Alpha correlations and clustering in the shell model

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Two-nucleon correlation functions

Alpha correlation functions

Alpha clustering

Neutron-proton pairing, ESNT, September 2023

Two-nucleon correlators

Two-nucleon correlation operators are defined as

$$\sum_{i<j} \delta(r - \hat{r}_{ij}), \qquad \sum_{i<j} \delta(R - \hat{R}_{ij})\delta(r - \hat{r}_{ij})$$

with

$$\hat{R}_{ij} = \frac{1}{2}|\hat{\vec{r}}_i + \hat{\vec{r}}_j|, \qquad \hat{r}_{ij} = |\hat{\vec{r}}_i - \hat{\vec{r}}_j|$$

Expectation value defines a probability density:

times dr $\rightarrow$ *probability of two nucleons separated by r;*

times drdR $\rightarrow$ *probability of two nucleons separated by r* and at a distance *R* from the centre of mass.

Two-nucleon correlation functions

Expectation values for a two-nucleon state

$$\langle abLST|\delta(r-\hat{r}_{12})|cdLST\rangle = \frac{1}{\sqrt{8}}r^2\sum_{\mathcal{N}\mathcal{L}nn'l}\tilde{a}^{n_al_an_bl_b}_{\mathcal{N}\mathcal{L}nl,LST}\tilde{a}^{n_cl_cn_dl_d}_{\mathcal{N}\mathcal{L}n'l,LST}R_{nl}\left(\frac{r}{\sqrt{2}}\right)R_{n'l}\left(\frac{r}{\sqrt{2}}\right)$$

and

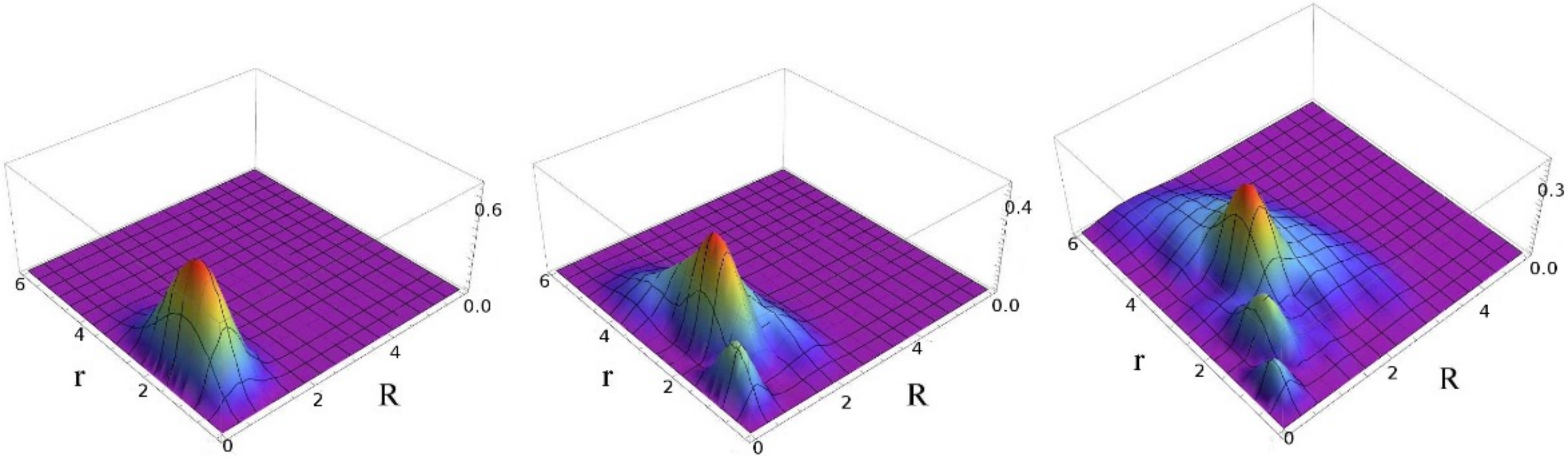
$$\begin{aligned}&\langle abLST|\delta(R-\hat{R}_{12})\delta(r-\hat{r}_{12})|cdLST\rangle\\&=R^2r^2\sum_{\mathcal{N}\mathcal{N}'\mathcal{L}}\sum_{nn'l}\tilde{a}^{n_al_an_bl_b}_{\mathcal{N}\mathcal{L}nl,LST}\tilde{a}^{n_cl_cn_dl_d}_{\mathcal{N}'\mathcal{L}n'l,LST}R_{\mathcal{N}\mathcal{L}}\left(\sqrt{2}R\right)R_{\mathcal{N}'\mathcal{L}}\left(\sqrt{2}R\right)R_{nl}\left(\frac{r}{\sqrt{2}}\right)R_{n'l}\left(\frac{r}{\sqrt{2}}\right)\end{aligned}$$

in terms of modified Talmi-Moshinsky brackets

$$\tilde{a}^{n_al_an_bl_b}_{\mathcal{N}\mathcal{L}nl,LST}\equiv\frac{1-(-)^{l+S+T}}{\sqrt{2(1+\delta_{n_an_b}\delta_{l_al_b})}}a^{n_al_an_bl_b}_{\mathcal{N}\mathcal{L}nl,L}$$

P. Mei & P. Van Isacker, Ann. Phys. (NY) **327** (2012) 1162

Two identical nucleons in *s* orbitals



(a) $|(0s)^2; 00\rangle$ (b) $|(1s)^2; 00\rangle$ (c) $|(2s)^2; 00\rangle$

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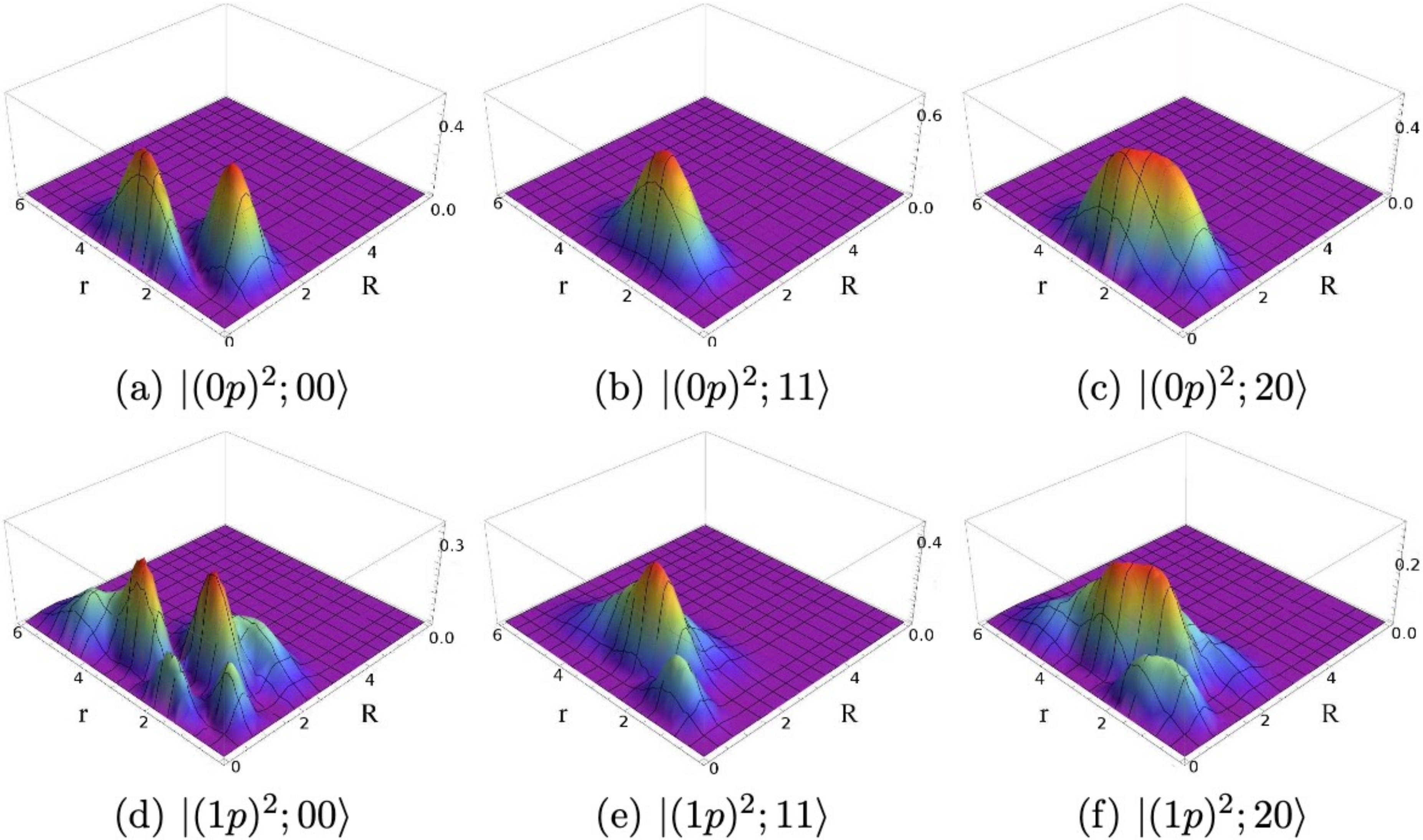
Two identical nucleons in *p* orbitals



(a) $|(0p)^2;00\rangle$ (b) $|(0p)^2;11\rangle$ (c) $|(0p)^2;20\rangle$

(d) $|(1p)^2;00\rangle$ (e) $|(1p)^2;11\rangle$ (f) $|(1p)^2;20\rangle$

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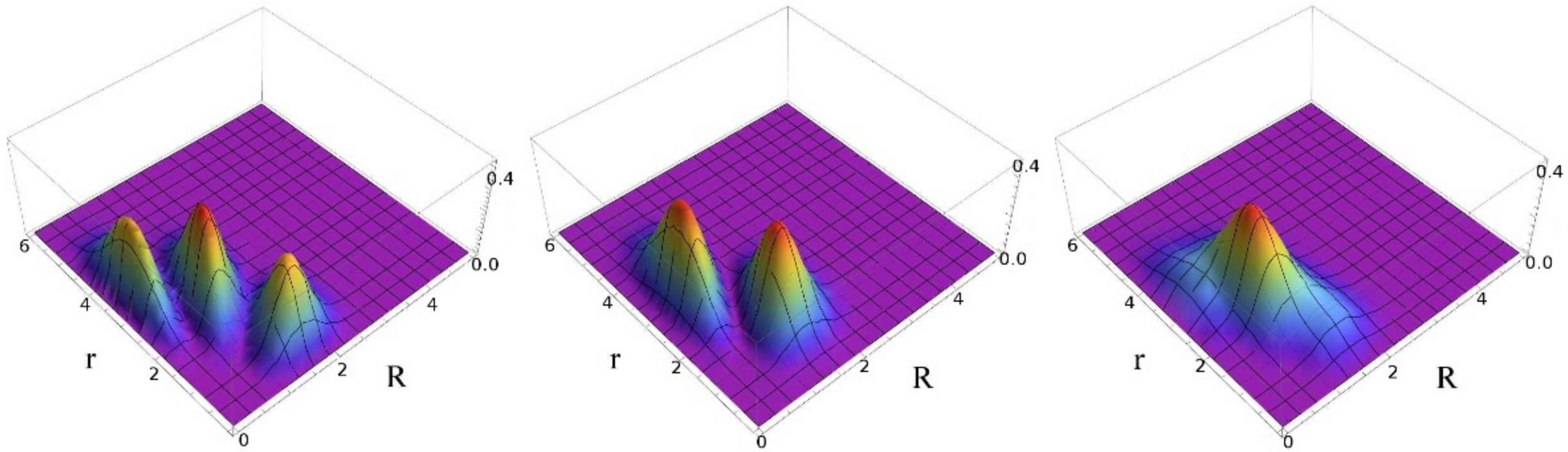
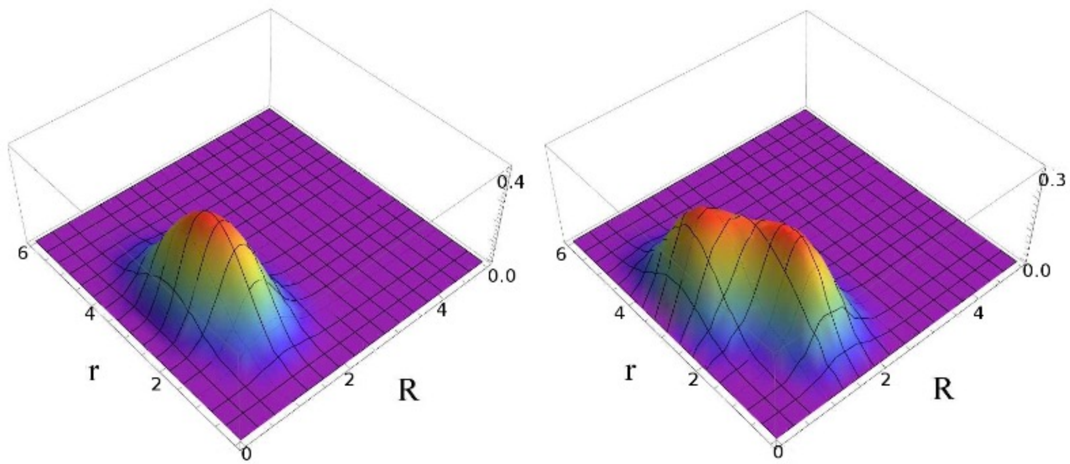
Two identical nucleons in 0*d* orbital



(a) $|(0d)^2;00\rangle$ (b) $|(0d)^2;11\rangle$ (c) $|(0d)^2;20\rangle$



(d) $|(0d)^2;31\rangle$ (e) $|(0d)^2;40\rangle$

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Four identical nucleons in 0p orbital

A $J = 0$ state has $(LS) = (00)$ or $(LS) = (11)$ with probability densities given by

$$\mathcal{P}_{p^4;00} = \frac{9}{1024\pi^4} \sum_{(ij)\neq(kl)} \sin^2\theta_{ij} \sin^2\theta_{kl} \cos^2\theta_{ij,kl},$$

and

$$\mathcal{P}_{p^4;11} = \frac{9}{2048\pi^4} \sum_{(ij)\neq(kl)} \sin^2\theta_{ij} \sin^2\theta_{kl} \sin^2\theta_{ij,kl}.$$

where θ_{ij} is the angle between $\bar{r}_i$ and $\bar{r}_j$, and $\theta_{ij,kl}$ is the angle between $\bar{r}_{ij}$ and $\bar{r}_{kl}$.

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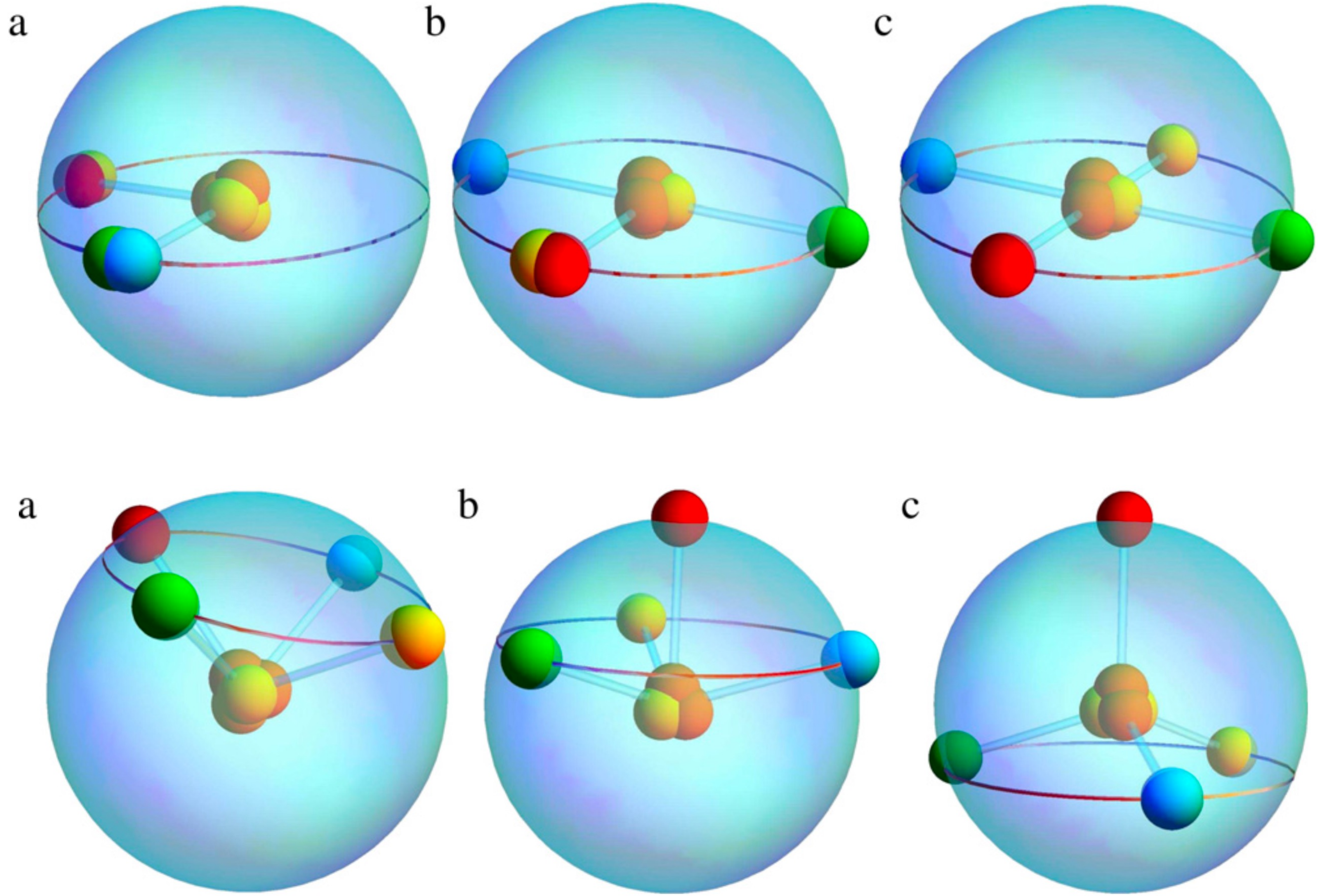
Four identical nucleons in 0*p* orbital



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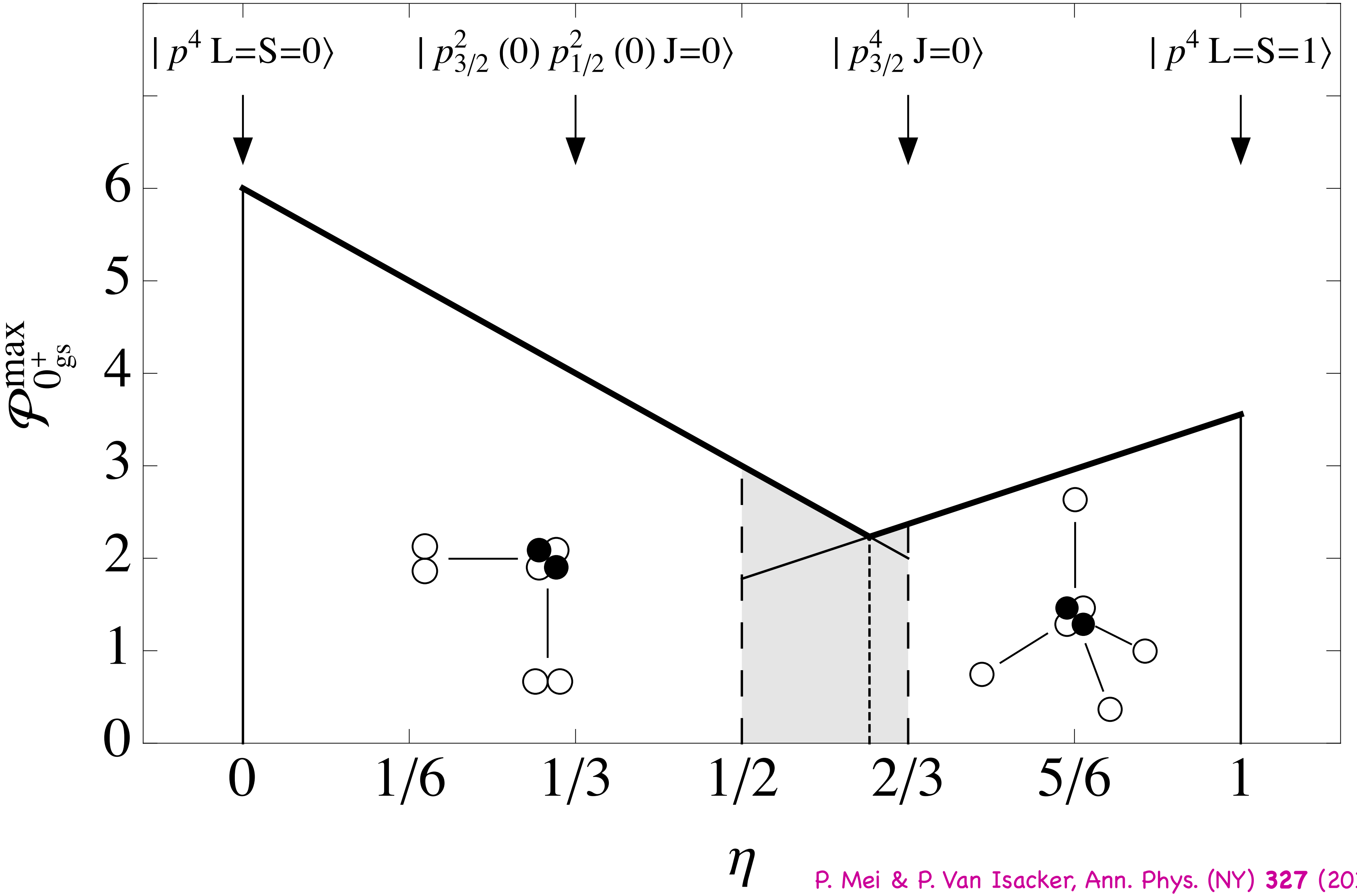
Example: ^{8}He



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Four-nucleon $(2\nu, 2\pi)$ correlators

Four-nucleon correlation operators are defined as

$$\hat{\Delta}_{ijkl}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) \equiv \delta(R_{\nu\pi} - \hat{R}_{ij\cdot kl})\delta(r_{\nu\nu} - \hat{r}_{ij})\delta(r_{\pi\pi} - \hat{r}_{kl})$$

with $(ij \in \nu, kl \in \pi)$

$$\hat{R}_{ij} = |\tfrac{1}{2}(\hat{\vec{r}}_i + \hat{\vec{r}}_j)|, \quad \hat{r}_{ij} = |\hat{\vec{r}}_i - \hat{\vec{r}}_j|,$$

$$\hat{R}_{kl} = |\tfrac{1}{2}(\hat{\vec{r}}_k + \hat{\vec{r}}_l)|, \quad \hat{r}_{kl} = |\hat{\vec{r}}_k - \hat{\vec{r}}_l|,$$

$$\hat{R}_{ijkl} = |\tfrac{1}{2}(\hat{\vec{R}}_{ij} + \hat{\vec{R}}_{kl})|, \quad \hat{R}_{ij\cdot kl} = |\hat{\vec{R}}_{ij} - \hat{\vec{R}}_{kl})|,$$

$$\hat{r}_{ijkl} = |\tfrac{1}{2}(\hat{\vec{r}}_{ij} + \hat{\vec{r}}_{kl})|, \quad \hat{r}_{ij\cdot kl} = |\hat{\vec{r}}_{ij} - \hat{\vec{r}}_{kl}|.$$

Geometry of $2\nu + 2\pi$

If

$r_{\nu\nu}$ *is the distance between the neutrons,*

$r_{\pi\pi}$ *is the distance between the protons,*

$R_{\nu\pi}$ *is the distance between the centres of mass of the neutrons and of the protons,*

$r_{\nu\pi}$ *is the distance between a neutron and a proton,*

one has the relation, valid for a tetrahedron,

$$4R^2_{\nu\pi} = 4r^2_{\nu\pi} - r^2_{\nu\nu} - r^2_{\pi\pi}$$

Coordinate transformation

$$
\begin{aligned}
&|ab(L_\nu S_\nu), cd(L_\pi S_\pi); LS\rangle \\
&= \sum_{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi} \sum_{n_\nu l_\nu n_\pi l_\pi} \tilde{a}^{n_a l_a n_b l_b}_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu, L_\nu S_\nu} \tilde{a}^{n_c l_c n_d l_d}_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi, L_\pi S_\pi} \\
&\quad \times |\mathcal{N}_\nu \mathcal{L}_\nu (\bar{R}'_{12}) n_\nu l_\nu (\bar{r}'_{12})(L_\nu S_\nu), \mathcal{N}_\pi \mathcal{L}_\pi (\bar{R}'_{34}) n_\pi l_\pi (\bar{r}'_{34})(L_\pi S_\pi); LS\rangle \\
&= - \sum_{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi} \sum_{n_\nu l_\nu n_\pi l_\pi} \tilde{a}^{n_a l_a n_b l_b}_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu, L_\nu S_\nu} \tilde{a}^{n_c l_c n_d l_d}_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi, L_\pi S_\pi} \sum_{\mathcal{L}_\rho l_\rho \mathcal{S}_\rho s_\rho} \begin{bmatrix} \mathcal{L}_\nu & l_\nu & L_\nu \\ \mathcal{L}_\pi & l_\pi & L_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & S_\nu \\ 1/2 & 1/2 & S_\pi \\ \mathcal{S}_\rho & s_\rho & S \end{bmatrix} \\
&\quad \times |\mathcal{N}_\nu \mathcal{L}_\nu (\bar{R}'_{12}) \mathcal{N}_\pi \mathcal{L}_\pi (\bar{R}'_{34})(\mathcal{L}_\rho \mathcal{S}_\rho), n_\nu l_\nu (\bar{r}'_{12}) n_\pi l_\pi (\bar{r}'_{34})(l_\rho s_\rho); LS\rangle \\
&= - \sum_{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi} \sum_{n_\nu l_\nu n_\pi l_\pi} \tilde{a}^{n_a l_a n_b l_b}_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu, L_\nu S_\nu} \tilde{a}^{n_c l_c n_d l_d}_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi, L_\pi S_\pi} \sum_{\mathcal{L}_\rho l_\rho \mathcal{S}_\rho s_\rho} \begin{bmatrix} \mathcal{L}_\nu & l_\nu & L_\nu \\ \mathcal{L}_\pi & l_\pi & L_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & S_\nu \\ 1/2 & 1/2 & S_\pi \\ \mathcal{S}_\rho & s_\rho & S \end{bmatrix} \\
&\quad \times \sum_{\mathcal{N} \mathcal{L} n l} a^{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi}_{\mathcal{N} \mathcal{L} n l, \mathcal{L}_\rho} |\mathcal{N} \mathcal{L} (\bar{R}'_{1234}) n l (\bar{R}'_{12\cdot 34})(\mathcal{L}_\rho \mathcal{S}_\rho), n_\nu l_\nu (\bar{r}'_{12}) n_\pi l_\pi (\bar{r}'_{34})(l_\rho s_\rho); LS\rangle,
\end{aligned}
$$

Four-nucleon correlation function

$$\langle ab(L_\nu S_\nu), cd(L_\pi S_\pi); LS|\hat{\Delta}_{1234}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi})|a'b'(L'_\nu S'_\nu), c'd'(L'_\pi S'_\pi); LS\rangle$$

$$= \frac{1}{8} R^2_{\nu\pi} r^2_{\nu\nu} r^2_{\pi\pi} \delta_{S_\nu S'_\nu} \delta_{S_\pi S'_\pi} \sum_{\mathcal{L}_\nu \mathcal{L}_\pi} \sum_{\mathcal{L}'_\nu \mathcal{L}'_\pi} \sum_{l_\nu l_\pi} \sum_{\mathcal{L}_\rho l_\rho} \begin{bmatrix} \mathcal{L}_\nu & l_\nu & L_\nu \\ \mathcal{L}_\pi & l_\pi & L_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix} \begin{bmatrix} \mathcal{L}'_\nu & l_\nu & L'_\nu \\ \mathcal{L}'_\pi & l_\pi & L'_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix}$$

$$\times \sum_{\mathcal{N}_\nu \mathcal{N}_\pi} \left[\sum_{n_\nu} \tilde{a}^{n_a l_a n_b l_b}_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu, L_\nu S_\nu} R_{n_\nu l_\nu} \left(\frac{r_{\nu\nu}}{\sqrt{2}} \right) \right] \left[\sum_{n_\pi} \tilde{a}^{n_c l_c n_d l_d}_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi, L_\pi S_\pi} R_{n_\pi l_\pi} \left(\frac{r_{\pi\pi}}{\sqrt{2}} \right) \right]$$

$$\times \sum_{\mathcal{N}'_\nu \mathcal{N}'_\pi} \left[\sum_{n'_\nu} \tilde{a}^{n'_a l'_a n'_b l'_b}_{\mathcal{N}'_\nu \mathcal{L}'_\nu n'_\nu l_\nu, L'_\nu S'_\nu} R_{n'_\nu l_\nu} \left(\frac{r_{\nu\nu}}{\sqrt{2}} \right) \right] \left[\sum_{n'_\pi} \tilde{a}^{n'_c l'_c n'_d l'_d}_{\mathcal{N}'_\pi \mathcal{L}'_\pi n'_\pi l_\pi, L'_\pi S'_\pi} R_{n'_\pi l_\pi} \left(\frac{r_{\pi\pi}}{\sqrt{2}} \right) \right]$$

$$\times \sum_{\mathcal{N}\mathcal{L}l} \left[\sum_{n} a^{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi}_{\mathcal{N}\mathcal{L}nl, \mathcal{L}_\rho} R_{nl}(R_{\nu\pi}) \right] \left[\sum_{n'} a^{\mathcal{N}'_\nu \mathcal{L}'_\nu \mathcal{N}'_\pi \mathcal{L}'_\pi}_{\mathcal{N}\mathcal{L}n'l, \mathcal{L}_\rho} R_{n'l}(R_{\nu\pi}) \right],$$

The α particle: $0\hbar\omega$

If all nucleons are in the 0*s* orbital, $(LST) = (000)$, the probability density is

$$P_{000}^{(0)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 e^{-R_{\nu\pi}^2 - \frac{1}{2} r_{\nu\nu}^2 - \frac{1}{2} r_{\pi\pi}^2}$$

The probability peaks at $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$ and $R_{\nu\pi} = 1$ or $r_{\nu\pi} = r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$.

$\rightarrow$ *The most probable geometry of an α particle is a Platonic tetrahedron.*

The α particle: $1\hbar\omega$

The isoscalar 1^- state $(LST) = (100)$ is spurious and has the probability density

$$P_{100}^{(1)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 e^{-R_{\nu\pi}^2 - \frac{1}{2} r_{\nu\nu}^2 - \frac{1}{2} r_{\pi\pi}^2}$$

The probability peaks at $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$ and $R_{\nu\pi} = 1$ or $r_{\nu\pi} = r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$.

$\rightarrow$ *The spurious state has the same density distribution as the $0\hbar\omega$ ground state since it has the same intrinsic structure.*

The α particle: $1\hbar\omega$

The isovector 1^- state $(LST) = (101)$ has the probability density

$$P^{(1)}_{101}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{16}{3\pi^{3/2}} R^4_{\nu\pi} r^2_{\nu\nu} r^2_{\pi\pi} e^{-R^2_{\nu\pi} - \frac{1}{2} r^2_{\nu\nu} - \frac{1}{2} r^2_{\pi\pi}}$$

The probability peaks at $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$ and $R_{\nu\pi} = \sqrt{2}$ or $r_{\nu\pi} = \sqrt{3}$.

The α particle: $1\hbar\omega$

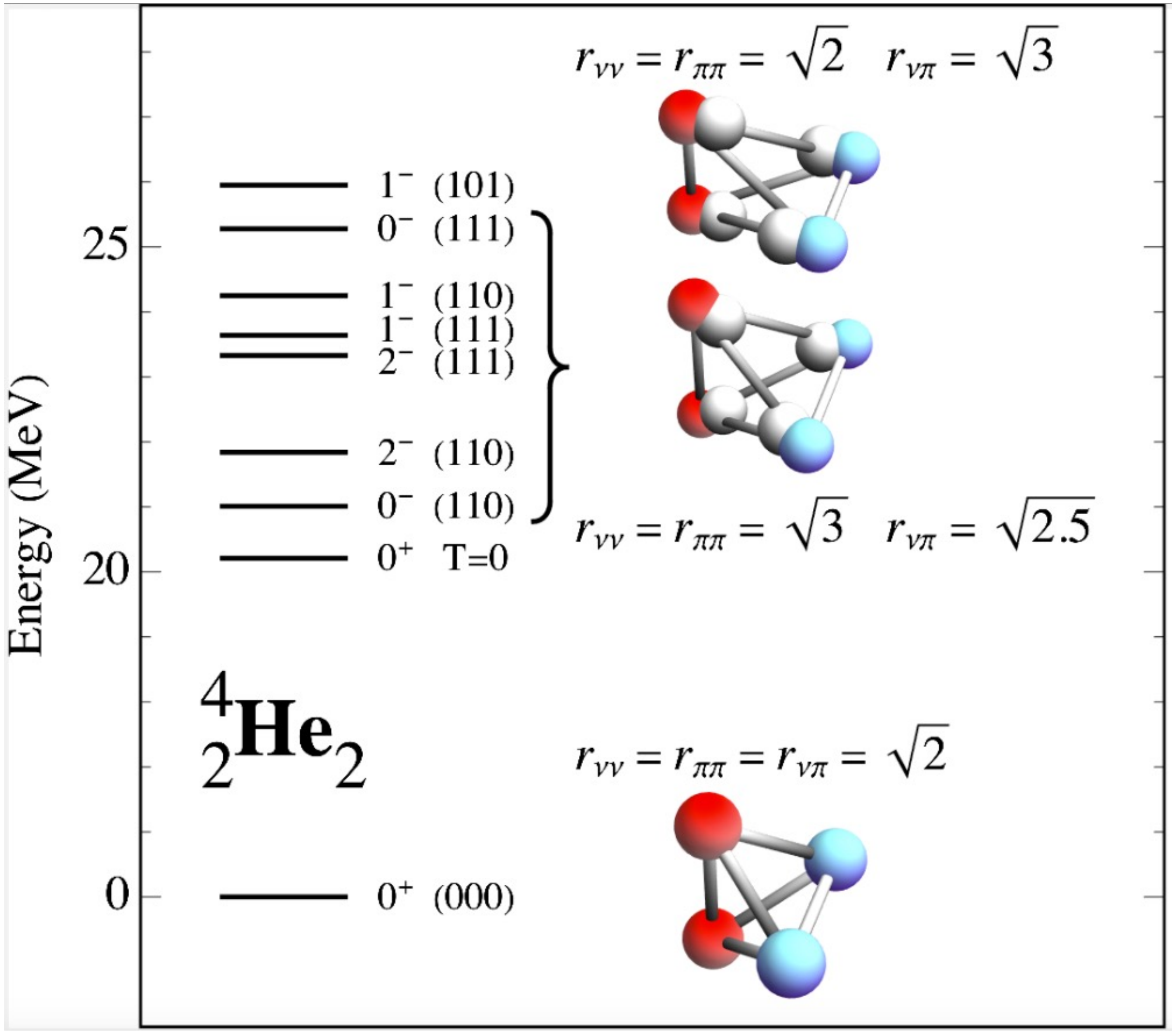
The isoscalar and isovector states with $(LS) = (11)$ have the probability density

$$P^{(1)}_{110}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{4}{3\pi^{3/2}} R^2_{\nu\pi} r^2_{\nu\nu} r^2_{\pi\pi} (r^2_{\nu\nu} + r^2_{\pi\pi}) e^{-R^2_{\nu\pi} - \frac{1}{2} r^2_{\nu\nu} - \frac{1}{2} r^2_{\pi\pi}}$$

$$P^{(1)}_{111}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{4}{3\pi^{3/2}} R^2_{\nu\pi} r^2_{\nu\nu} r^2_{\pi\pi} (r^2_{\nu\nu} + r^2_{\pi\pi}) e^{-R^2_{\nu\pi} - \frac{1}{2} r^2_{\nu\nu} - \frac{1}{2} r^2_{\pi\pi}}$$

The probability peaks at $r_{\nu\nu} = r_{\pi\pi} = \sqrt{3}$ and $R_{\nu\pi} = 1$ or $r_{\nu\pi} = \sqrt{2.5}$.

The α particle: $0\hbar\omega + 1\hbar\omega$



The α particle: $0\hbar\omega + 2\hbar\omega$

The tensor force mixes the $(LS) = (00)$ ground state with $(LS) = (22)$, with probability density

$$P^{(2)}_{220}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{9\pi^{3/2}} R^2_{\nu\pi} r^4_{\nu\nu} r^4_{\pi\pi} e^{-R^2_{\nu\pi} - \frac{1}{2} r^2_{\nu\nu} - \frac{1}{2} r^2_{\pi\pi}}$$

The probability peaks at $r_{\nu\nu} = r_{\pi\pi} = 2$ and $R_{\nu\pi} = 1$ or $r_{\nu\pi} = \sqrt{3}$.

The α particle: $0\hbar\omega + 2\hbar\omega$

A schematic Hamiltonian

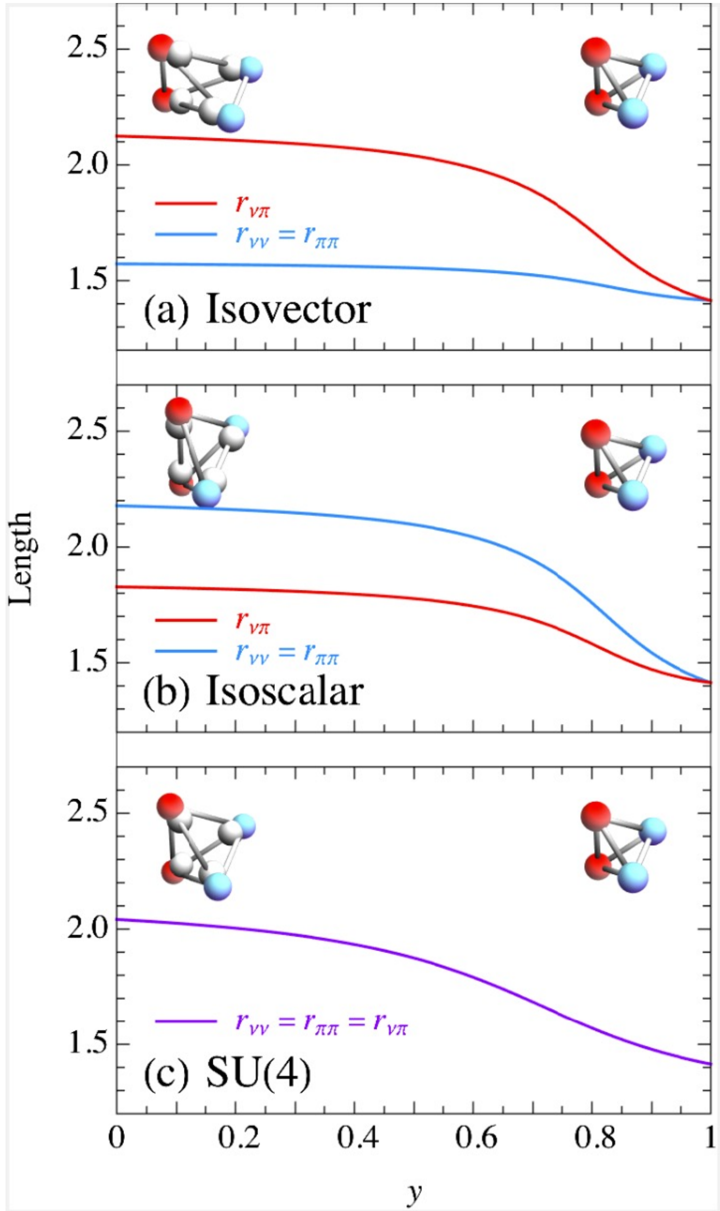
$$\hat{H} = \hbar\omega\ \hat{n} - 4\pi \sum_{T=0,1} a'_T \sum_{i<j} \delta(\bar{r}_i - \bar{r}_j)\delta(\bar{r}_i - R_0)$$

Orbitals split by $\hbar\omega$ and SDI with isoscalar and isovector strengths a_0 and a_1.

The α particle: $0\hbar\omega + 2\hbar\omega$

If $y = 0$, orbitals are degenerate, $\hbar\omega = 0$.

If $y = 1$, $\hbar\omega \gg a_T$ and we recover s^4 solution.



Alpha clustering

Consider a system of n_ν neutrons and n_π protons in the shell model.

Problem: Can we quantify the probability of formation of an α particle in an arbitrary shell-model state for that system?

Approximation: Assume throughout that the α particle coincides with two neutrons and two protons in the $0s$ orbital.

Discussions with Augusto Macchiavelli, 2021

α-particle probability for $(2\nu, 2\pi)$

Consider two neutrons and two protons in orbitals Ω_ν and Ω_π, respectively.

The two neutrons (protons) have the oscillator quanta $N_{q\nu}$ ($N_{q\pi}$) associated with them, which differ from zero ($\neq \alpha$ particle).

Conjecture: The α-particle component of the $(2\nu, 2\pi)$ configuration is obtained by imposing the same *intrinsic* structure as the α particle. This is achieved if all quanta are put in the centre-of-mass coordinate and none in other degrees of freedom.

Coordinate transformation

$$
\begin{aligned}
&|a_\nu b_\nu(\Gamma_{2\nu}), a_\pi b_\pi(\Gamma_{2\pi}); \Gamma_4\rangle \\
&= \sum_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu} \sum_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi} \prod_{\rho=\nu,\pi} \tilde{a}^{n_{a\rho} l_{a\rho} n_{b\rho} l_{b\rho}}_{\mathcal{N}_\rho \mathcal{L}_{2\rho} n_\rho l_\rho, L_{2\rho} S_{2\rho}} \\
&\quad \times |\mathcal{N}_\nu \mathcal{L}_\nu(\bar{R}'_\nu) n_\nu l_\nu(\bar{r}'_\nu)(\Gamma_{2\nu}), \mathcal{N}_\pi \mathcal{L}_\pi(\bar{R}'_\pi) n_\pi l_\pi(\bar{r}'_\pi)(\Gamma_{2\pi}); \Gamma_4\rangle \\
&= -\sum_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu} \sum_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi} \prod_{\rho=\nu,\pi} \tilde{a}^{n_{a\rho} l_{a\rho} n_{b\rho} l_{b\rho}}_{\mathcal{N}_\rho \mathcal{L}_\rho n_\rho l_\rho, L_{2\rho} S_{2\rho}} \sum_{\mathcal{R}\mathcal{S}rs} \begin{bmatrix} \mathcal{L}_\nu\, 1/2 & l_\nu\, 1/2 & \Gamma_{2\nu} \\ \mathcal{L}_\pi\, 1/2 & l_\pi\, 1/2 & \Gamma_{2\pi} \\ \mathcal{R}\mathcal{S} & rs & L_4 S_4 \end{bmatrix} \\
&\quad \times |\mathcal{N}_\nu \mathcal{L}_\nu(\bar{R}'_\nu) \mathcal{N}_\pi \mathcal{L}_\pi(\bar{R}'_\pi)(\mathcal{R}\mathcal{S}), n_\nu l_\nu(\bar{r}'_\nu) n_\pi l_\pi(\bar{r}'_\pi)(rs); \Gamma_4\rangle \\
&= -\sum_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu} \sum_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi} \prod_{\rho=\nu,\pi} \tilde{a}^{n_{a\rho} l_{a\rho} n_{b\rho} l_{b\rho}}_{\mathcal{N}_\rho \mathcal{L}_\rho n_\rho l_\rho, L_{2\rho} S_{2\rho}} \sum_{\mathcal{R}\mathcal{S}rs} \begin{bmatrix} \mathcal{L}_\nu\, 1/2 & l_\nu\, 1/2 & \Gamma_{2\nu} \\ \mathcal{L}_\pi\, 1/2 & l_\pi\, 1/2 & \Gamma_{2\pi} \\ \mathcal{R}\mathcal{S} & rs & \Gamma_4 \end{bmatrix} \\
&\quad \times \sum_{\mathcal{N}\mathcal{L}nl} a^{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi}_{\mathcal{N}\mathcal{L}nl, \mathcal{R}} |\mathcal{N}\mathcal{L}(\bar{R}'_{\nu\pi}) nl(\bar{r}'_{\nu\pi})(\mathcal{R}\mathcal{S}), n_\nu l_\nu(\bar{r}'_\nu) n_\pi l_\pi(\bar{r}'_\pi)(rs); \Gamma_4\rangle
\end{aligned}
$$

α-particle probability for $(2\nu, 2\pi)$

The α-particle probability of the state $|a_\nu b_\nu(L_2S_2), a_\pi b_\pi(L_2S_2); L_4 = S_4 = 0\rangle$ equals

$$\left(\tilde{a}^{n_{a\nu}l_{a\nu}n_{b\nu}l_{b\nu}}_{\mathcal{N}_\nu L_2 00, L_2 S_2}\right)^2 \left(\tilde{a}^{n_{a\pi}l_{a\pi}n_{b\pi}l_{b\pi}}_{\mathcal{N}_\pi L_2 00, L_2 S_2}\right)^2 \left(a^{\mathcal{N}_\nu L_2 \mathcal{N}_\pi L_2}_{\mathcal{N}000,0}\right)^2$$

The α-particle probability of the state $|a_\nu b_\nu(J_2), a_\pi b_\pi(J_2); J_4 = 0\rangle$ equals

$$\sum_{S_2}\left(\sum_{L_2} \tilde{a}^{n_{a\nu}l_{a\nu}j_{a\nu}n_{b\nu}l_{b\nu}j_{b\nu}}_{\mathcal{N}_\nu L_2 00, L_2 S_2 J_2} \tilde{a}^{n_{a\pi}l_{a\pi}j_{a\pi}n_{b\pi}l_{b\pi}j_{b\pi}}_{\mathcal{N}_\pi L_2 00, L_2 S_2 J_2} \begin{bmatrix} L_2 & S_2 & J_2 \\ L_2 & S_2 & J_2 \\ 0 & 0 & 0 \end{bmatrix} a^{\mathcal{N}_\nu L_2 \mathcal{N}_\pi L_2}_{\mathcal{N}000,0}\right)^2$$

General α-particle probability

The α-particle probability of the state

$$|\Phi\Gamma_n\rangle = \sum_{\Lambda_{n\nu}\Lambda_{n\pi}} b_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n}|\Lambda_{n\nu},\Lambda_{n\pi};\Gamma_n\rangle$$

is

$$p_\alpha \equiv \langle\Phi\Gamma_n|\hat{P}_\alpha|\Phi\Gamma_n\rangle^2 = \sum_{\Lambda_{r\nu}\Lambda_{r\pi}}\sum_{\mathcal{S}=0,1}\left(\sum_{\Lambda_{n\nu}\Lambda_{n\pi}} b_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n} d^{\mathcal{S}\Lambda_{r\nu}\Lambda_{r\pi}}_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n}\right)^2$$

$$d^{\mathcal{S}\Lambda_{r\nu}\Lambda_{r\pi}}_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n} = \sum_{\Gamma_2}[\Lambda_{2\nu},\Lambda_{r\nu}|\}\Lambda_{n\nu}][\Lambda_{2\pi},\Lambda_{r\pi}|\}\Lambda_{n\pi}]\begin{bmatrix}\Gamma_2 & \Gamma_{r\nu} & \Gamma_{n\nu}\\ \Gamma_2 & \Gamma_{r\pi} & \Gamma_{n\pi}\\ 00 & \Gamma_n & \Gamma_n\end{bmatrix} c^{\mathcal{S}00}_{\Lambda_{2\nu}\Lambda_{2\pi}}$$

$$c^{\mathcal{S}00}_{\Lambda_{2\nu}\Lambda_{2\pi}} = -\tilde{a}^{n_{a\nu}l_{a\nu}n_{b\nu}l_{b\nu}}_{\mathcal{N}_\nu L_2 00, L_2 S_2}\tilde{a}^{n_{a\pi}l_{a\pi}n_{b\pi}l_{b\pi}}_{\mathcal{N}_\pi L_2 00, L_2 S_2}\begin{bmatrix}1/2 & 1/2 & S_2\\ 1/2 & 1/2 & S_2\\ \mathcal{S} & \mathcal{S} & 0\end{bmatrix} a^{\mathcal{N}_\nu L_2 \mathcal{N}_\pi L_2}_{\mathcal{N}000,0}$$

Dependence on mass number A

The oscillator length b in a nucleus with mass number A differs from that in the α particle.

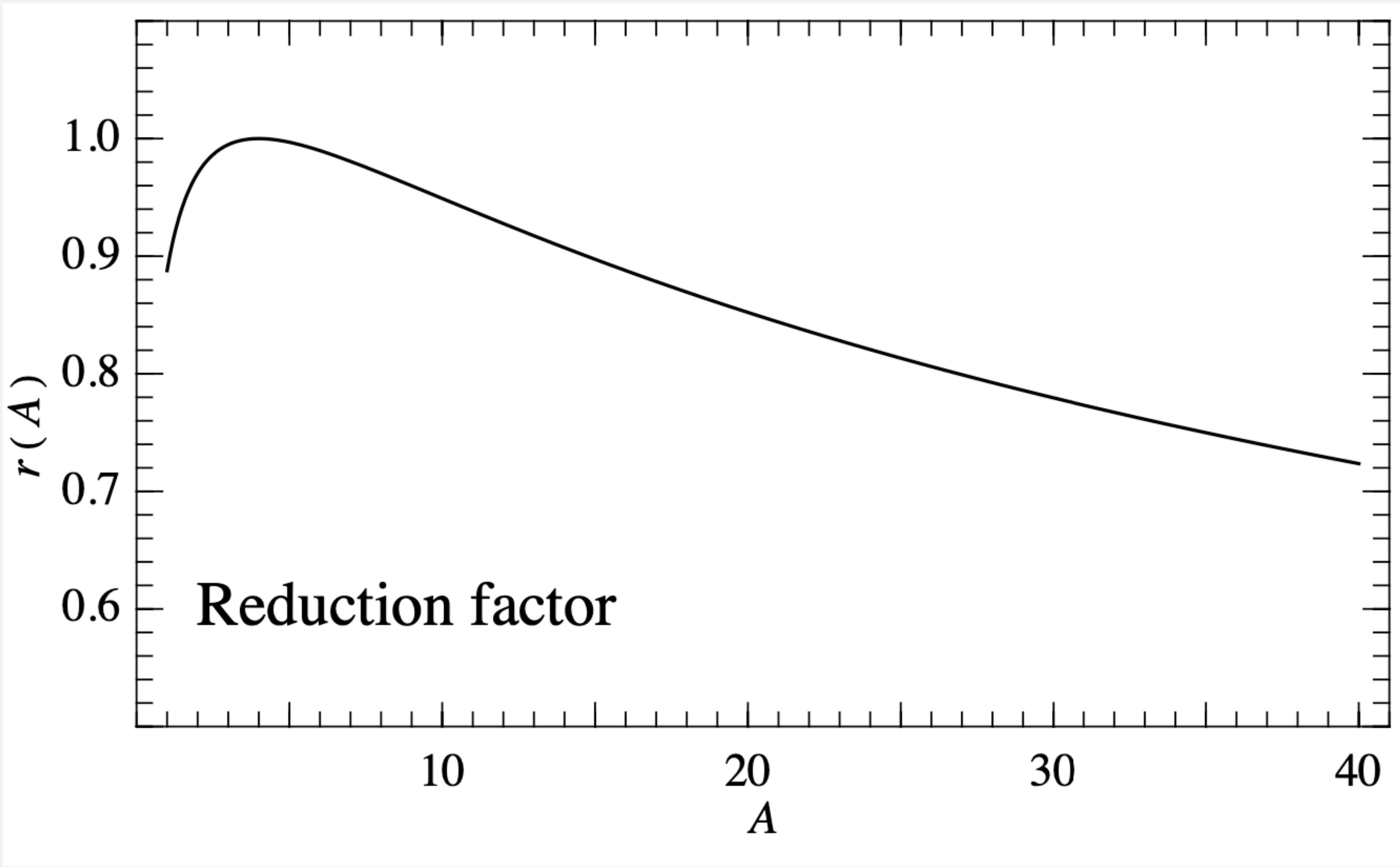
The overlap between 0s wave functions with lengths b_A and b_4 is

$$\frac{4}{(b_A b_4)^{3/2}\sqrt{\pi}}\int_0^\infty \exp\left[-\left(\frac{b_A^2+b_4^2}{2b_A^2 b_4^2}\right)r^2\right]r^2 dr = \left(\frac{2b_A b_4}{b_A^2+b_4^2}\right)^{3/2}$$

The reduction factor is therefore

$$r(A) \equiv \left(\frac{2b_A b_4}{b_A^2+b_4^2}\right)^{9/2} \approx \frac{64}{A^{3/4}}\left(\frac{1}{1+(4/A)^{1/3}}\right)^{9/2}$$

Dependence on mass number A



Example: the *sd* shell

A schematic Hamiltonian

$$\hat{H} = -g \sum_{i=1}^{A} \bar{l}_i \cdot \bar{s}_i - 4\pi \sum_{T=0,1} a'_T \sum_{i<j=1}^{A} \delta(\bar{r}_i - \bar{r}_j)\delta(r_i - R_0)$$

SDI with isoscalar and isovector strengths a_0 and a_1 and spin-orbit term with strength g.

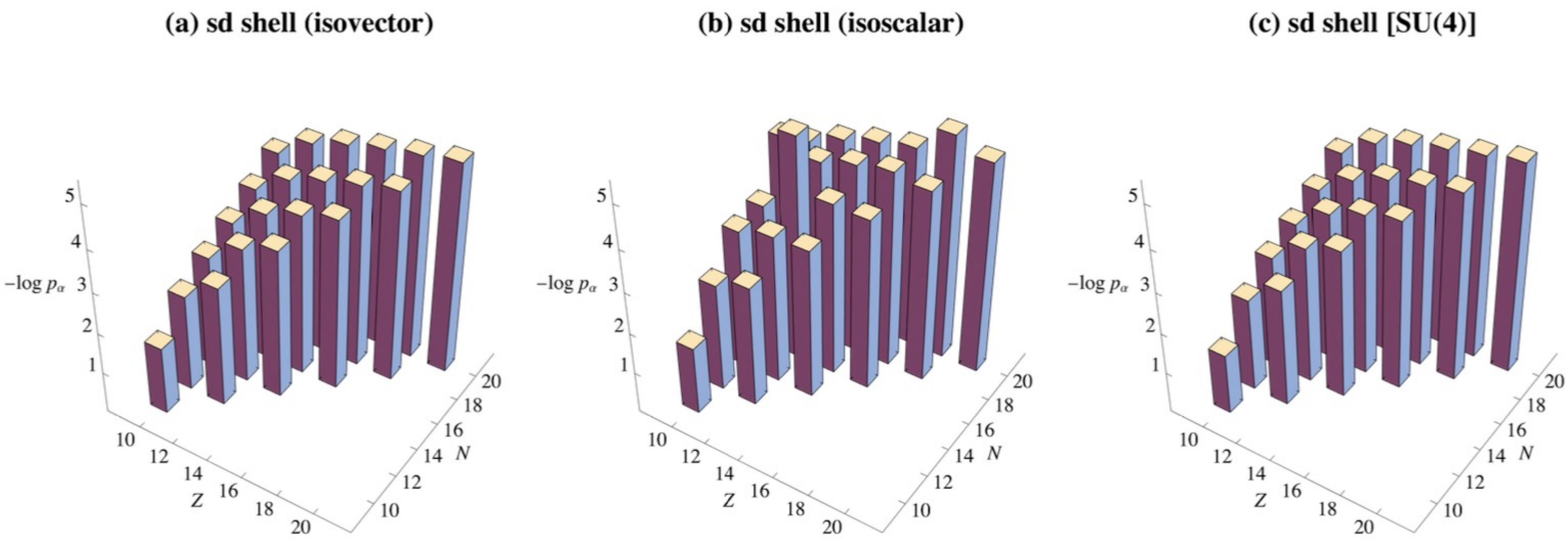
Take $g = 0 \rightarrow LS$-coupled eigenstates.

Example: the *sd* shell

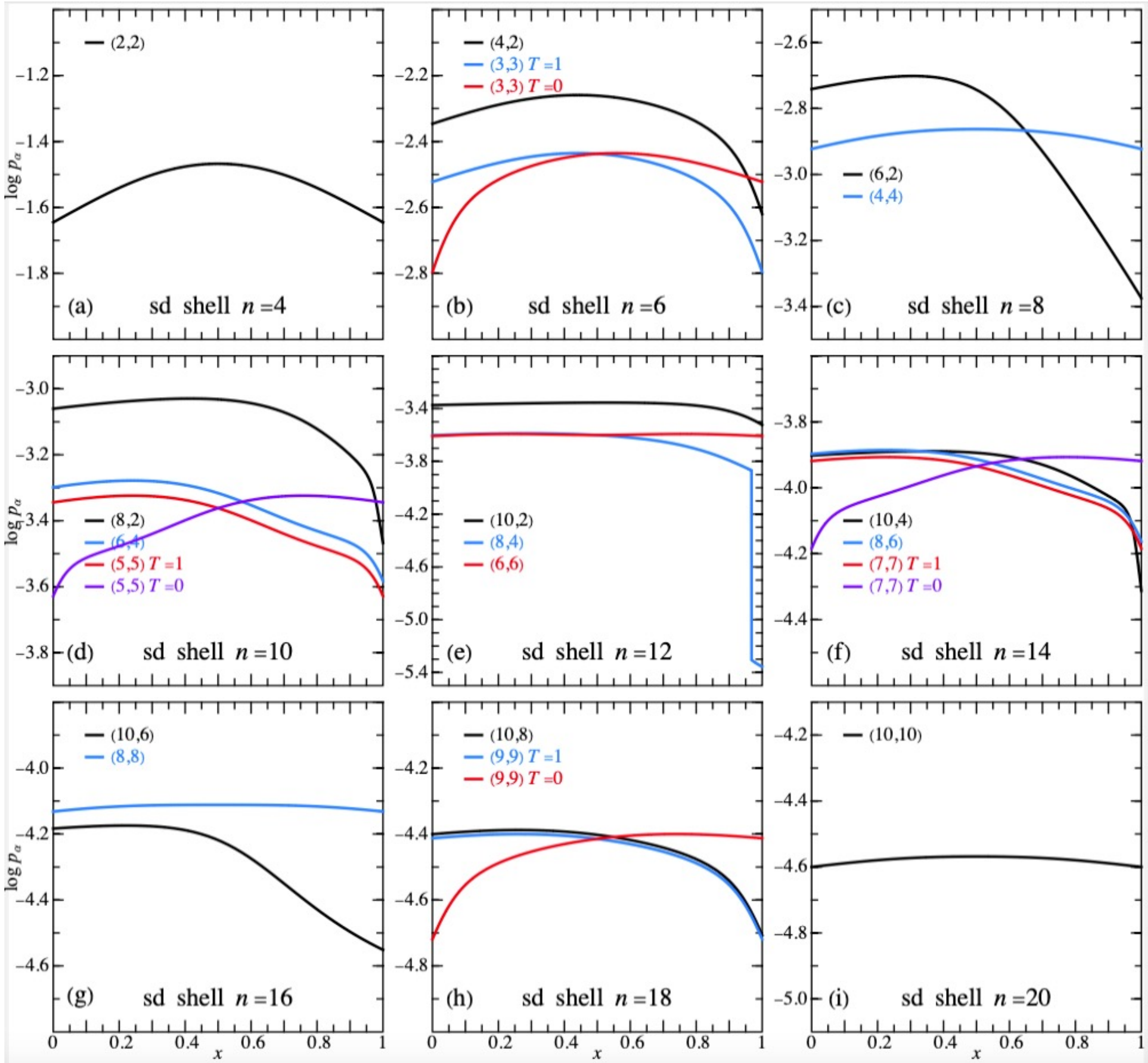
p_α decreases with valence nucleon number.

In ^{20}Ne $p_\alpha = 0.034$ if $a_0 = a_1$ [SU(4)] and $p_\alpha = 0.023$ if $a_0 = 0$ or $a_1 = 0$.

In ^{40}Ca $p_\alpha \sim 10^{-5}$.



Example: the *sd* shell



Cluster states and SU(3)

ON THE CONNECTION BETWEEN THE CLUSTER MODEL AND THE SU_3 COUPLING SCHEME FOR PARTICLES IN A HARMONIC OSCILLATOR POTENTIAL

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Abstract: It is shown that the cluster model of Wildermuth and Kanellopoulos provides an alternative description of certain states in the SU_3 coupling scheme of Elliott.

Example: ^{8}Be

Example: $^{8}_{4}Be_{4}$ $\quad p^4 \quad$ states with $(LS)=(00)$

SU(3) analysis

$$U(12) \supset U(3) \otimes \left(SU(4) \supset SU_S(2) \otimes SU_T(2)\right)$$

$[1111]$ $\quad$ $[4]$ (40), $L = \boxed{0}, 2, 4$ $\quad$ $[1111]$ $\quad (ST) = \boxed{(00)}$

$[31]$ (21), $L = 1, 2, 3$ $\quad$ $[211]$ $\quad (ST) = (01)(10)(11)$

$[22]$ (02), $L = \boxed{0}, 2$ $\quad$ $[22]$ $\quad (ST) = \boxed{(00)}(11)\boxed{(02)}(20)$

Three states have $(LS)=(00)$

$$(\lambda\mu)=(40): \quad \frac{\sqrt{5}}{3}\,|(L_\nu S_\nu)=(L_\pi S_\pi)=(00)\rangle + \frac{2}{3}\,|(L_\nu S_\nu)=(L_\pi S_\pi)=(22)\rangle \qquad T=0$$

$$(\lambda\mu)\ (02): \quad -\frac{2}{3}\,|(L_\nu S_\nu)=(L_\pi L_\pi)=(00)\rangle + \frac{\sqrt{5}}{3}\,|(L_\nu S_\nu)=(L_\pi S_\pi)=(22)\rangle \qquad T=0,2$$

α-particle probability

	$s=0$	$s=1$
$d^{s}_{(00)(00)(00)} =$	$\frac{1}{8}\sqrt{\frac{5}{6}}$	$\frac{1}{8}\sqrt{\frac{5}{2}}$
$d^{s}_{(22)(22)(00)} =$	$\frac{1}{4\sqrt{6}}$	$\frac{1}{4\sqrt{2}}$

$$\Rightarrow \begin{cases} p_\alpha\,[(\lambda\mu)=(40)] = \frac{3}{32} \\ p_\alpha\,[(\lambda\mu)=(02)] = 0 \end{cases}$$