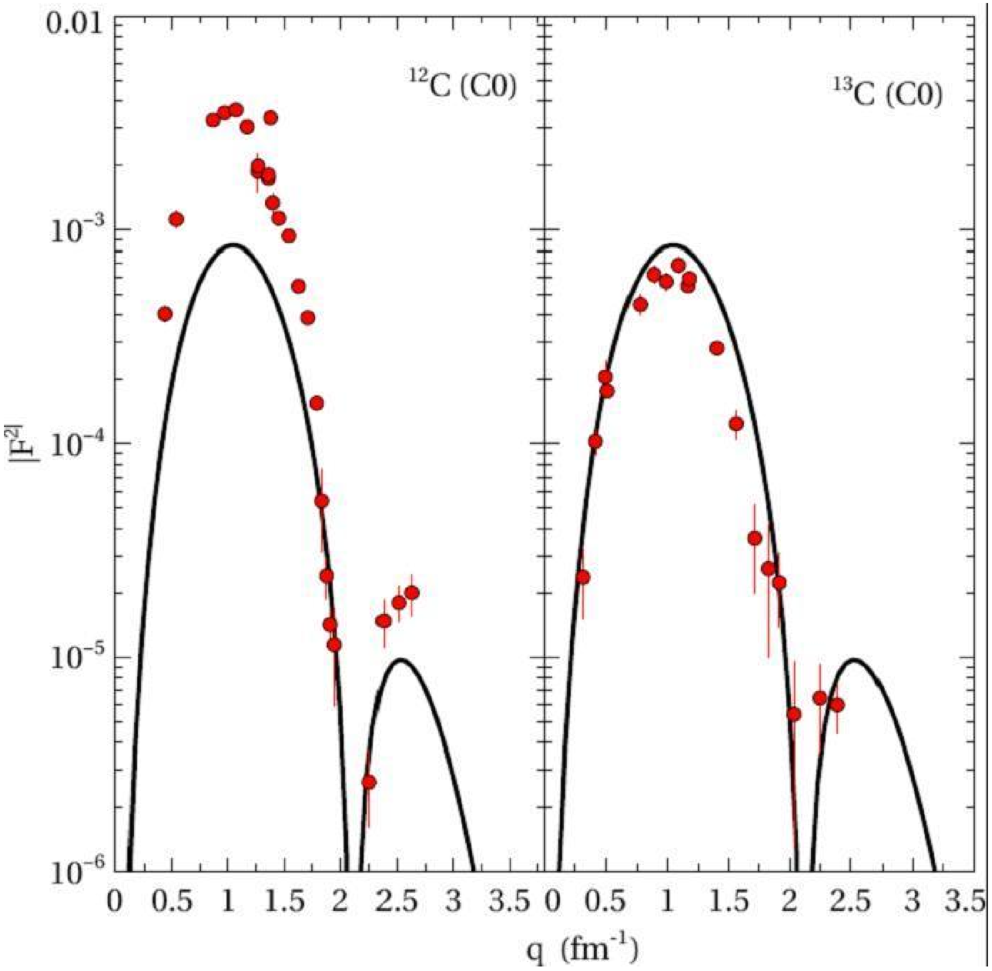
$$\left|F_{C\lambda}(^{12}\mathrm{C}, q; 0^+, 0 \rightarrow L^P, K)\right|^2 = \sum_{I,K'} \left|F_{C\lambda}(^{13}\mathrm{C}, q; \tfrac{1}{2}^-, \tfrac{1}{2} \rightarrow I^{P'}, K')\right|^2_{\mathrm{c}}$$



Factor de forma elástico C0

Factor de forma inelástico C2

Analogue of Hoyle state in ^{13}C



$^{12}\mathrm{C}$: $F(0_1^+ \rightarrow 0_2^+)$

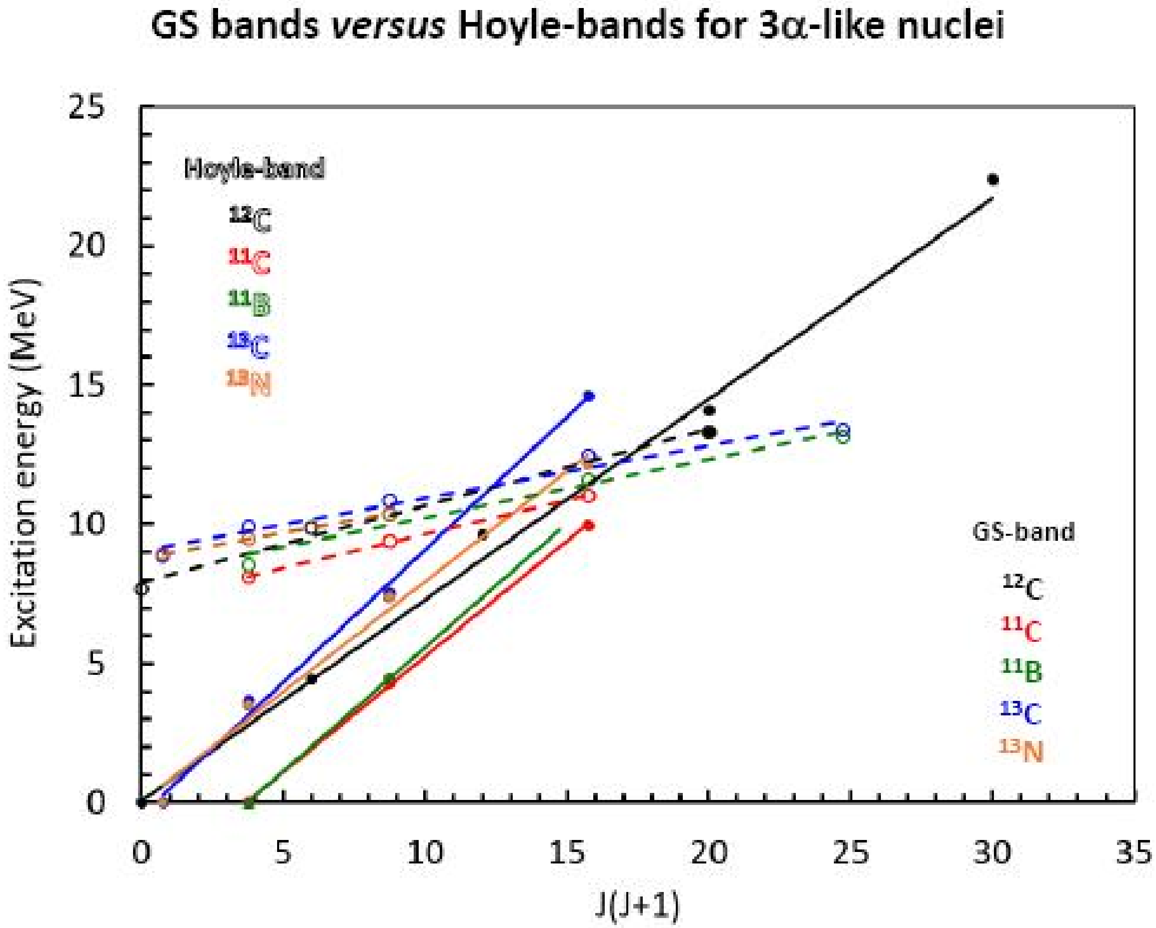
0_2^+ @ 7.65 MeV

$^{13}\mathrm{C}$: $F(1/2_1^- \rightarrow 1/2_2^-)$

$1/2_2^-$ @ 8.86 MeV

Santana & Bijker, Phys. Lett. B 843, 138026 (2023)

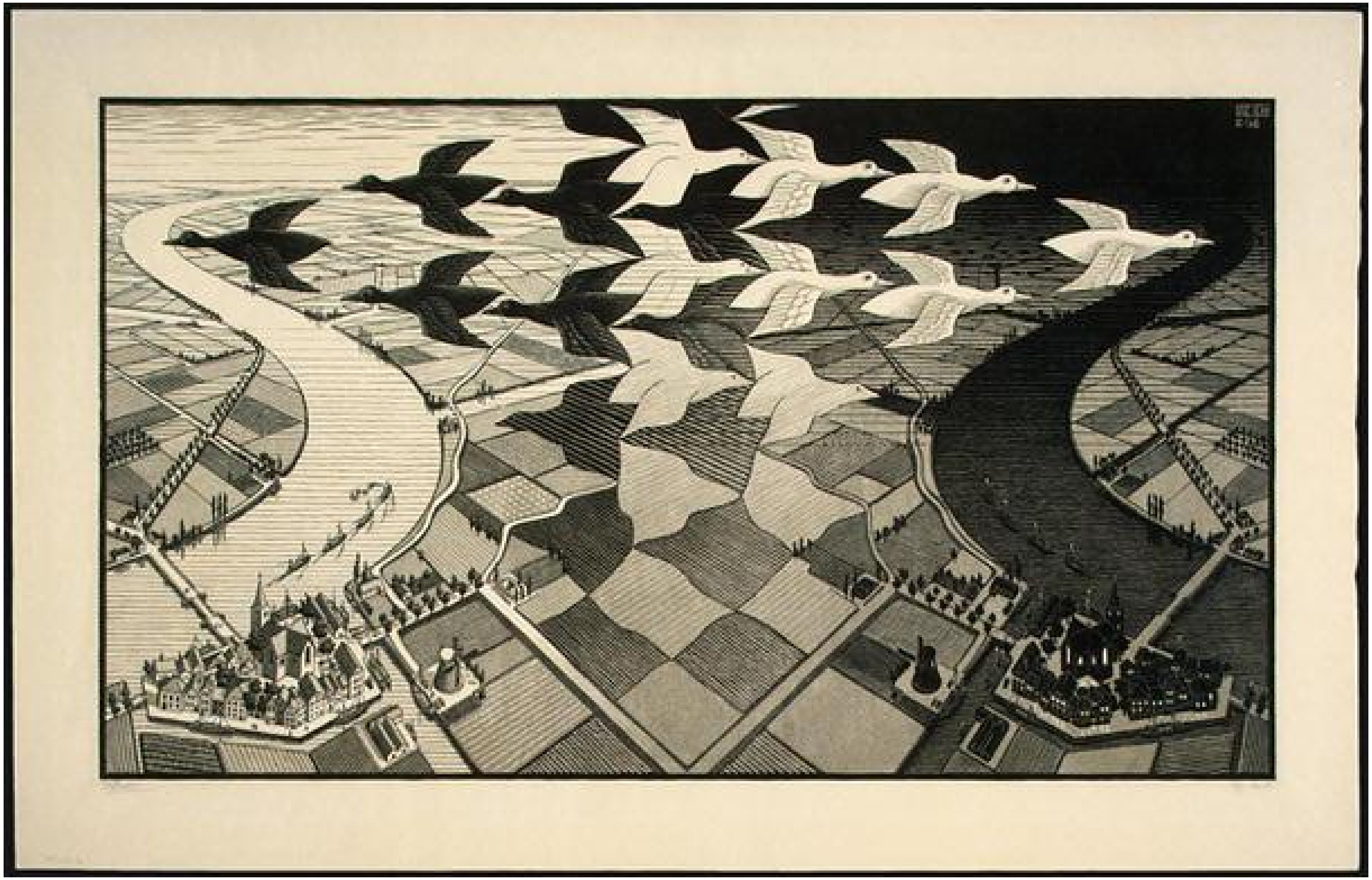
Energy Systematics



Courtesy
Ivano Lombardo, 2019

Coincides in the identification of the analogue of the Hoyle state in ^{13}C

$4n+1$	Nuclei	References
$n=2$	^{9}Be, ^{9}B	NPA 973, 1 (2018)
$n=3$	^{13}C	PRL 122, 162501 (2019)
		EPJ-ST 229, 2353 (2020)
		PLB 843, 138026 (2023)
	^{13}N	B.Sc. Thesis, Villavicencio (2025)
$n=5$	^{21}Ne, ^{21}Na	NPA 1010, 122193 (2021)
$4n+2$	**Nuclei**	**References**
$n=2$	^{10}Be	JPCS 2619, 012006 (2023)
	^{10}B, ^{10}C	in progress, Omar Díaz



Summary and Conclusions

- Cluster Shell Model: ^{13}C
- Symmetries
- Rotational bands: fingerprints of a triangular configuration of three alpha particles plus a neutron
- Large electric transitions
- Form factor to identify the analogue of Hoyle state in ^{13}C

- Benchmark for microscopic studies of nuclear clustering

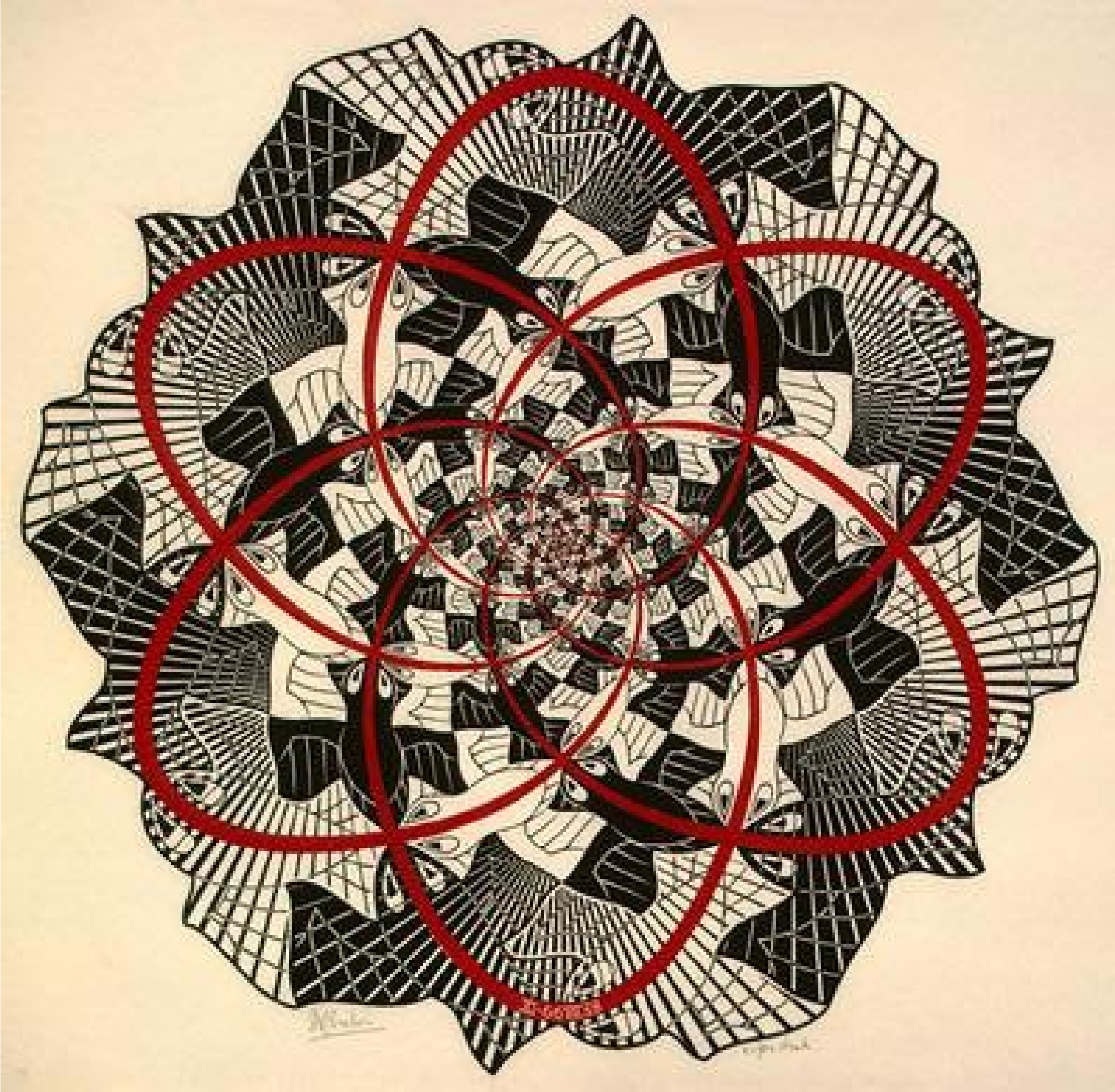
Bijker & Iachello, PRL 122, 162501 (2019)

Who's Who?

- UNAM, Mexico
- Adrian Santana
- Omar Díaz

- UABJ, Mexico
- Emiliano Villavicencio

- Yale
- Valeria Della Rocca
- Francesco Iachello

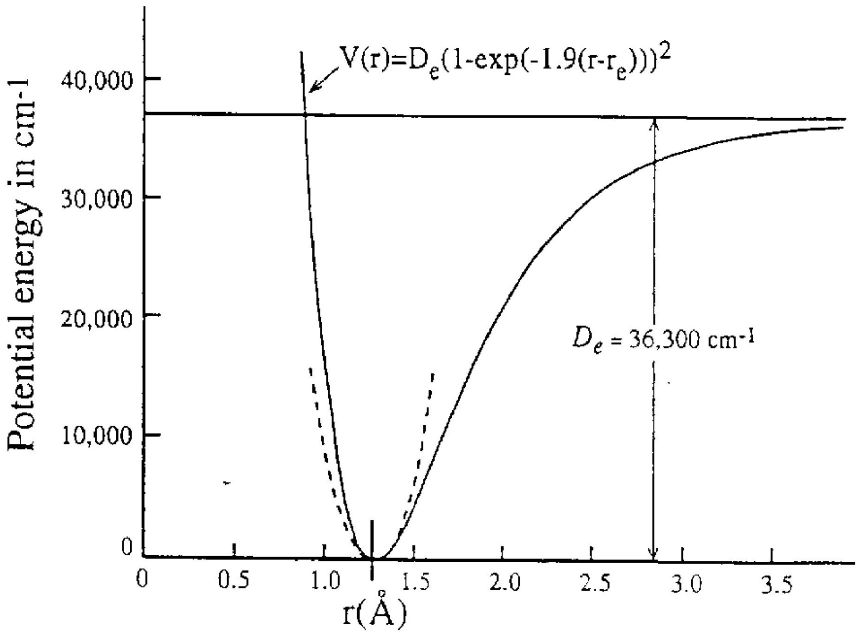


Pauli Principle

- ACM: effective α-α interaction of the Morse type

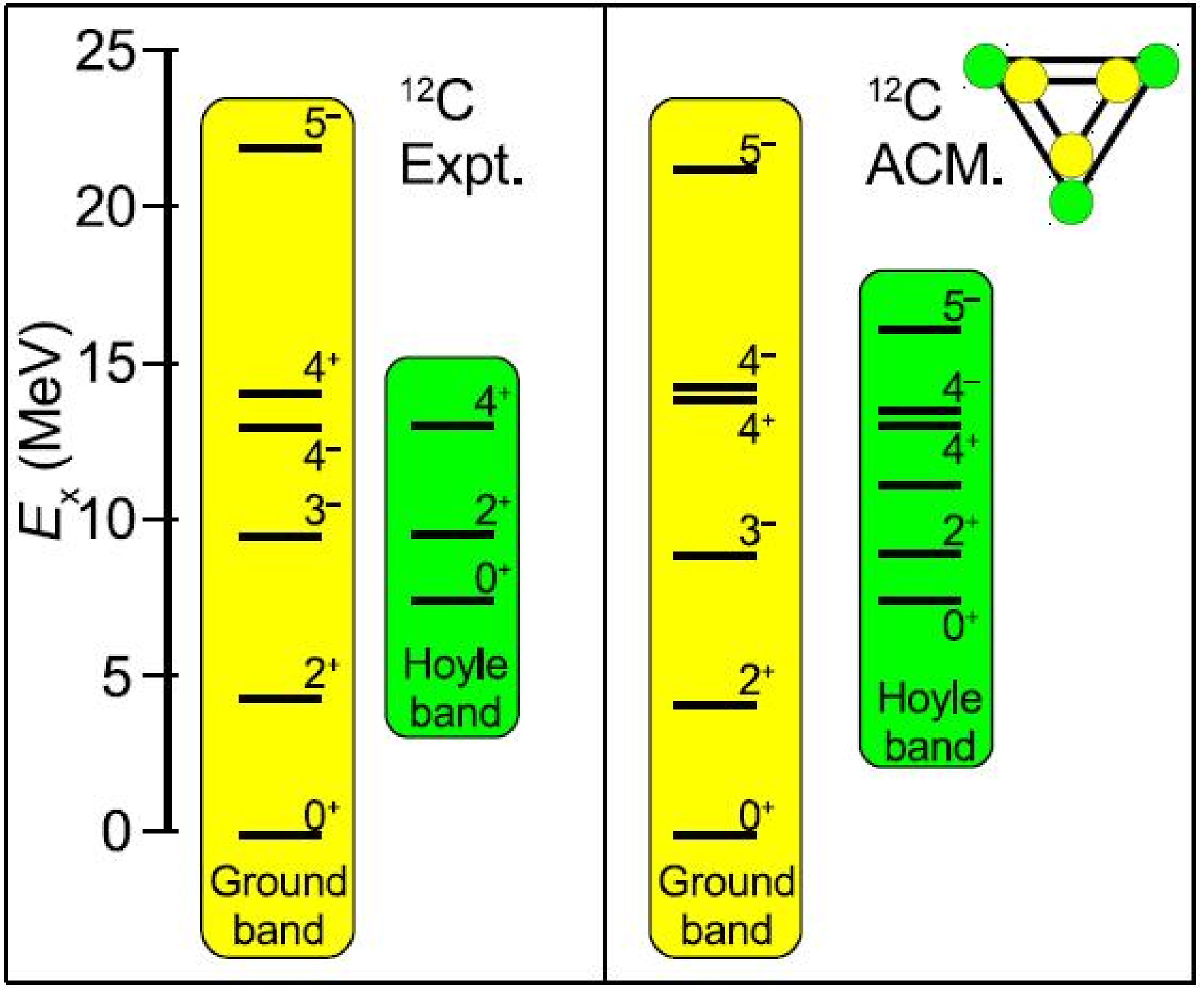
- CSM similar to Nilsson model

- Nucleon moves in the deformed field generated by the cluster of α particles



Experimental Studies ^{12}C

gs	3^-	Kokalova et al, PRC 87, 057307 (2013)
gs	4^-	Freer et al, PRC 76, 034320 (2007)
		Kirsebom et al, PRC 81, 064313 (2010)
gs	5^-	Marín-Lámbarri et al, PRL 113, 012502 (2014)
Hoyle	2^+	Itoh et al, PRC 84, 054308 (2011)
		Freer et al, PRC 86, 034320 (2012)
		Zimmerman et al, PRL 110, 152502 (2013)
Hoyle	4^+	Freer et al, PRC 83, 034314 (2011)
Hoyle	$3^-, 4^-$	Some evidence for negative parity strengths between 11 and 14 MeV
		Freer et al, PRC 76, 034320 (2007)



$$\langle r^2 \rangle^{1/2}_{\rm gs} = 2.47 \text{ fm}$$
$$\langle r^2 \rangle^{1/2}_{\rm H} = 3.45 \text{ fm}$$

Some evidence for negative parity strength between 11-14 MeV !

Freer et al, PRC 76, 034320 (2007)

t: experimentally observed states currently assigned to the grou

Tzany Kokalova, JPCS 569, 012010 (2014)

Estimate of Hoyle Radius

- Moments of inertia and radii of ground state (i=gs) and Hoyle band (i=H)

$$\frac{1}{2\mathcal{I}_i} = \frac{1}{Am\beta_i^2(1+2/\alpha\beta_i^2)}$$

$$\left\langle r^2 \right\rangle_i^{1/2} = \sqrt{\beta_i^2 + 3/2\alpha}$$

- Radii

$$\left\langle r^2 \right\rangle_{\mathrm{gs}}^{1/2} = 2.47 \text{ fm}$$

$$\left\langle r^2 \right\rangle_{\mathrm{H}}^{1/2} = 3.45 \text{ fm}$$

4. Four-body Clusters: Spherical Top

$$
\begin{aligned}
H_{4,\mathrm{vib}} &= \xi_1 \, (R^2 \, s^\dagger s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger - b_3^\dagger \cdot b_3^\dagger) \, (\mathrm{h.c.}) \\
&+\xi_2 \left[(-2\sqrt{2} \, b_1^\dagger \cdot b_3^\dagger + 2 b_1^\dagger \cdot b_2^\dagger) \, (\mathrm{h.c.}) \right. \\
&\qquad \left. + \, (-2\sqrt{2} \, b_2^\dagger \cdot b_3^\dagger + (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)) \, (\mathrm{h.c.}) \right] \\
&+\xi_3 \left[(2 b_1^\dagger \cdot b_3^\dagger + 2\sqrt{2} \, b_1^\dagger \cdot b_2^\dagger) \, (\mathrm{h.c.}) \right. \\
&\qquad + (2 b_2^\dagger \cdot b_3^\dagger + \sqrt{2} \, (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)) \, (\mathrm{h.c.}) \\
&\qquad \left. + (b_1^\dagger \cdot b_1^\dagger + b_2^\dagger \cdot b_2^\dagger - 2 b_3^\dagger \cdot b_3^\dagger) \, (\mathrm{h.c.}) \right]
\end{aligned}
$$

$R^2 = 0$	:	anharmonic oscillator
$R^2 = 1, \, \xi_1 > 0, \, \xi_2 = \xi_3 = 0$	:	deformed oscillator
$R^2 \neq 0, \, \xi_1, \, \xi_2, \, \xi_3 > 0$	:	spherical top

Equilibrium Shape

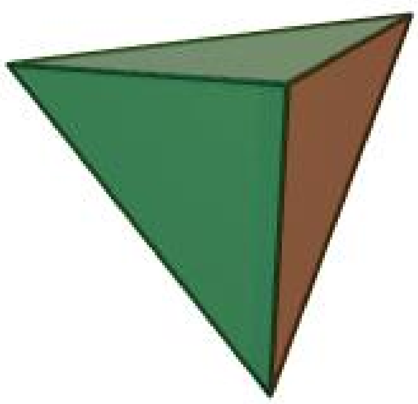
- Three coordinates have equal length

$$q_{1,0} = q_{2,0} = q_{3,0} = \sqrt{2R^2/(1+R^2)}$$

- and are perpendicular

$$\theta_{12,0} = \theta_{23,0} = \theta_{31,0} = \pi/2$$

- Regular tetrahedron
- Platonic solids



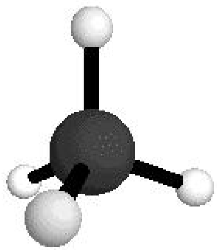
Vibrations

- Vibrational excitations of a spherical top with tetrahedral symmetry

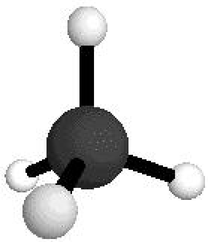
$$E_{4,\mathrm{vib}} = \omega_1(\nu_1 + \frac{1}{2}) + \omega_2(\nu_2 + 1) + \omega_3(\nu_3 + \frac{3}{2})$$

- Frequencies

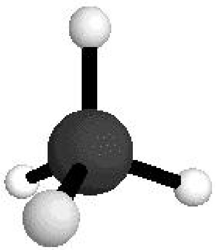
$$\omega_1 = 4NR^2\xi_1 \ , \quad \omega_2 = \frac{8NR^2}{1+R^2}\xi_2 \ , \quad \omega_3 = \frac{8NR^2}{1+R^2}\xi_3$$



$\nu_1(A_1)$



$\nu_2(E)$



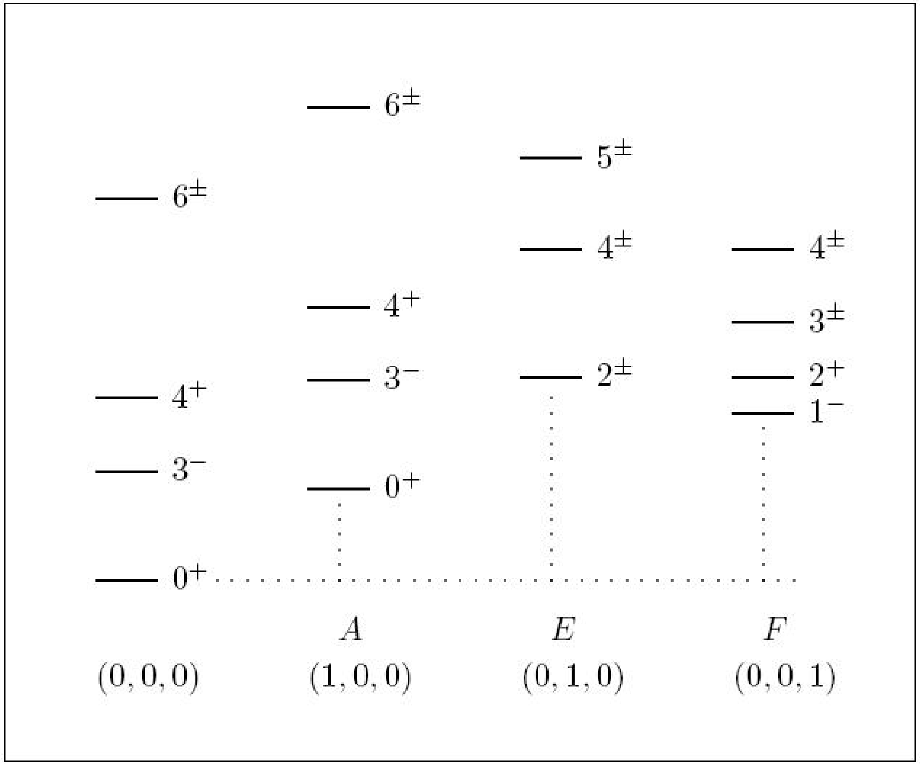
$\nu_3(F_2)$

Rotations

- Hamiltonian $H_4 = H_{4,\text{vib}} + H_{4,\text{rot}}$
- Angular momentum: L is exact symmetry of $H_{4,\text{rot}} = \kappa_1 \vec{L} \cdot \vec{L} + \kappa_2 (\vec{L} \cdot \vec{L} - \vec{I} \cdot \vec{I})^2$
- Angular momentum in index space: I is good quantum number if $\xi_2 = \xi_3$
- Rotational excitations of ground state vibrational band have L=I
- Rotational energies $E_{3,\text{rot}} = \kappa_1 L(L+1)$

Energy Spectrum

$$E_4 = \omega_1(\nu_1 + \frac{1}{2}) + \omega_2(\nu_2 + 1) + \omega_3(\nu_3 + \frac{3}{2}) + \kappa_1 L(L+1)$$

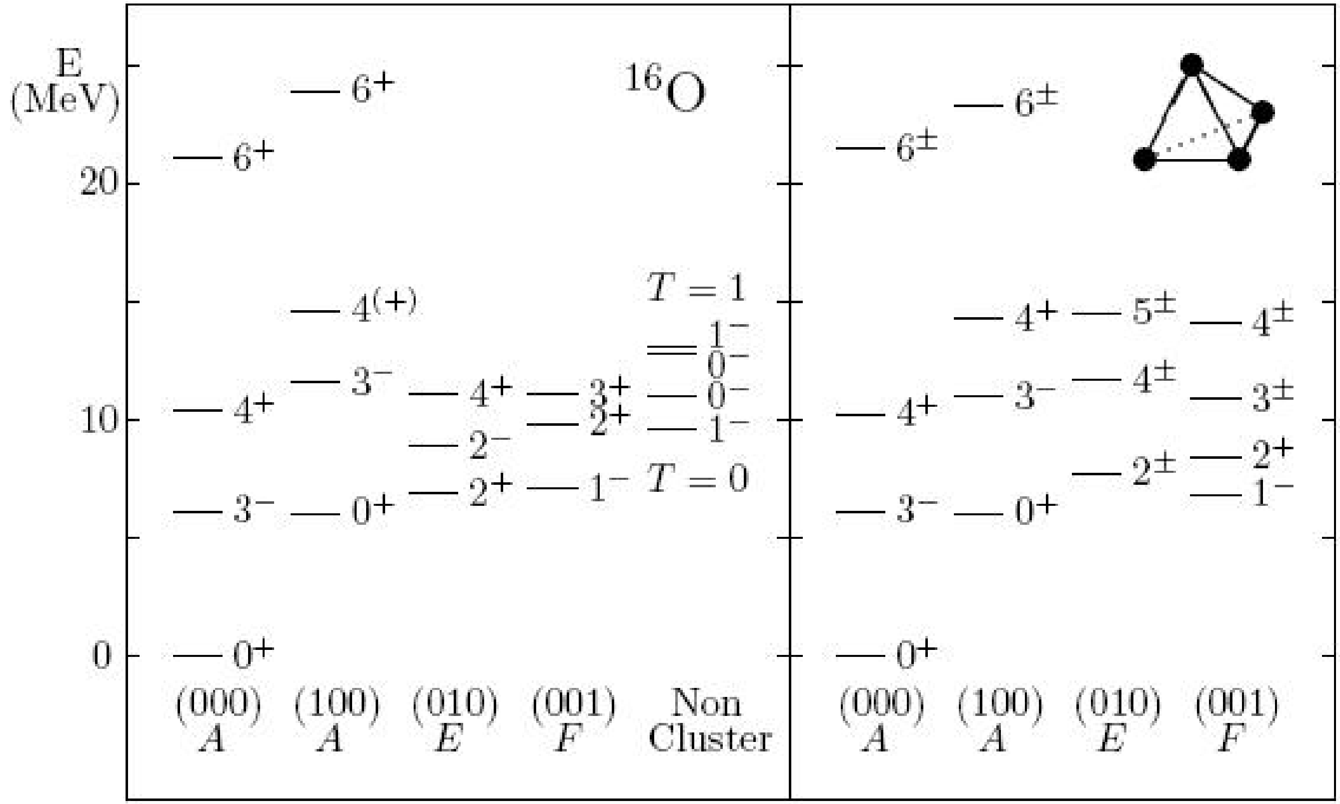


ACM for ^{16}O

- Lowlying $L_i^P = 0_2^+$ state interpreted as a

$(1, 0, 0)$

$3^-, 4^+, \ldots$



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RC 25, 1108

Figure 5: Comparison between the observed spectrum of ^{16}O (left) and the theoretical spectrum (right). The levels are organized in columns corresponding to the ground state band and the three vibrational bands with A, E and F symmetry of a spherical top with tetrahedral

Electric Transitions

$$B(EL;0\to L) \;=\; \left(\frac{Ze}{4}\right)^2 \beta^{2L}\,\frac{2L+1}{4\pi}\left[4+12P_L(-\frac{1}{3})\right]$$

$$\begin{aligned}
B(E1;0\to 1) &= 0\\
B(E2;0\to 2) &= 0\\
B(E3;0\to 3) &= (Ze)^2\,\frac{7}{4\pi}\,\frac{5}{9}\,\beta^6\\
B(E4;0\to 4) &= (Ze)^2\,\frac{9}{4\pi}\,\frac{7}{27}\,\beta^8\\
B(E5;0\to 5) &= 0\\
B(E6;0\to 6) &= (Ze)^2\,\frac{13}{4\pi}\,\frac{32}{81}\,\beta^{12}
\end{aligned}$$

ACM for four-body systems

9 relative degrees of freedom: Jacobi vectors

$$\begin{aligned}
\vec{\rho}_1 &= (\vec{r}_1 - \vec{r}_2)/\sqrt{2} \\
\vec{\rho}_2 &= (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6} \\
\vec{\rho}_3 &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 - 3\vec{r}_4)/\sqrt{12}
\end{aligned}$$

Introduce three dipole bosons
and an auxiliary scalar boson

$$b^{\dagger}_{1,m} \, , \, b^{\dagger}_{2,m} \, , \, b^{\dagger}_{3,m} \, , \, s^{\dagger}$$

such that the total number of bosons is conserved

$$N = n_1 + n_2 + n_3 + n_s$$

Permutation symmetry

$$P(12)\,|\psi_t\rangle \;=\; \begin{cases} +\,|\psi_t\rangle & t = A_1, E_\lambda, F_{2\lambda}, F_{2\eta}, F_{1\rho} \quad n_\rho \text{ even} \\ -\,|\psi_t\rangle & t = A_2, E_\rho, F_{1\lambda}, F_{1\eta}, F_{2\rho} \quad n_\rho \text{ odd} \end{cases}$$

Separate basis states with n_ρ even and odd

$$\langle\psi_t|\,P(1234)\,|\psi_t\rangle \;=\; \langle\psi_t|\,\mathrm{e}^{i\pi(b_\rho^\dagger b_\rho + b_\lambda^\dagger b_\lambda + b_\eta^\dagger b_\eta)}\,\mathrm{e}^{\theta_1(b_\rho^\dagger b_\lambda - b_\lambda^\dagger b_\rho)}\,\mathrm{e}^{\theta_2(b_\lambda^\dagger b_\eta - b_\eta^\dagger b_\lambda)}\,|\psi_t\rangle$$

$$\theta_1 = \arctan\sqrt{3} \qquad \theta_2 = \arctan\sqrt{8}$$

Change of oscillator coordinates: Talmi-Moshinsky brackets

	A_1	E_λ	$F_{2\lambda}$	$F_{2\eta}$	$F_{1\rho}$	A_2	E_ρ	$F_{1\lambda}$	$F_{1\eta}$	$F_{2\rho}$
$\langle\psi_t\|\,P(12)\,\|\psi_t\rangle$	1	1	1	1	1	-1	-1	-1	-1	-1
$\langle\psi_t\|\,P(1234)\,\|\psi_t\rangle$	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$