

$A \leq 3$ nuclei in finite volume with next-to-leading order pionless EFT

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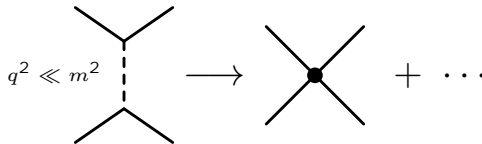
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Effective Field Theory (EFT)

- | **Effective field theory** (EFT) enables to systematically describe physical phenomena at momenta Q , while the underlying theory is valid at a higher mass scale M_{hi} , $Q \ll M_{hi}$.
- | The interactions are written in terms of **low energy degrees of freedom**, while retaining the symmetries of the underlying theory.
- | The details of the short-range dynamics are encoded in the **interaction strengths** (low energy constants, LECs).

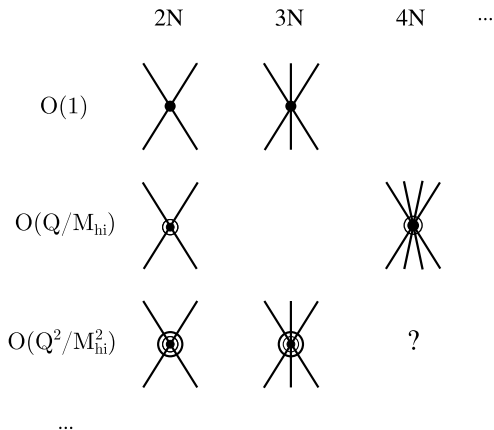
\neq EFT

- | EFTs are used in nuclear physics to realize **QCD** in terms of **hadrons**, instead of quarks and gluons.
- | In **=EFT**, nucleons are the sole degrees of freedom and the pion exchange is integrated out.



W. Hammer, S. König, and U. van Kolck, Rev. Mod. Phys. **92**, 025004 (2020).

∇ EFT power counting



W. Hammer, S. König, and U. van Kolck, Rev. Mod. Phys. **92**, 025004 (2020).

∇ EFT at LO

- | The leading-order (LO) potential:

$$\hat{V}_{LO} = \sum_{i<j} V_{S=1;T=0}(\mathbf{r}_{ij}) + V_{S=0;T=1}(\mathbf{r}_{ij}) + \sum_{i<j<k} V_3(\mathbf{r}_{ij}; \mathbf{r}_{jk})$$

$$V_{S;T}(\mathbf{r}_{ij}) = C_{S;T}^{(0)}(\Lambda) \hat{P}_{S;T} G_{\Lambda}(\mathbf{r}_{ij})$$

$$V_3(\mathbf{r}_{ij}; \mathbf{r}_{jk}) = D_1^{(0)}(\Lambda) \hat{P}_{1=2;1=2} G_{\Lambda}(\mathbf{r}_{ij}) G_{\Lambda}(\mathbf{r}_{jk})$$

- | Regularization scheme:

$$G_{\Lambda}(r) = \frac{\Lambda^3}{8^{3=2}} \exp\left[-\frac{\Lambda^2}{4} r^2\right]$$

The leading order is **solved exactly**.

$\not\propto$ EFT at NLO

- | Next-to-leading order (NLO) potential:

$$V_{\text{NLO}} = \tilde{V}_{\text{LO}} + \sum_{i < j} C_{0,1}^{(1)}(\Lambda) \hat{P}_{0,1} + C_{1,0}^{(1)}(\Lambda) \hat{P}_{1,0} \cdot G_{\Lambda}(\mathbf{r}_{ij}) \vec{\nabla}_{ij}^2 + \overleftarrow{\nabla}_{ij}^2 G_{\Lambda}(\mathbf{r}_{ij})$$

Due to the Wigner bound, the NLO terms are **treated perturbatively**.

$\not\propto$ EFT in finite-volume

Periodicity:

Inside a box of size $L \times L \times L$ with periodic boundary conditions, the potential is periodicised,

$$V_L(\mathbf{r}_1; \mathbf{r}_2; \dots) = \sum_{\mathbf{n}_1; \mathbf{n}_2; \dots} V(\mathbf{r}_1 + L\mathbf{n}_1; \mathbf{r}_2 + L\mathbf{n}_2; \dots)$$

Therefore, the Gaussian regulator is modified,

$$G_{\Lambda, \text{p.b.c.}}(r) = \frac{\Lambda^3}{8} \sum_{\mathbf{q} \in 2\mathbb{Z}^3} \exp\left[-\frac{\Lambda^2}{4}(\mathbf{r} - L\mathbf{q})^2\right]$$

SVM in free-space

- | We solve numerically the N -body Schrödinger equation with the **stochastic variational method** (SVM).
- | The wave function is expanded over a **correlated Gaussian basis**,

$$\Psi = \sum_i c_i \hat{A} \{ G_i(A_i; \mathbf{r}) \}_{SM_S IM_I}$$

- | The correlated Gaussians are given by,

$$G_i(A_i; \mathbf{r}) = \exp \left[-\frac{1}{2} \mathbf{r}^T A_i \mathbf{r} \right]$$

\mathbf{r} is a vector of all single-particle coordinates,

$$\mathbf{r} = (\mathbf{r}_1; \mathbf{r}_2; \dots; \mathbf{r}_N)^T$$

SVM in free-space

- | The **energy** and the **coefficients** c_i are obtained by solving the generalized eigenvalue problem,

$$\mathcal{H}\mathbf{c} = E\mathcal{N}\mathbf{c}$$

The diagram shows the equation $\mathcal{H}\mathbf{c} = E\mathcal{N}\mathbf{c}$ at the top. Two arrows point downwards from this equation to the matrix elements $\mathcal{H}_{ij} = \langle G_i | H | G_j \rangle$ on the left and $\mathcal{N}_{ij} = \langle G_i | G_j \rangle$ on the right.

$$\mathcal{H}_{ij} = \langle G_i | H | G_j \rangle \quad \mathcal{N}_{ij} = \langle G_i | G_j \rangle$$

- | The **non-linear parameters** A_j are chosen stochastically to minimize the energy.

SVM in finite-volume

- | In **box-SVM**, the wave-function is periodic in the box size,

$$\Psi_L(\mathbf{r}_1; \mathbf{r}_2; \dots) = \Psi_L(\mathbf{r}_1 + L\mathbf{n}_1; \mathbf{r}_2 + L\mathbf{n}_2; \dots):$$

- | Periodicised Gaussian basis function:

$$G_L(A_i; \mathbf{r}) = \sum_{\mathbf{b} \in \mathbb{Z}^{3N}} \exp \left[-\frac{1}{2} (\mathbf{r} - L\mathbf{b})^T A_i (\mathbf{r} - L\mathbf{b}) \right]$$

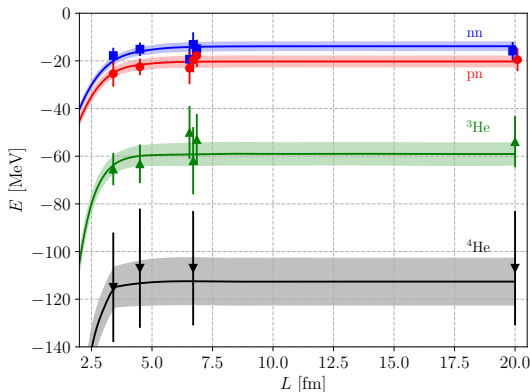
X. Yin and D. Blume, Phys. Rev. A **87**, 063609 (2013).

Extrapolation of LQCD results

- Finite-volume \neq EFT can be fitted using **LQCD results**.
- The EFT is then solved in free space to obtain **physical observables**.

M. Eliyahu, B. Bazak and N. Barnea, Phys. Rev. C **102**, 044003 (2020).

W. Detmold and P. E. Shanahan, Phys. Rev. D **103**, 074503 (2021).



Symbols represent NPLQCD results for $m_\pi = 806$ MeV, from S.R. Beane *et al.*, Phys. Rev. D **87**, 034506 (2013). Curves represent \neq EFT results obtained by M. Eliyahu *et al.*

Lüscher formalism

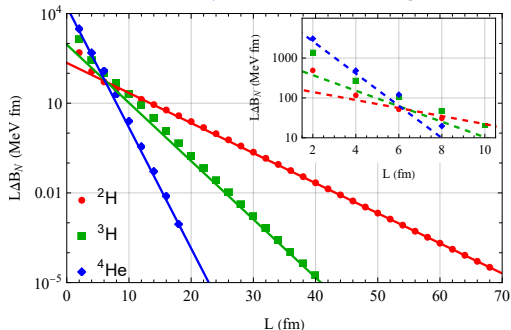
Scale separation - $R \ll \lambda \ll L$

Lüscher two-body **bound state** formula:

$$B_2(L) = B_2^{\text{free}} + \frac{6}{L} \frac{|\mathcal{A}_2|^2}{2} e^{-\rho_2 L} + \mathcal{O}(e^{-\rho_2 L}); \quad \rho_2 = \frac{1}{2} \sqrt{2m(B_2^{\text{free}} - E)}$$

M. Lüscher, Commun. Math. Phys. **104**, 177 (1986).

Generalization to an N -body bound state, S. König and D. Lee, Phys. Lett. B **779**, 9 (2018).



The energy shift ΔB_N due to the box, multiplied by the box size L , as function of L .

R. Yaron *et al.*,
Phys. Rev. D **106**
014511 (2022).

Lüscher formalism

Lüscher two-body **scattering state** formula:

$$k_L \cot \delta_0 = \frac{1}{L} S \left(\frac{Lk_L}{2} \right); \quad S(\eta) \equiv \lim_{\Omega \rightarrow 1} \frac{1}{j_{2Z}^2} \frac{1}{|\mathcal{J}|^2 - 4} \Omega^A$$

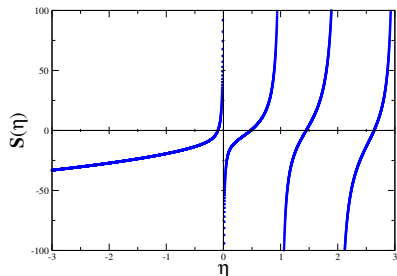
M. Lüscher, Commun. Math. Phys. **105**, 153 (1986); Nucl. Phys. B **354**, 531 (1991).

Effective range expansion:

$$k \cot \delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \mathcal{O}(k^4)$$

Binding energy:

$$B_2 \approx \frac{1}{ma_0^2} \left(1 - \frac{r_0}{a_0} \right)$$



S. R. Beane *et al.*, Phys. Lett. B **585** (2004).

$\not\propto$ EFT at NLO for nuclei in finite volume

W. Detmold *et al.*, **Constraint of pionless EFT using two-nucleon spectra from lattice QCD**. Phys. Rev. D **108**, 034509 (2023).

- | Naive power counting at NLO.
- | Non-perturbative inclusion of the NLO.

$\not\equiv$ EFT at NLO for nuclei in finite volume

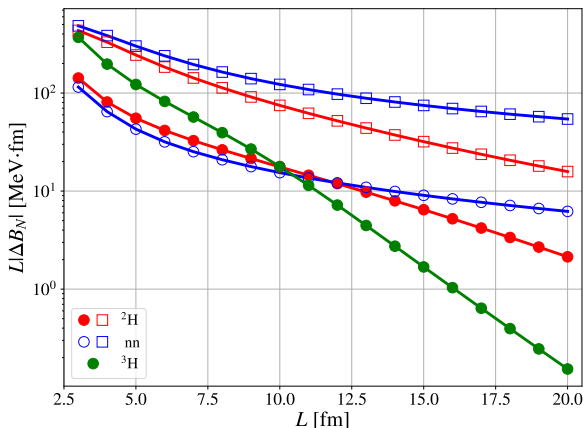
- | Fit finite-volume $\not\equiv$ EFT at NLO to finite-volume data, varying the box size.
- | Solve the same EFT in free-space and calculate S -wave observables.
- | Compare with the Lüscher formalism, fit to the same finite-volume data as the EFT was fitted to, and benchmark with known results.

Perturbative application of next-to-leading order pionless EFT for $A \leq 3$ nuclei in a finite volume.

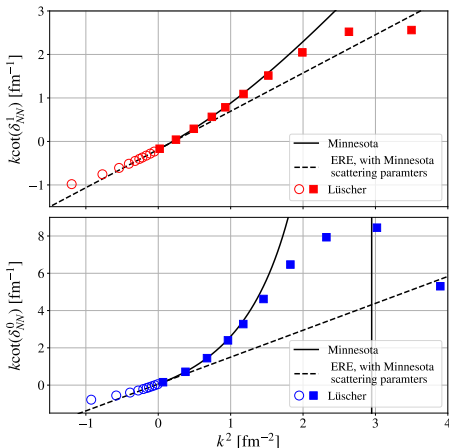
Tafat Weiss-Attia, Martin Schäfer and Betzalel Bazak,
arXiv:2402.04817. Submitted to Phys. Rev. D.

Input data

The A_1^+ spectra of $A \leq 3$ nuclei in a box with periodic boundary conditions, interacting via the phenomenological NN Minnesota potential:



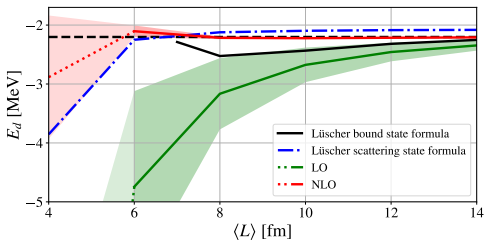
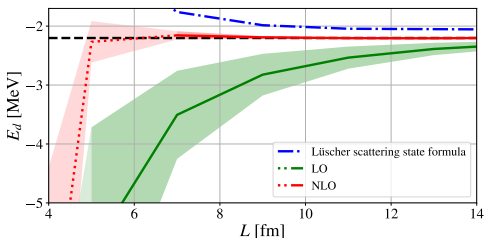
NN s -wave phase shifts with the Lüscher formalism



The NN s -wave phase shifts, presented as $k \cot(\delta_{NN})$, for the deuteron (upper panel) and dineutron (lower panel) channels plotted as a function k^2 .

Deuteron binding energy results

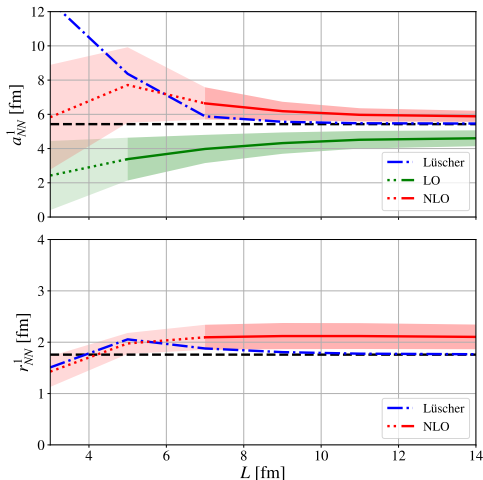
Upper panel -
using ground and
excited state
energies from one
box size L .



Lower panel -
using ground state
energies from two
adjacent boxes,
 $\langle L \rangle \pm 1$.

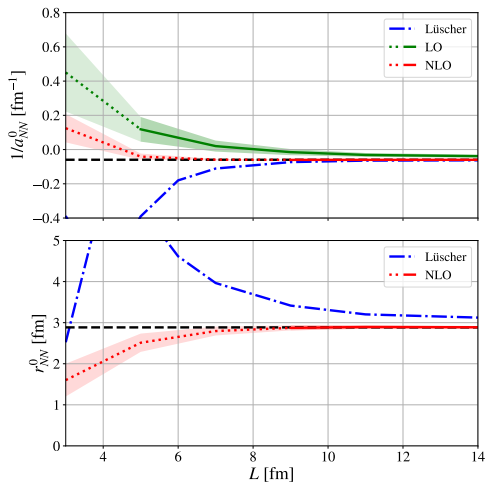
The free-space deuteron binding energy E_d extracted from finite-volume energies, as function of the box size, L (upper panel), or average box, $\langle L \rangle$ (lower panel), used.

NN spin-triplet scattering length and effective range



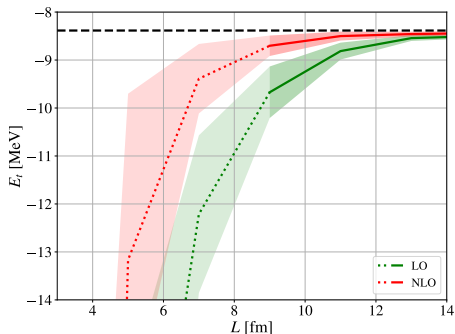
The free-space NN spin-triplet scattering length (upper panel) and effective range (lower panel) extracted from finite-volume energies, as function of the box size used.

NN spin-singlet scattering length and effective range

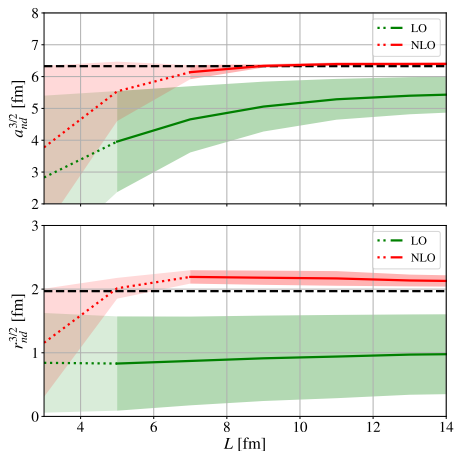


The free-space NN spin-singlet scattering length (upper panel) and effective range (lower panel) extracted from finite-volume energies, as function of the box size used.

Triton channel and nd scattering results



The free-space triton binding energy E_t extracted from finite-volume energies, as function of the box size used.



The free-space nd spin-quartet scattering length (upper panel) and effective range (lower panel) extracted from finite-volume energies, as function of the box size used.

Robustness of the methods

Propagation of uncertainties, assuming a 5% statistical uncertainty in the input data.

	$L = 5$			$L = 7$	
	Minnesota	Lüscher	NLO	Lüscher	NLO
E_d [MeV]	-2.2	-0.8(7)	-2.5(8)	-1.8(4)	-2.2(4)
a_{NN}^1 [fm]	5.4	8.4(3.1)	6.1(8)	5.9(5)	6.3(6)
r_{NN}^1 [fm]	1.7	2.1(3)	1.8(1)	1.9(2)	2.0(2)
$1=a_{NN}^0$ [fm ⁻¹]	-0.058	-0.4(3)	-0.02(2)	-0.11(5)	-0.05(2)
r_{NN}^0 [fm]	2.885	5.8(3.2)	2.3(2)	4(1)	2.7(3)

a_{NN}^1 with $L = 5$ fm: Lüscher $\sim 37\%$, =EFT at NLO $\sim 13\%$.

r_{NN}^0 with $L = 7$ fm: Lüscher $\sim 26\%$, =EFT at NLO $\sim 10\%$.

Conclusions and summary

- | =EFT up to NLO and the Lüscher formalism are employed to obtain **predictions** for free-space binding energies and S -wave scattering parameters of two- and three-nucleon systems, based on finite-volume spectrum.
- | =EFT at NLO generally yields accurate predictions when box sizes of $L = 5;7$ fm are used, whereas Lüscher predictions typically require larger box sizes of $L \geq 9$ fm.
- | =EFT proves to be a powerful extrapolation tool, and allows the extraction of physical observations with a limited amount of input data.