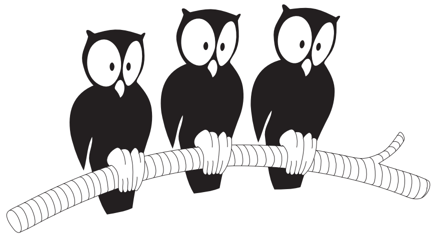
(Bold) Diagrammatic Monte Carlo

Or: Large-order summation of connected Feynman diagrams for strongly correlated fermions

Kris Van Houcke

Ecole Normale Supérieure



CEA, June 5, 2023

With

Félix Werner (LKB-ENS)

Riccardo Rossi (CCQ -> EPFL->SU)

Boris Svistunov & Nikolay Prokof’ev (Umass Amherst)

Takahiro Ohgoe (Tokyo)

Gabriele Spada (ENS -> Trento)

Fedor Simkovic, Renaud Garioud, Michel Ferrero (Collège de France & Polytechnique)

Unbiased approaches for fermionic $\mathcal{N}$-body problems

strongly interacting fermions:
- *electrons: solids, molecules*
- *nucleons: nuclei, neutron stars*
- *QCD*

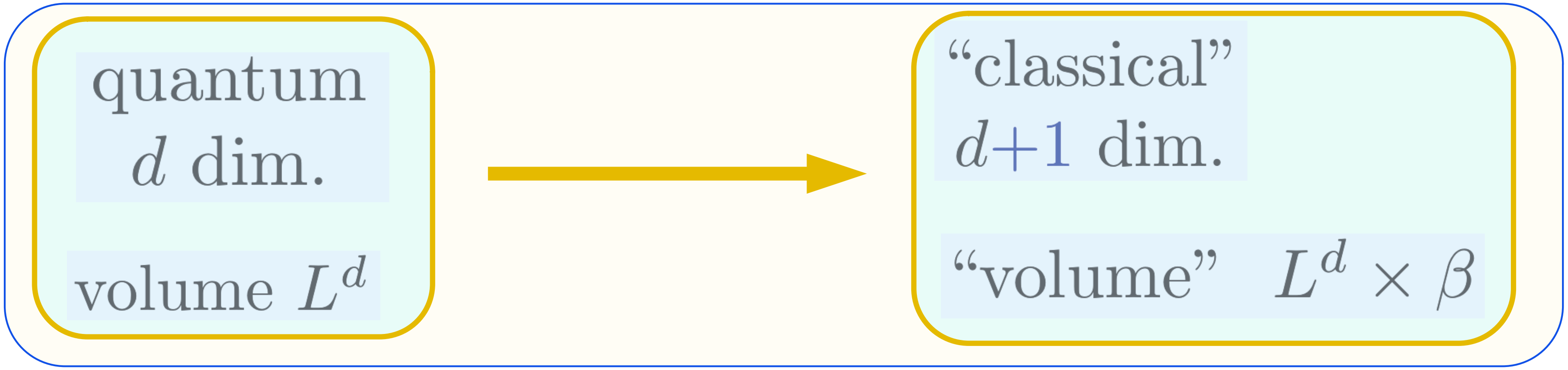
theoretical challenge:

reliable & accurate predictions for large $\mathcal{N}$, including $\mathcal{N} \to \infty$

computing / sampling wavefunction / partition-function: *hard*

Tensor network: 1D:  2D: harder
3D: ? continuous space: ?

"bulk" Quantum Monte Carlo



- path integral $\mathbf{r}_1(\tau), \ldots, \mathbf{r}_\mathcal{N}(\tau)$
- **A**uxiliary **F**ield **QMC** *(lattice QCD)*
- **D**eterminental **D**iagrammatic **MC** *(CT-INT)*

"bulk" Quantum Monte Carlo

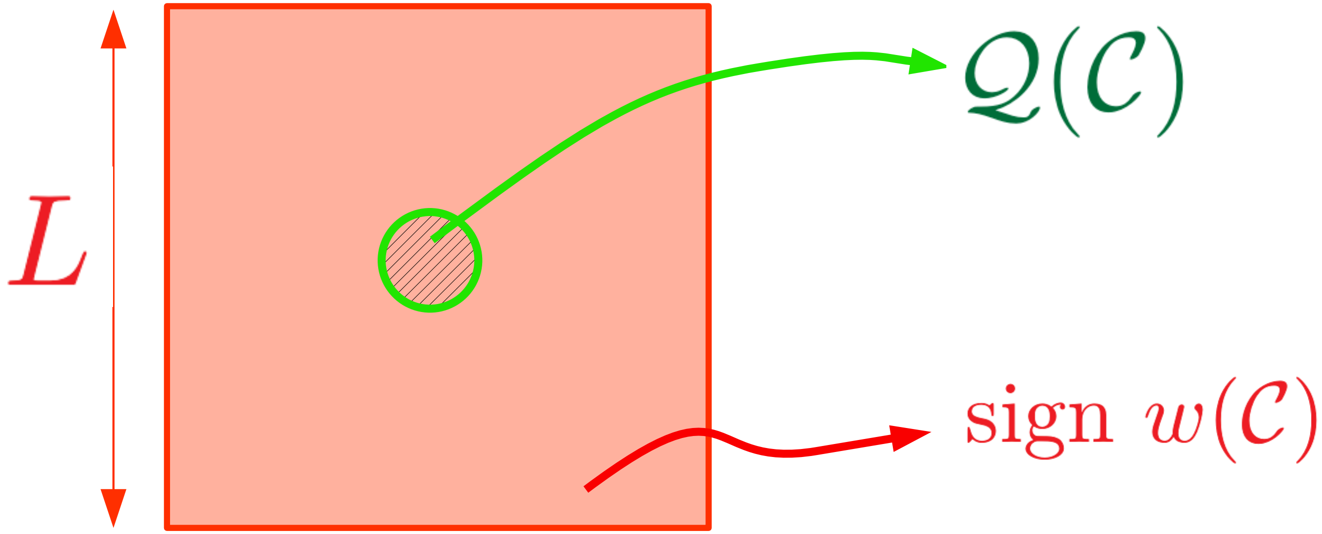
quantum d dim. — volume L^d → "classical" $d+1$ dim. — "volume" $L^d \times \beta$

- path integral $\mathbf{r}_1(\tau), \ldots, \mathbf{r}_{\mathcal{N}}(\tau)$
- **Auxiliary Field QMC** *(lattice QCD)*
- **Determinental Diagrammatic MC** *(CT-INT)*

except in special cases

fermion sign problem: $t_{\mathrm{CPU}} \sim e^{\#\beta\mathcal{N}}$

$$Q = \frac{\sum_{\mathcal{C}} \mathcal{Q}(\mathcal{C})\, w(\mathcal{C})}{\sum_{\mathcal{C}} w(\mathcal{C})}$$



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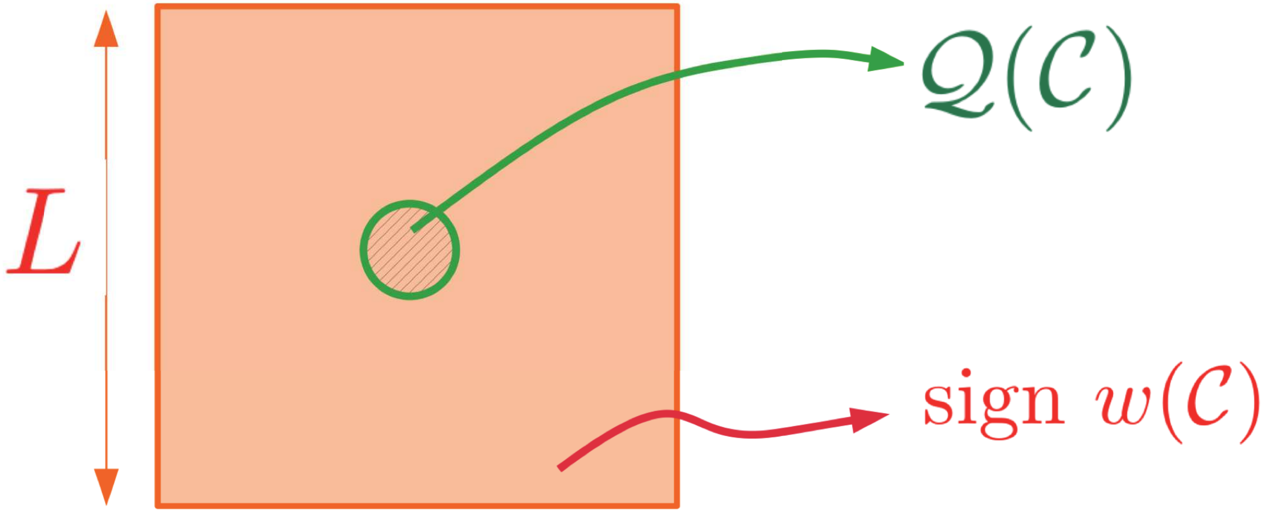

BUT: physics is local

Intensive Q are well-defined for $L = \infty$

Diagrammatic MC with ***connected diagrams***

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sum all connected Feynman diagrams of order $N \leq N_{\mathrm{max}}$

truncation error $\epsilon_{\mathrm{sys}} \underset{N_{\mathrm{max}} \to \infty}{\longrightarrow} 0$

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strong cancellations between diagrams
$\Rightarrow$ ***large-order contributions reduced***
lattice: series often converge

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flexibility **of diagrammatic technique**:

- change the *starting point* (order zero)
- reorganize expansion (use *dressed* propagators / vertices)

$\Rightarrow$ *non-perturbative*

$\Rightarrow$ ***low orders already good approx.***

Diagrammatic MC with ***connected diagrams***

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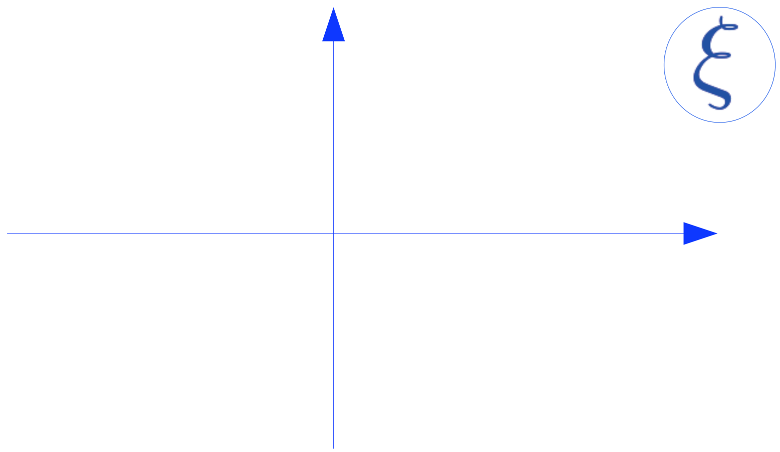

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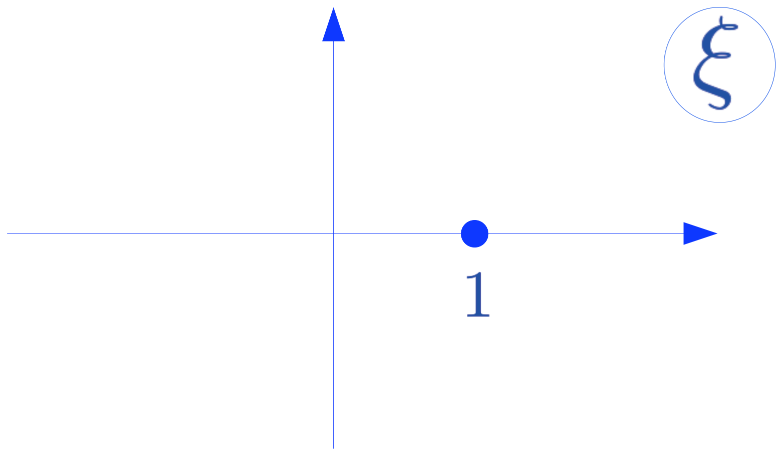

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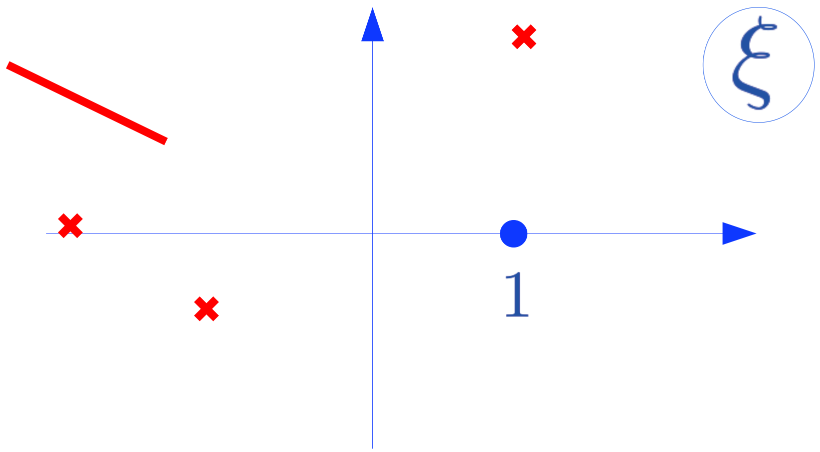

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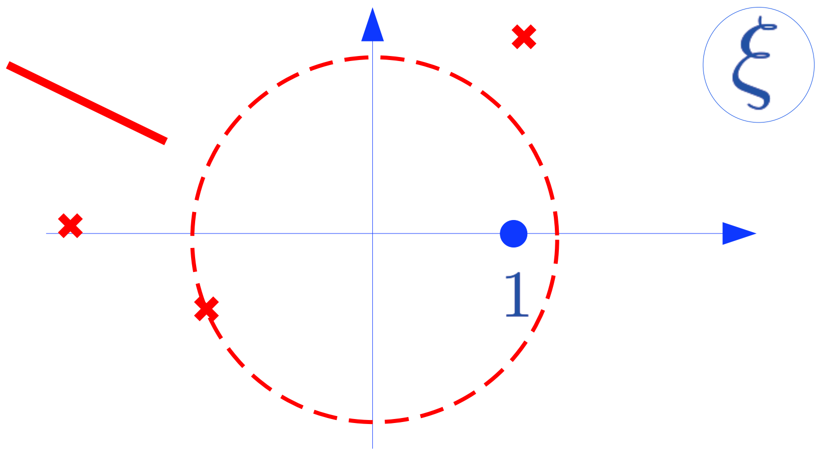

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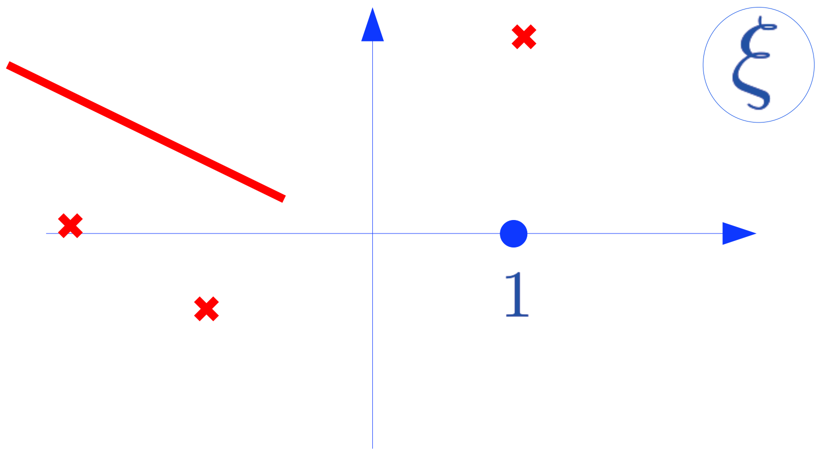

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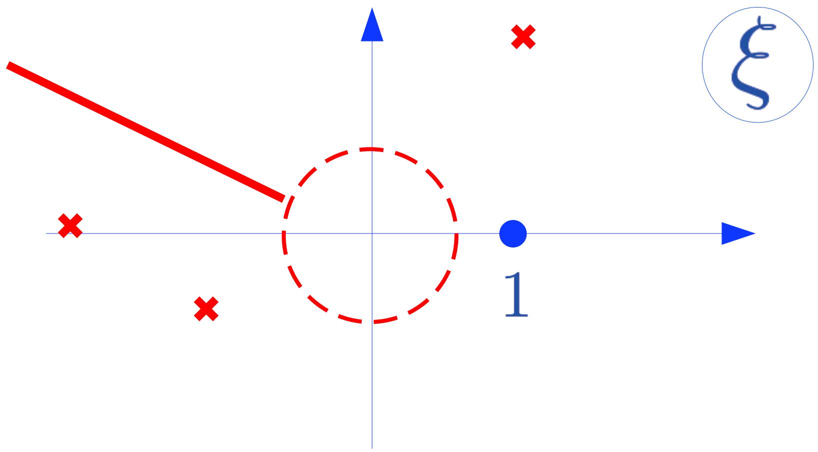

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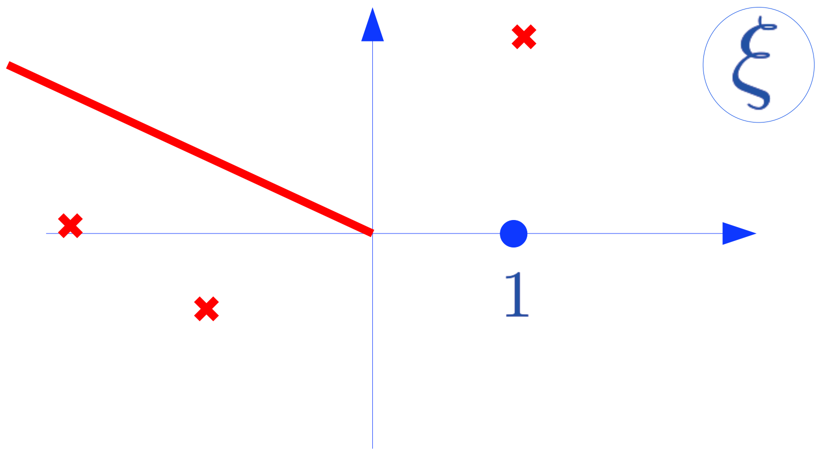

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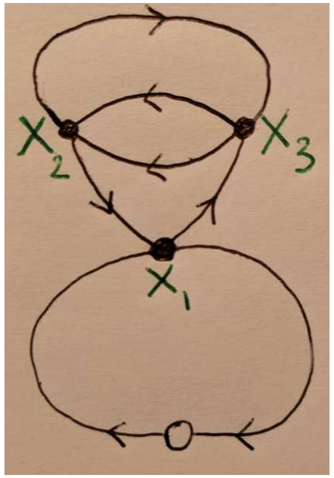
sum of all order-N ***connected*** *Feynman diag.*

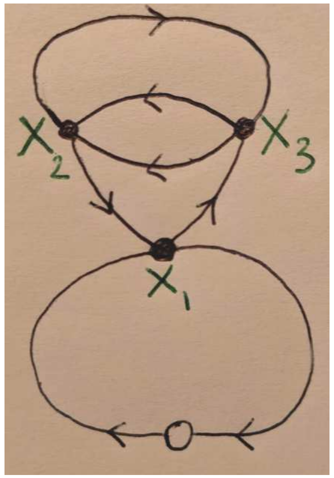


$$Q_N = \sum_{\substack{\text{connected} \\ \text{topologies } \mathcal{T}}} \int dX_1 \ldots dX_N \, \underbrace{\mathcal{D}(\mathcal{T}; X_1 \ldots X_N)}_{\xrightarrow[|X_i| \to \infty]{} 0}$$

$$X := (\vec{r}, \tau)$$

$$\int dX := \sum_{\vec{r}} \int_0^\beta d\tau$$

$\Longrightarrow$ **well-defined in thermo. limit $\mathcal{N} = \infty$**



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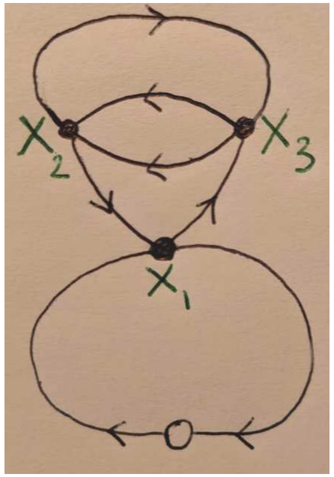

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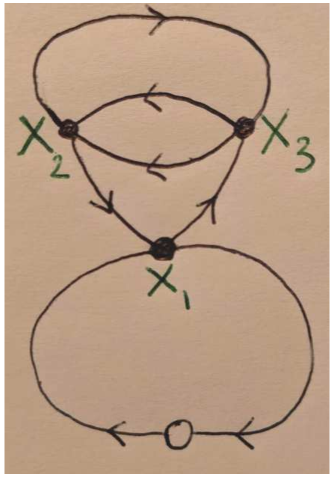

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Monte Carlo algorithms

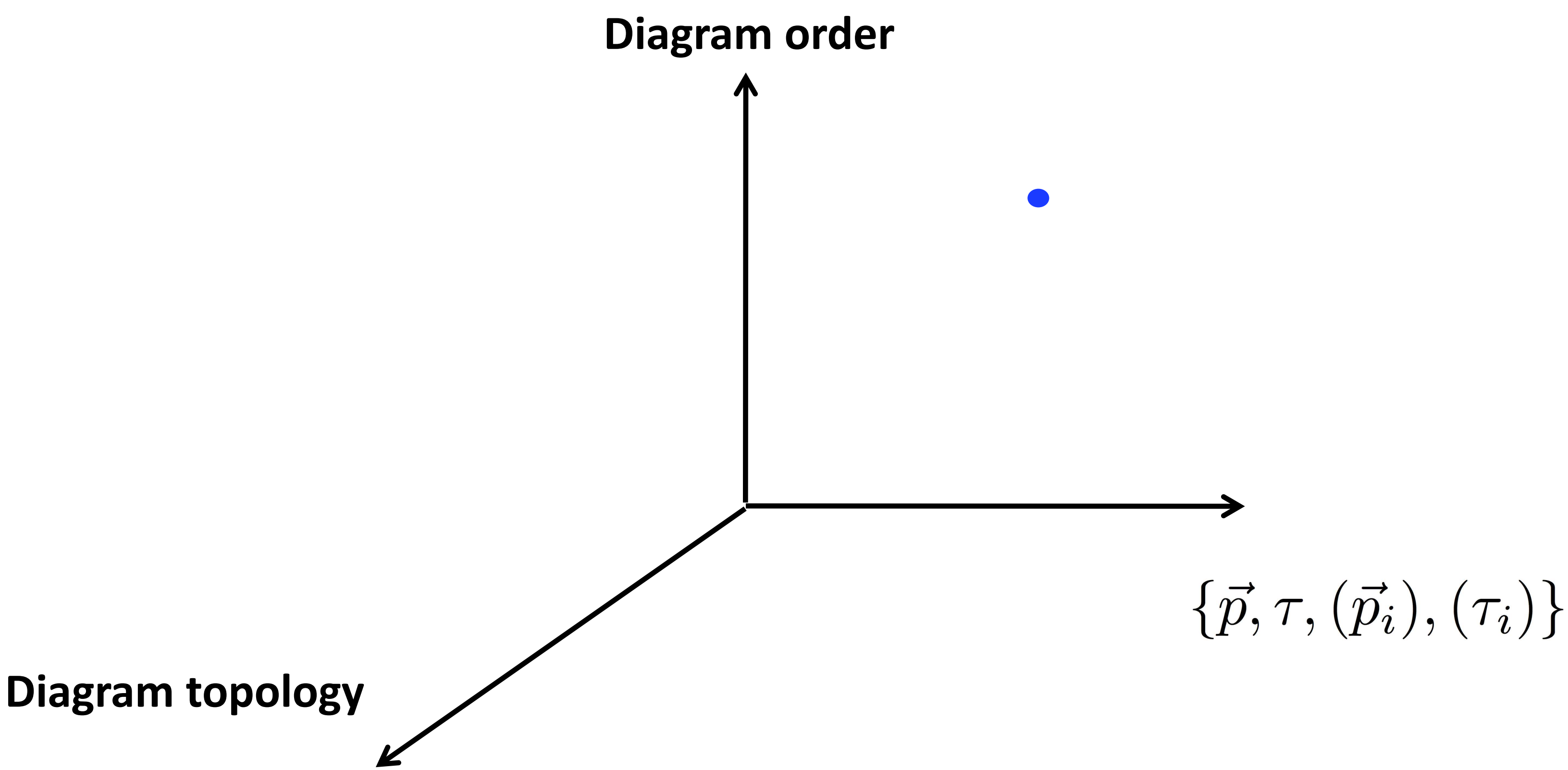
- DiagMC [Van Houcke *et al.* 2010]

  configuration: $\mathcal{C} = (\mathcal{T}, X_1, \dots, X_N)$ probability: $P(\mathcal{C}) \propto |\mathcal{D}(\mathcal{T}; X_1 \dots X_N)|$

- CDet [Rossi 2017, Rossi *et al.* 2020]

  $\mathcal{C} = (X_1, \dots, X_N)$ $P(\mathcal{C}) \propto \left| \sum_{\mathcal{T}} \mathcal{D}(\mathcal{T}; X_1 \dots X_N) \right|$

DiagMC



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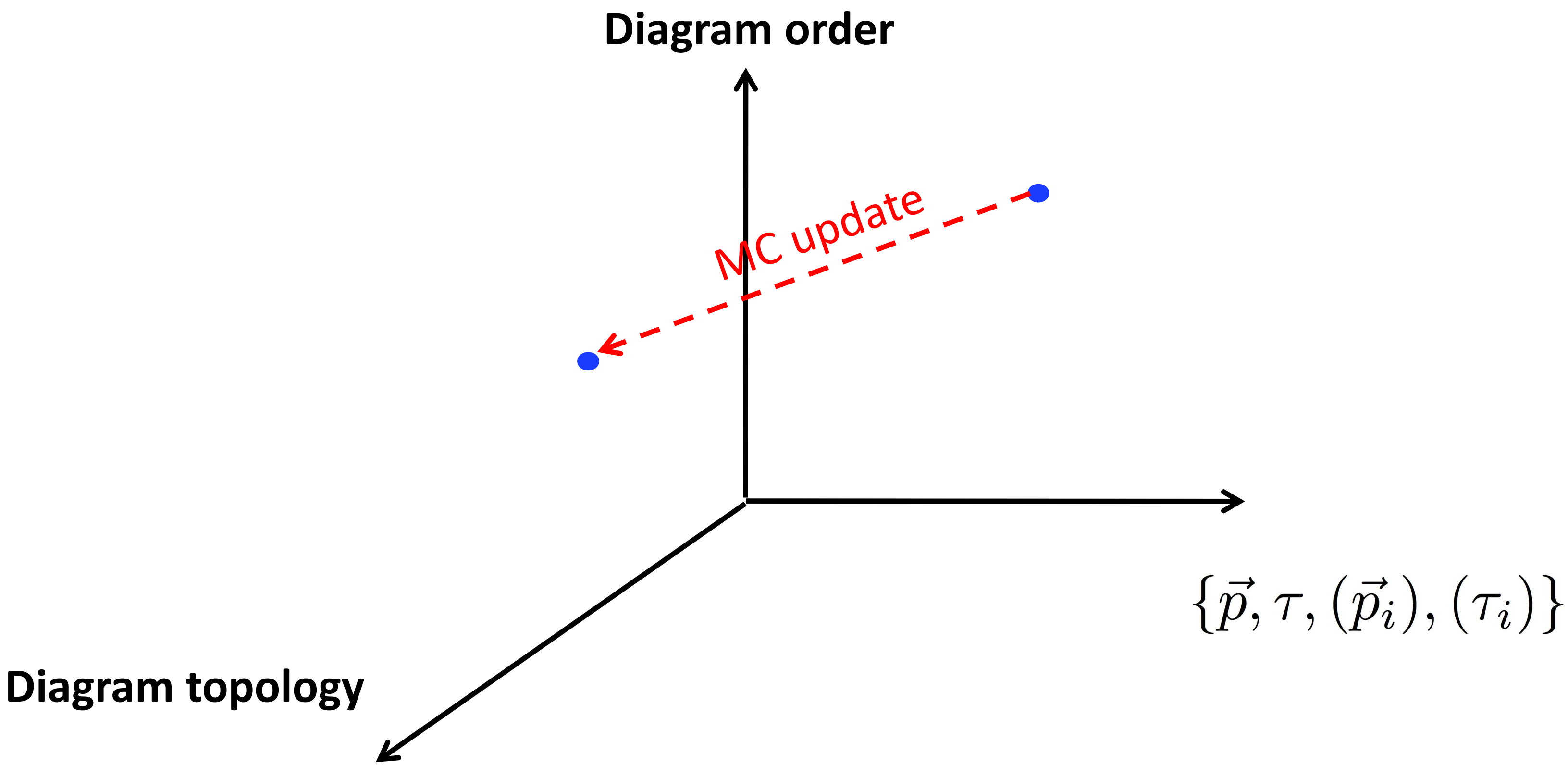
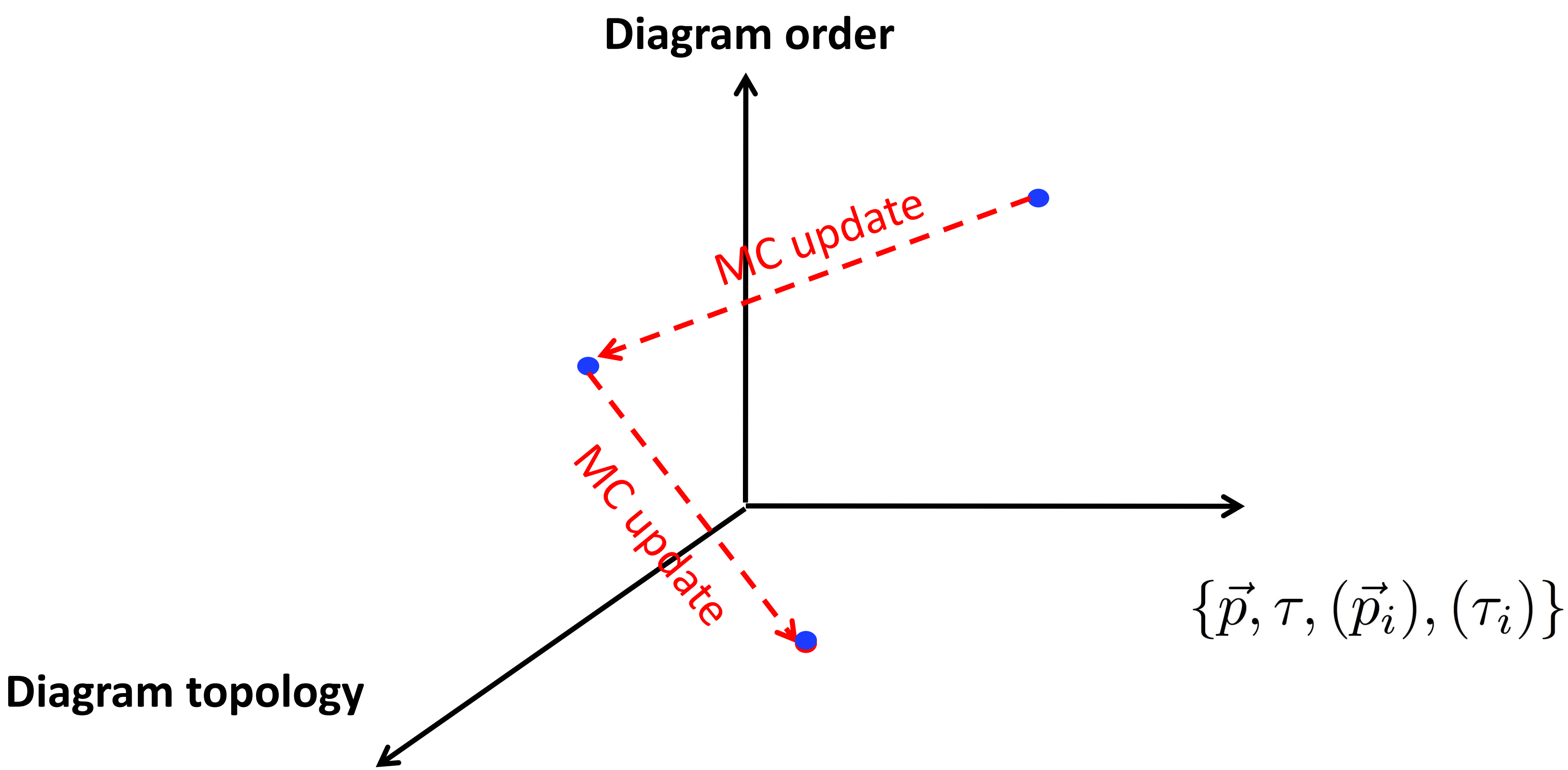


Diagram order
MC update
$\{\vec{p}, \tau, (\vec{p}_i), (\tau_i)\}$
Diagram topology

DiagMC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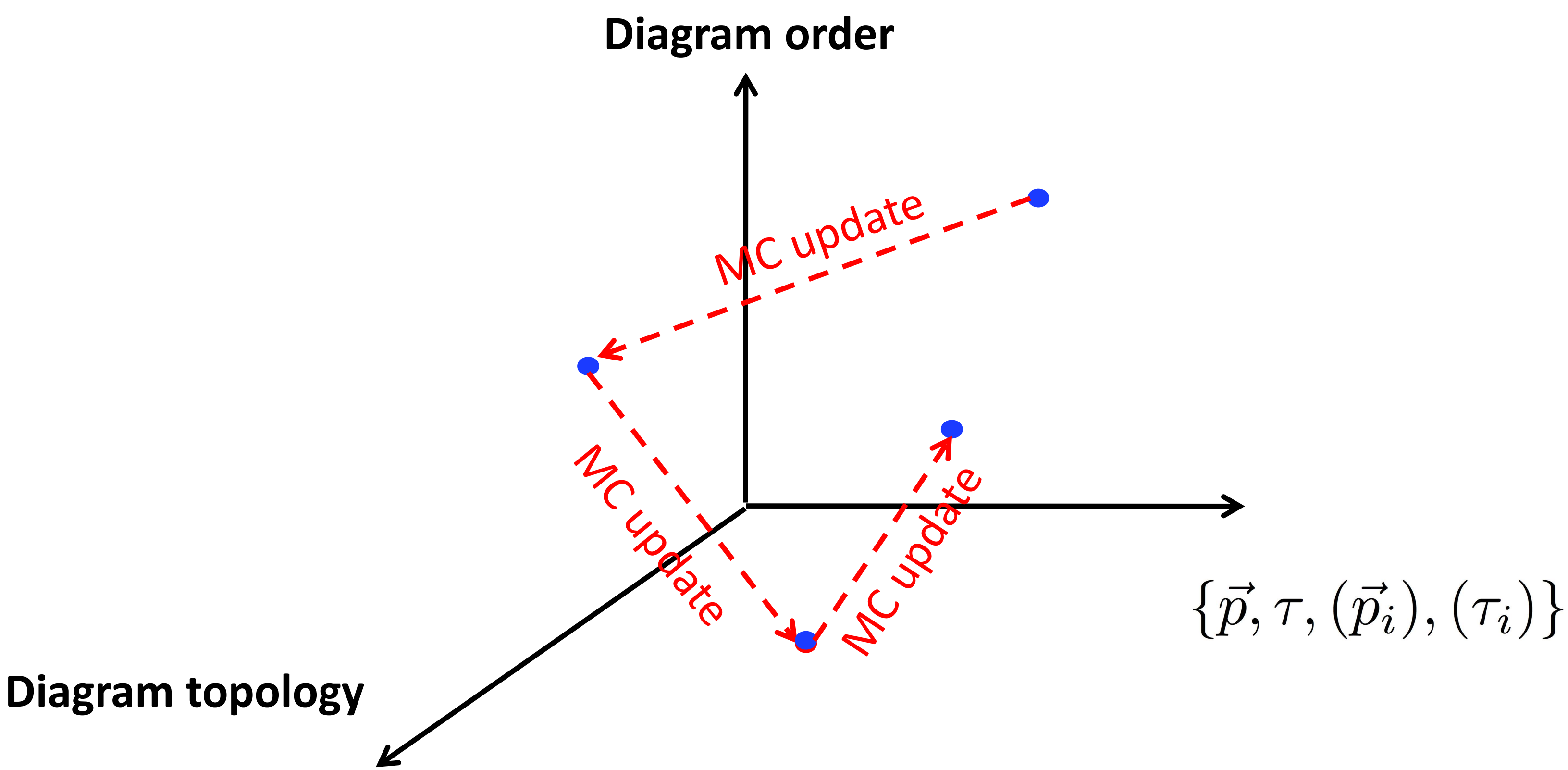

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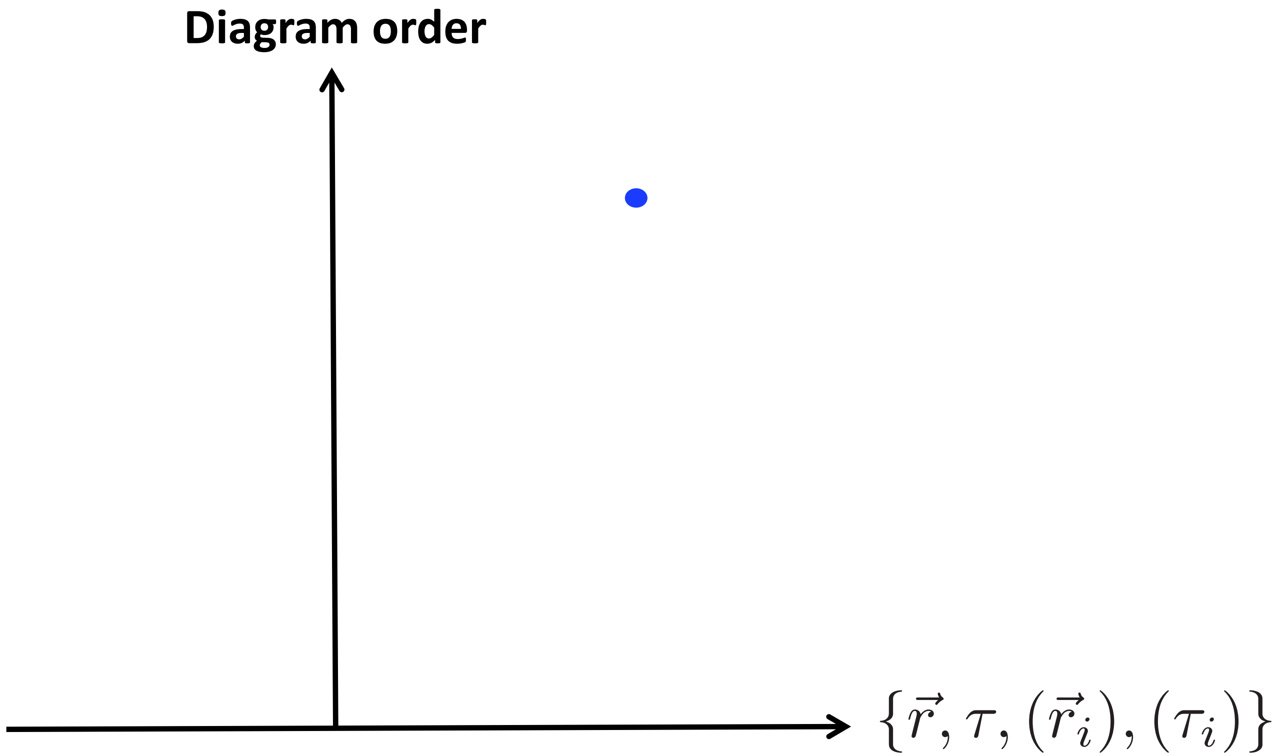

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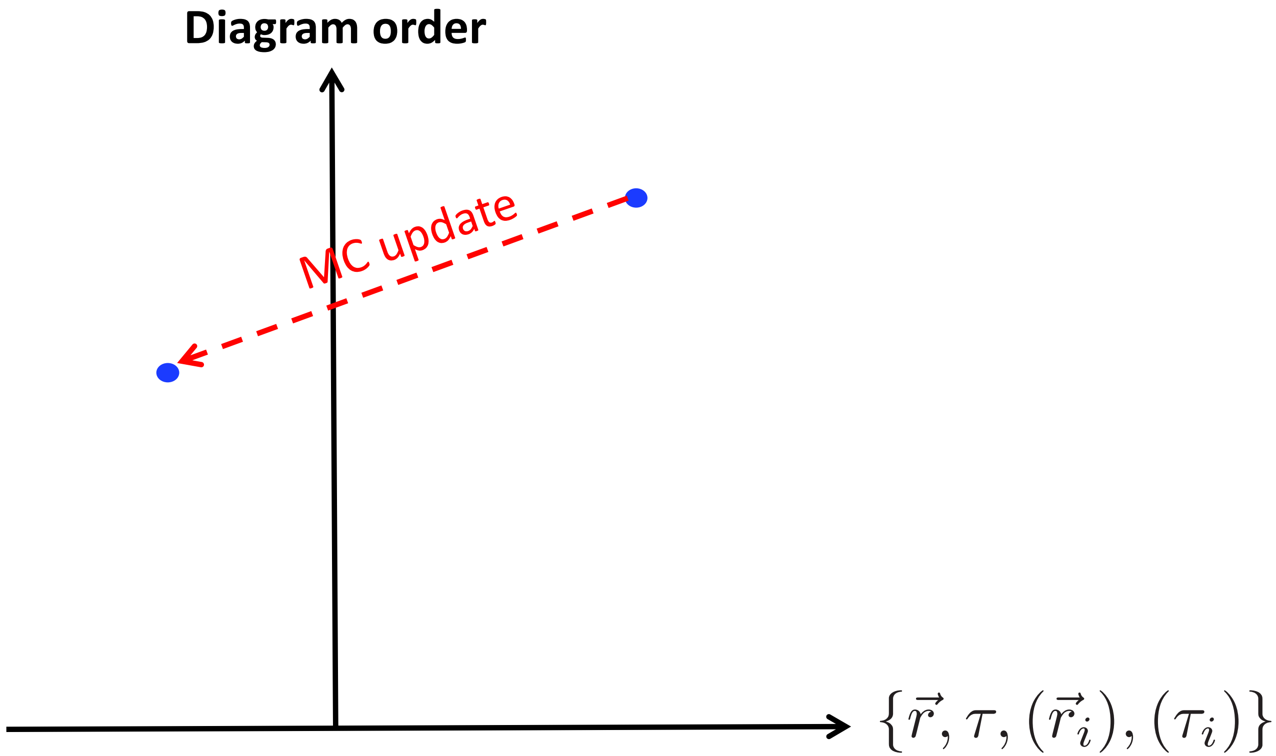

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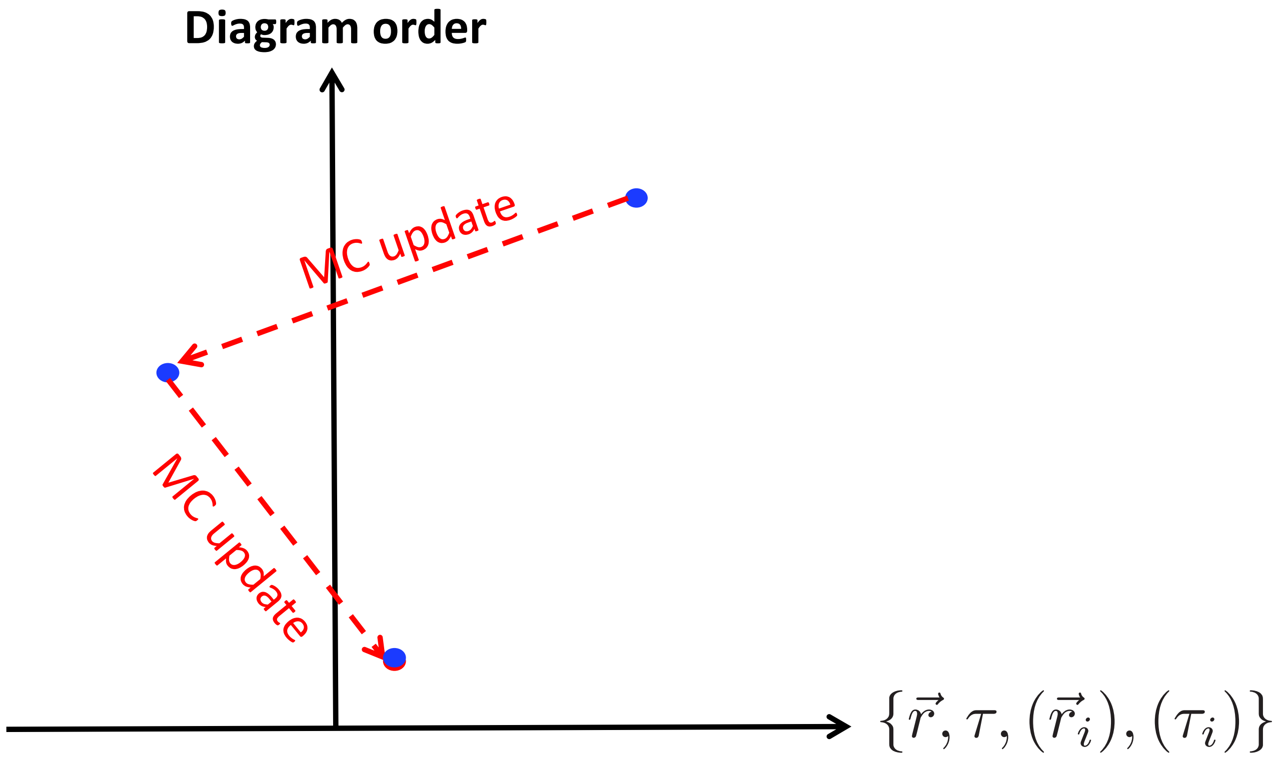

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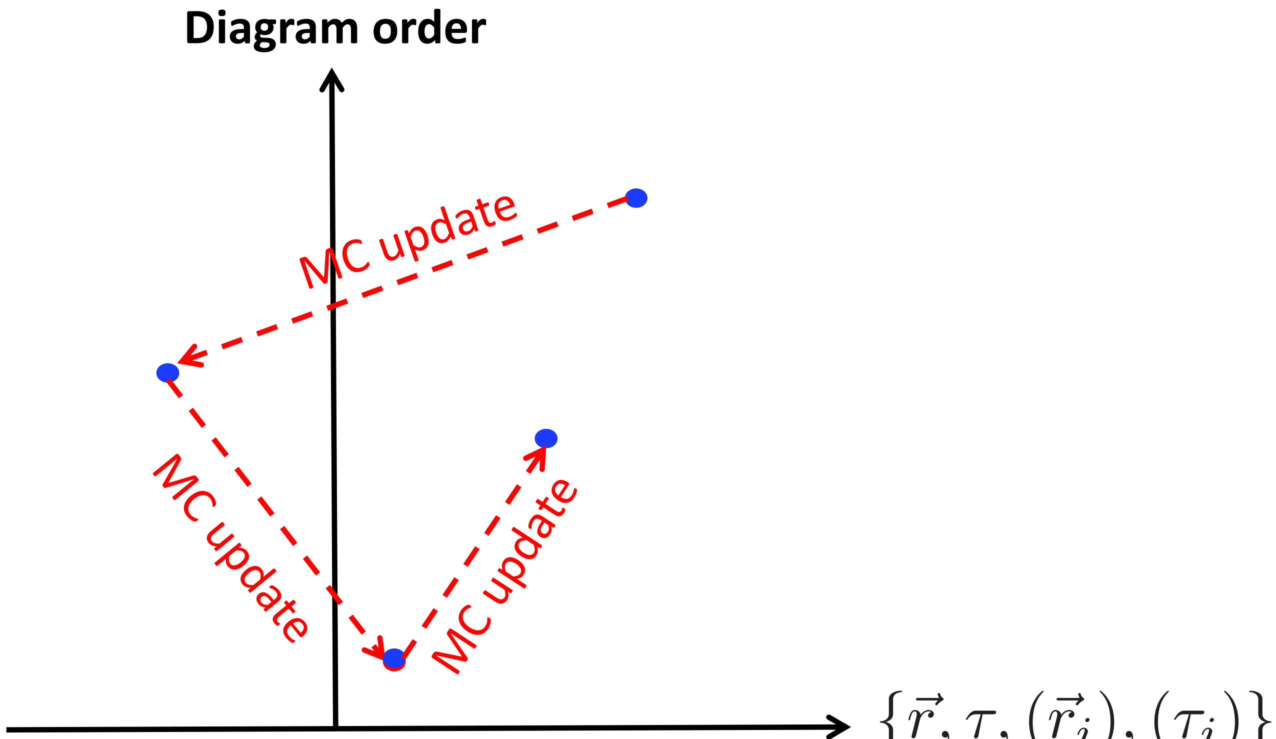
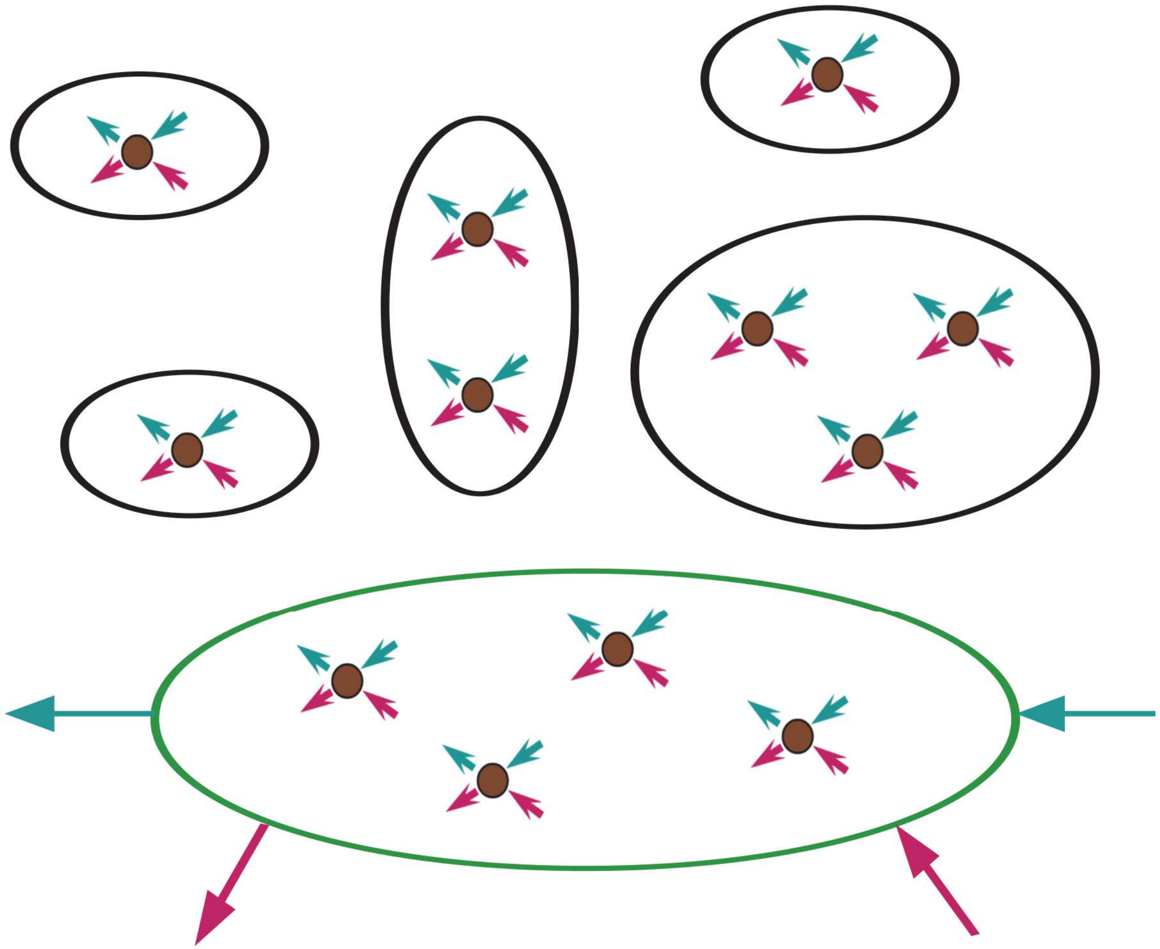




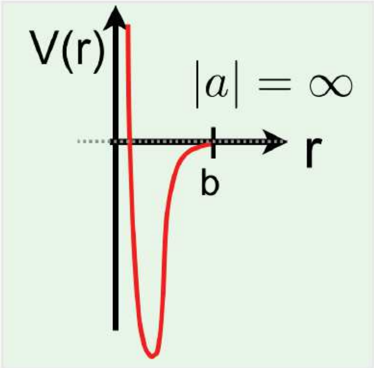
$$c_E(V) = a_E(V) - \sum_{S \subsetneq V} c_E(S) a_{\emptyset}(V \setminus S)$$

Unitary Fermi gas

Unitary Fermi gas

Spin-$\frac{1}{2}$ fermions, 3D **continuous** space, interactions $\left\{ \begin{array}{l} \text{zero range} \\ \text{scattering length} \quad a = \infty \end{array} \right.$

Universality hypothesis:



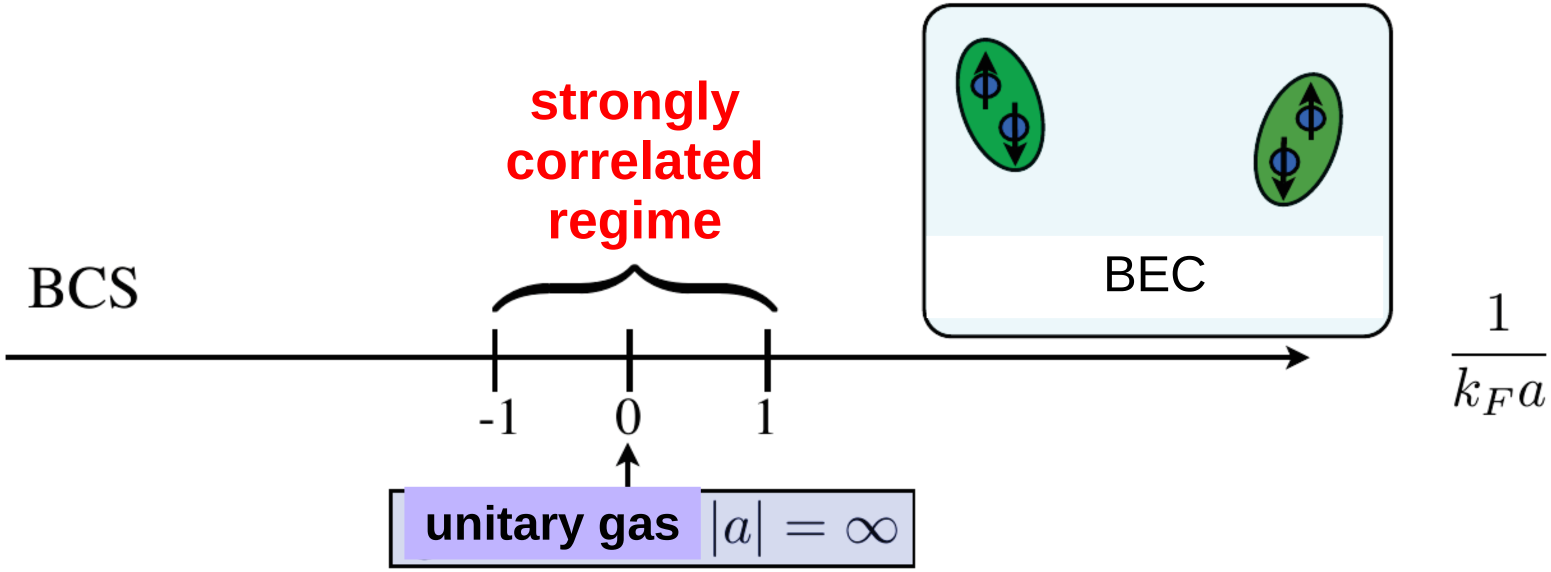
Zero-range limit: $\left\{ \begin{array}{r} n^{-1/3} \gg b \\ \lambda \gg b \end{array} \right.$ $\qquad \lambda \equiv \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$

$\Longrightarrow$ Properties do not dependent on $V(r)$

$(N_\uparrow = N_\downarrow)$ $\boxed{n(T,\mu)\lambda^3 = \text{universal function of } \beta\mu}$

Construction from Hubbard model:

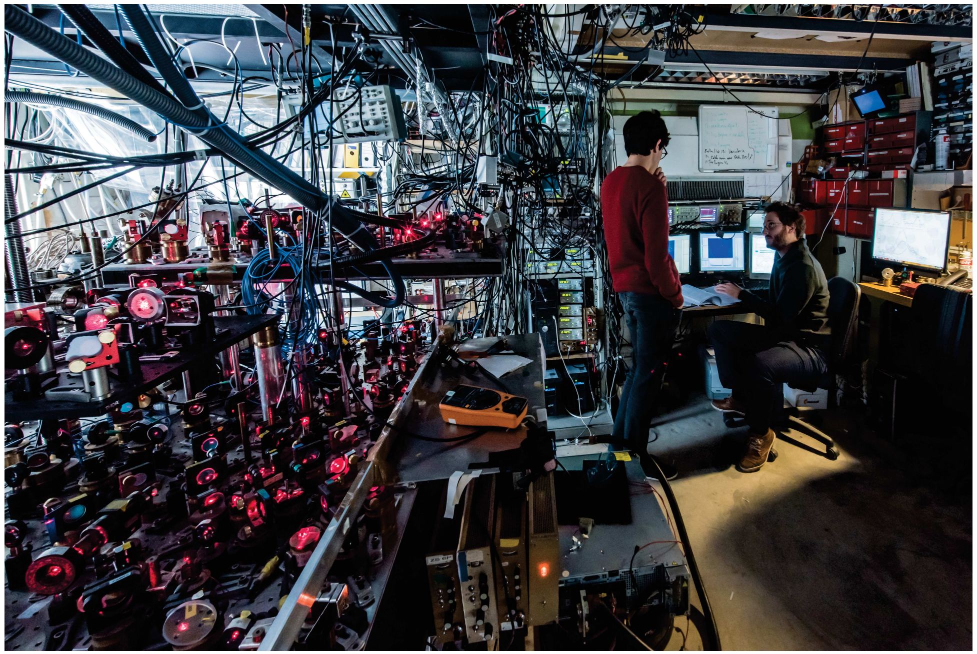
- $\frac{U}{t} = -7.913552\ldots$ (appearance of 2-body bound state)
- thermodynamic limit
- filling $\to 0$ with $\frac{T}{T_F}$ fixed (= continuum limit)



cold atom experiments
fermionic atom, 2 internal states, Feshbach resonance

accurate comparison

theory
zero-range



also relevant for neutron stars

widely studied

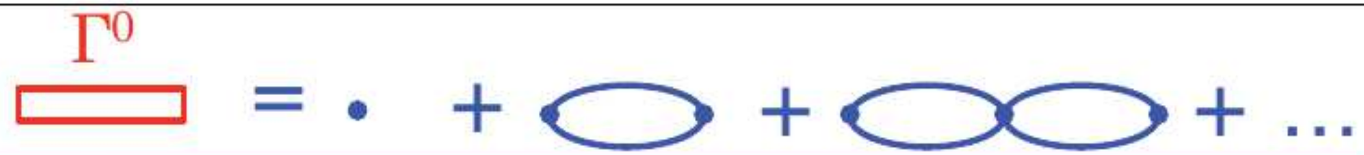
Experiments:

Jin, Thomas, Salomon, Chevy, Grimm, Ketterle, Zwierlein, Vale, Roati, Zaccanti, Esslinger, Brantut, Köhl, Sagi, Hecker Denschlag, Chen&Pan …

Theory:

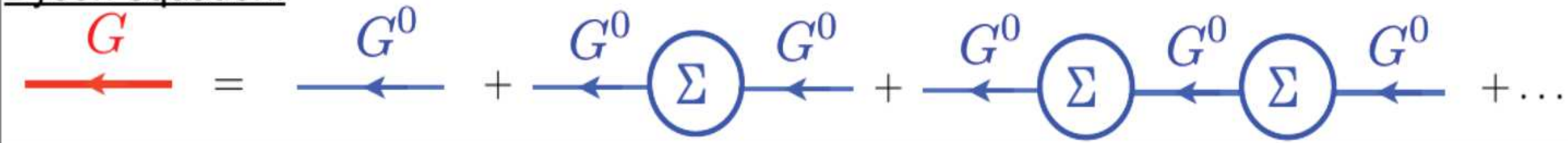
Leggett, Haussmann, Zwerger, Randeria, Sa de Melo, Strinati, Pieri, Giorgini, Stringari, Combescot, Leyronas, Shlyapnikov, Petrov, Levin, Son, Nishida, Hu, Liu, Bulgac, Drut, Kaplan, Gezerlis, Carlson, Gandolfi, Tan, Urban, Forbes, Alhassid, Ohashi, Castin, Chevy, Enss, Hofmann, Radzihovsky, Sheehy, Parish, Levinsen, Bruun, Massignan ……..

Ladder summation:



$\Rightarrow \Gamma^0$ is well-defined in the continuum limit, which can be taken analytically

Dyson equation:

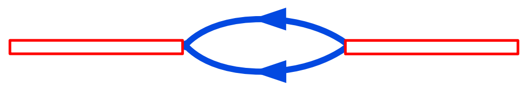


Self-energy:

sum all diagrams up to order ~9
by Diag MC

(“***ladder* scheme**”)

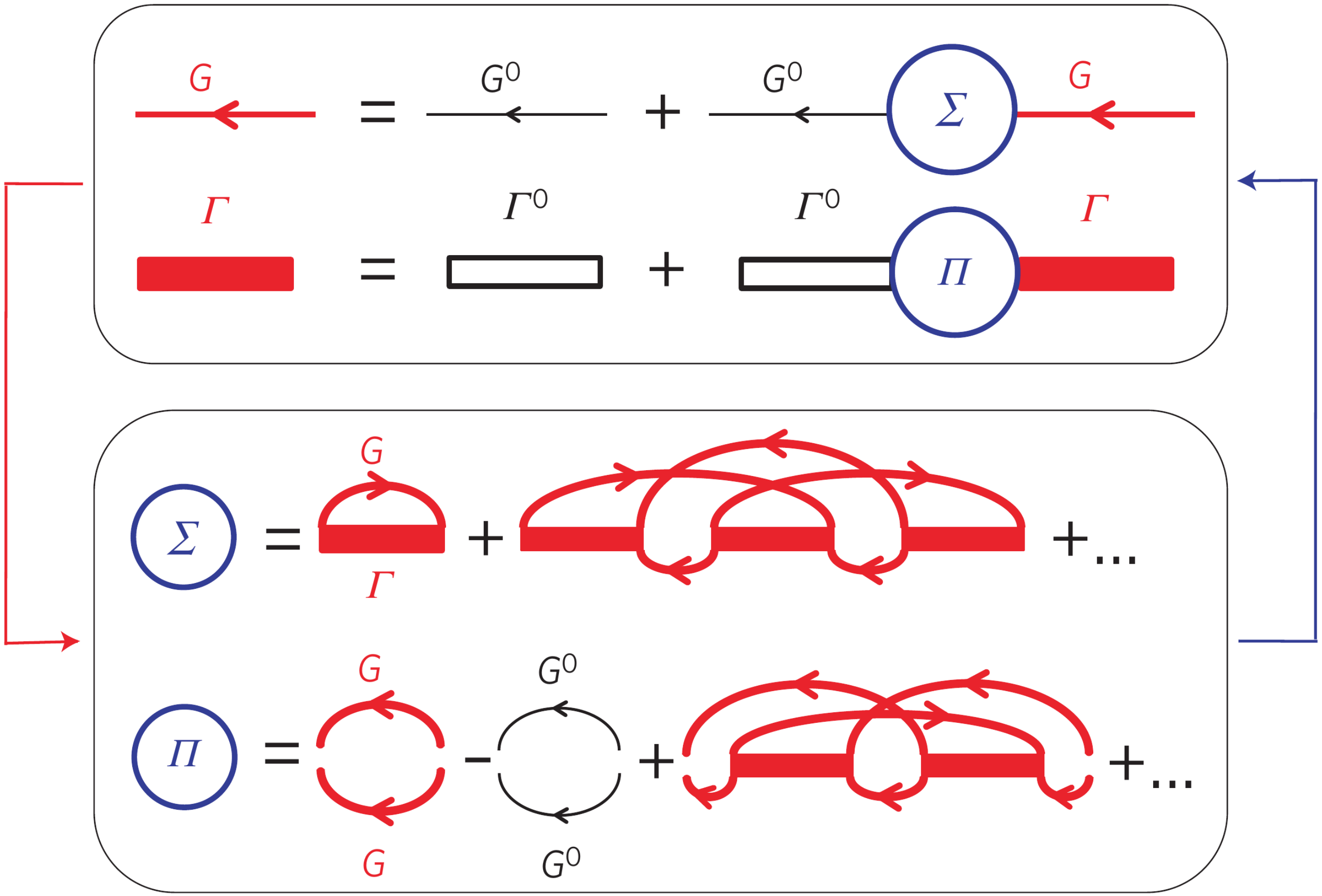
avoid double-counting:



forbidden

Bold scheme

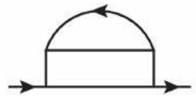
self-consistent

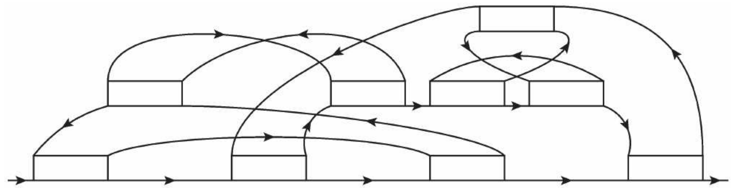


$$Q = \Sigma \text{ or } \Pi$$

$$Q = \sum_{n=0}^{\infty} a_n$$

sum of all order-n diagrams

$a_1 =$ 

$a_9 =$  $+ \dots$

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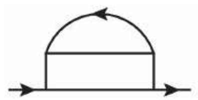
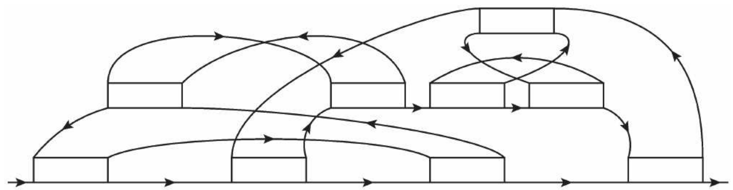
$a_1 =$ 

$a_9 =$  $+ \dots$

Problem: *zero convergence radius*

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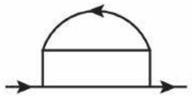
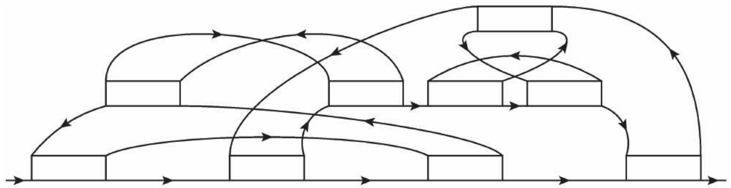
sum of all order-n diagrams

$a_1 =$ 

$a_9 =$  $+ \ldots$

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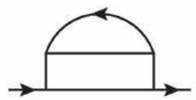
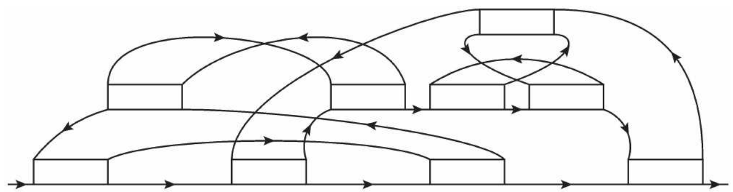
$a_1 =$ 

$a_9 =$  $+ \dots$

Problem: *zero convergence radius*

Solution:

$$z_{\text{here}} \equiv \xi_{\text{before}}$$

construct $Q(z)$ / $\begin{cases} \underset{z \to 0+}{\text{Taylor}} [Q(z)] \hat{=} \sum_{n=0}^{\infty} a_n z^n \\ Q(z=1) = Q_{\text{phys}} \end{cases}$

$$\{a_n\} \xrightarrow[\text{resummation}]{} Q(1)$$

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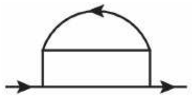
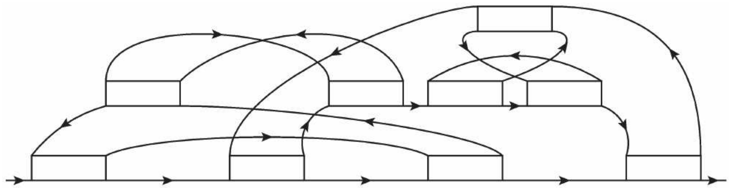
$a_1 =$ 

$a_9 =$  $+ \dots$

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$$\{a_n\} \xrightarrow[\text{resummation}]{} Q(1)$$

- instanton method ⟶ large-order behavior, branch cuts of Q(z)
- Conformal Borel (Nevanlinna theorem)

$$Q(z) \longleftarrow Z(z) = \int \mathcal{D}\eta \underbrace{\int \mathcal{D}\varphi \; e^{-S^{(z)}[\eta,\varphi]}}_{e^{-S_B^{(z)}[\eta]}}$$

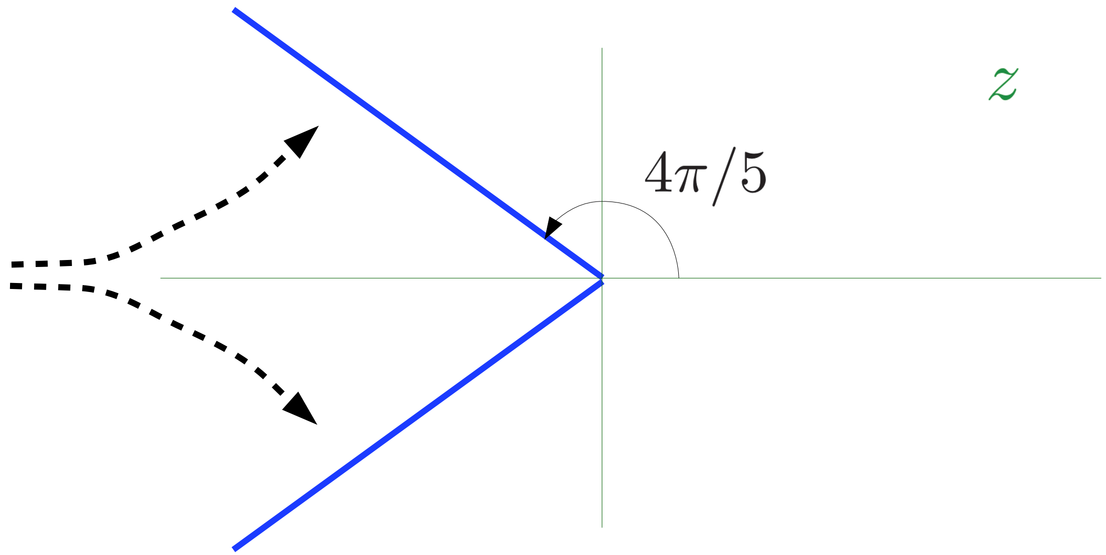
ladder scheme:

$$S^{(z)} = -\int d^3r \int_0^\beta d\tau \Big(\sum_{\sigma=\uparrow,\downarrow} \bar{\varphi}_\sigma G_{0,\sigma}^{-1} \varphi_\sigma + \bar{\eta}\Gamma_0^{-1}\eta - z\bar{\eta}\Pi_0\eta + \sqrt{z}(\bar{\eta}\varphi_\downarrow\varphi_\uparrow + \bar{\varphi}_\uparrow\bar{\varphi}_\downarrow\eta) \Big)$$

quasi-local approximation for $|\eta| \to \infty, \; z \to 0$

$$\frac{\delta S_B^{(z)}[\eta]}{\delta\eta} = 0$$ instanton

$$\text{Disc } Q(z) \underset{|z|\to 0}{\sim} \exp\left[-\left(\frac{A}{|z|}\right)^5 \right]$$



$$a_N \underset{N\to\infty}{\sim} (N/5)! \; A^{-N} \; \cos\left(\frac{4\pi}{5}N\right)$$

Conformal Borel transformation

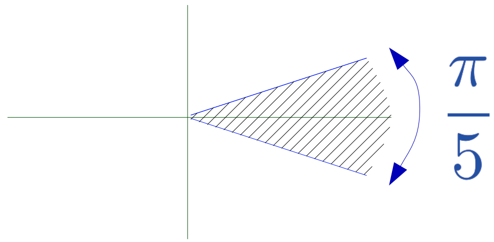
$$a_N \underset{N\to\infty}{\sim} (N/5)!\; A^{-N} \cos\left(\frac{4\pi}{5} N\right)$$

Borel transform : $\quad B(z) := \sum_{N=0}^{\infty} \frac{a_N}{(N/5)!} z^N \qquad |z| < A$

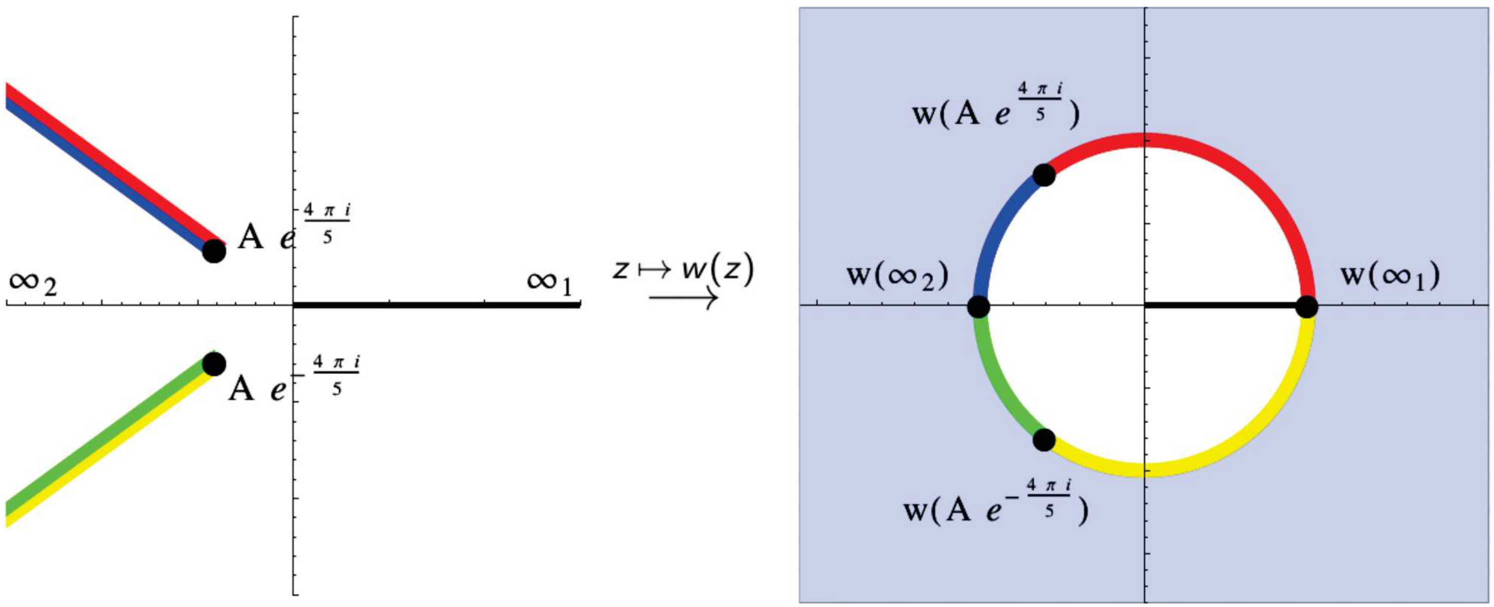
inverse Borel transform : $\quad Q(1) \overset{?}{=} \int_0^{\infty} dz\, z^4\, e^{-z^5} B(z)$

Yes, because [Nevanlinna theorem, 1919]:

- $Q(z)$ analytic in
- $\frac{1}{N!} \left| \frac{d^N Q(z)}{dz^N} \right| \lesssim (N/5)!$ in



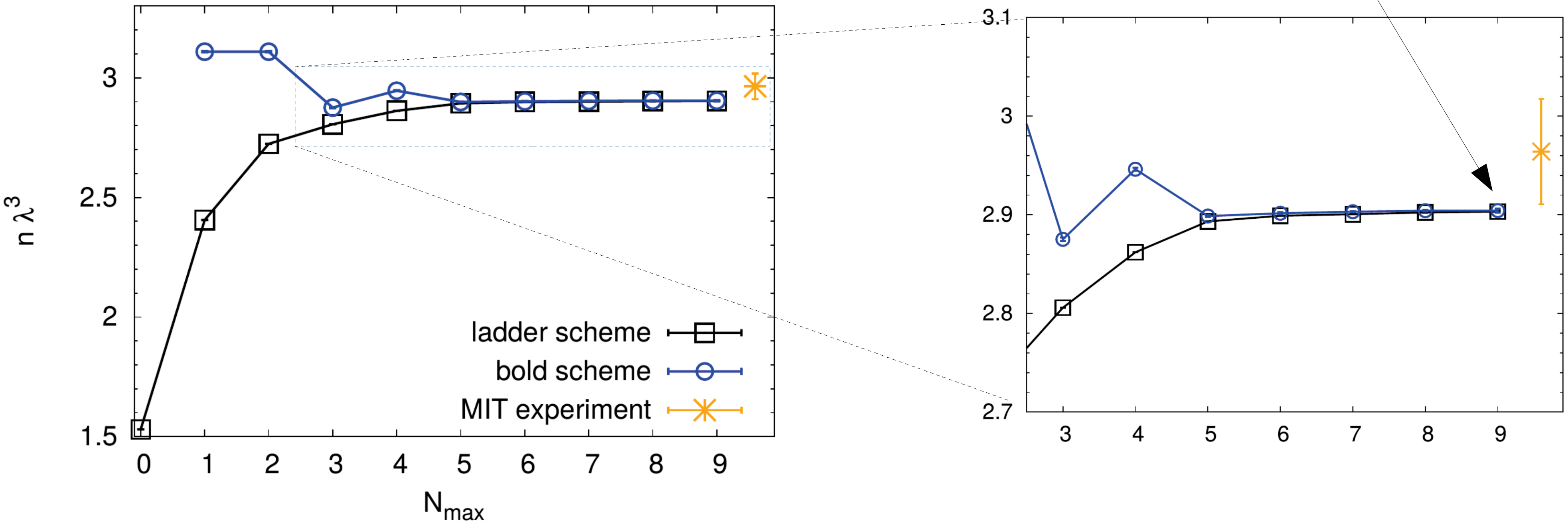
conformal mapping
$\int_0^{\infty} dz = \int_0^1 dw$



Equation of state

$$\mu = 0 \qquad \left(\frac{T}{T_F} \approx 0.6\right)$$

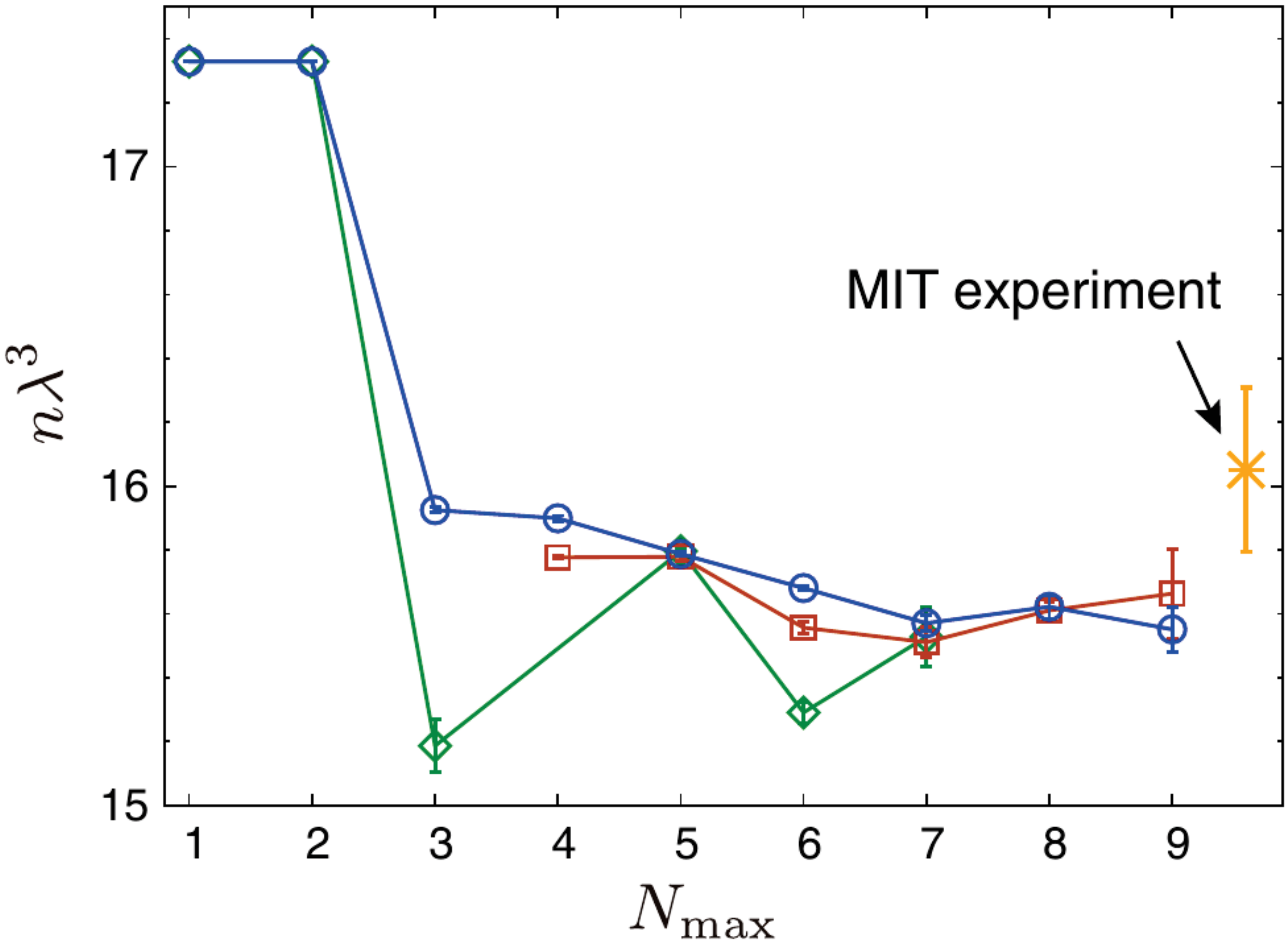
precision < 0.1%



Equation of state

$\beta\mu = 2 \qquad (T/T_F \approx 0.2)$

Bold scheme



Contact parameter $\mathcal{C}$

$$\langle \hat{n}_\uparrow(\mathbf{r})\, \hat{n}_\downarrow(\mathbf{0}) \rangle \underset{r\to 0}{\sim} \frac{\mathcal{C}}{(4\pi\, r)^2}$$

Measure all particle positions, in a unit volume.
Number of pairs of separation $< \epsilon$

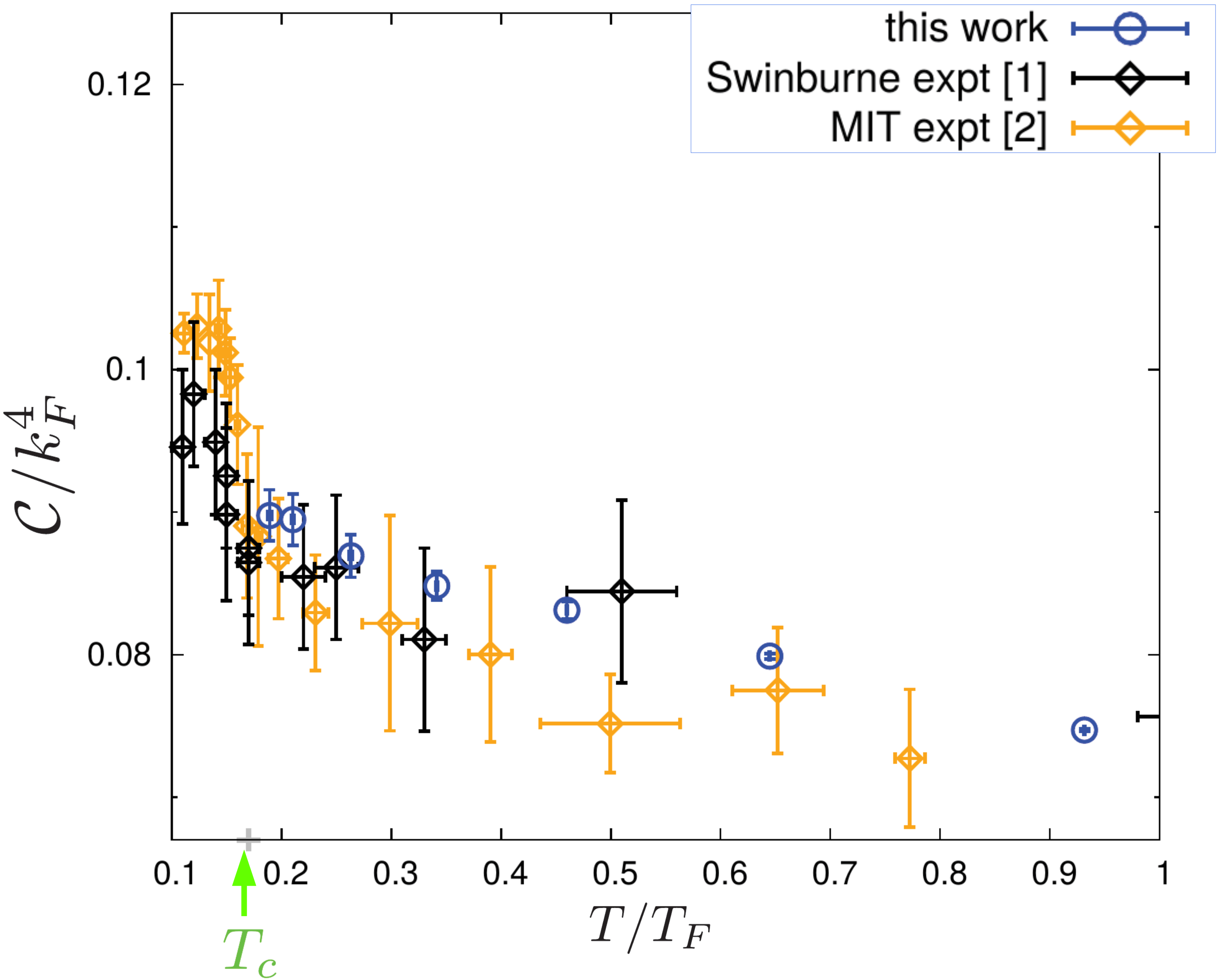
$$\underset{\epsilon\to 0}{\sim} \mathcal{C}\,\epsilon\,\frac{1}{4\pi}$$

$$n_\sigma(\mathbf{k}) \underset{k\to\infty}{\sim} \frac{\mathcal{C}}{k^4}$$

$$\mathcal{C} = \frac{4\pi m}{\hbar^2} \left.\frac{\partial p}{\partial(1/a)}\right|_{T,\mu}$$

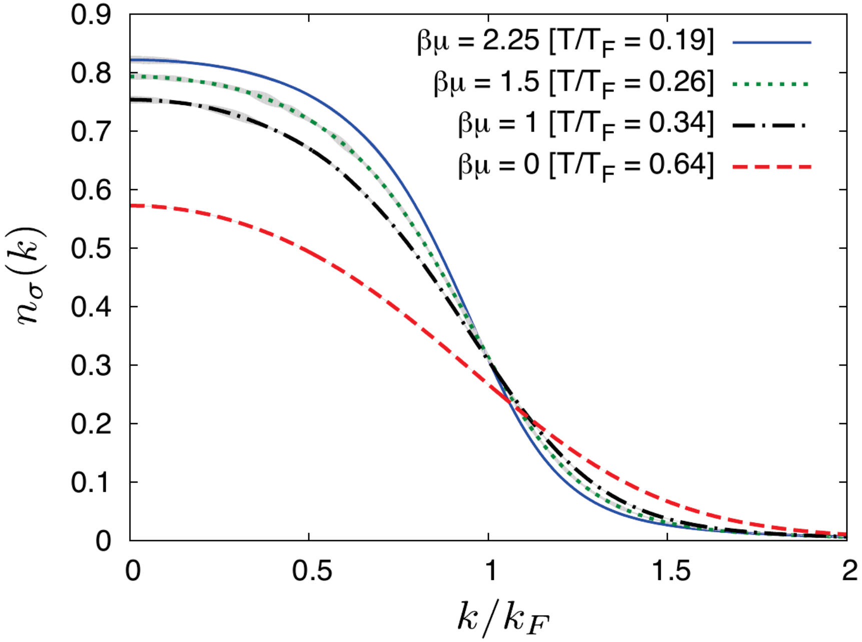
[S. Tan, Ann. Phys. 2008]

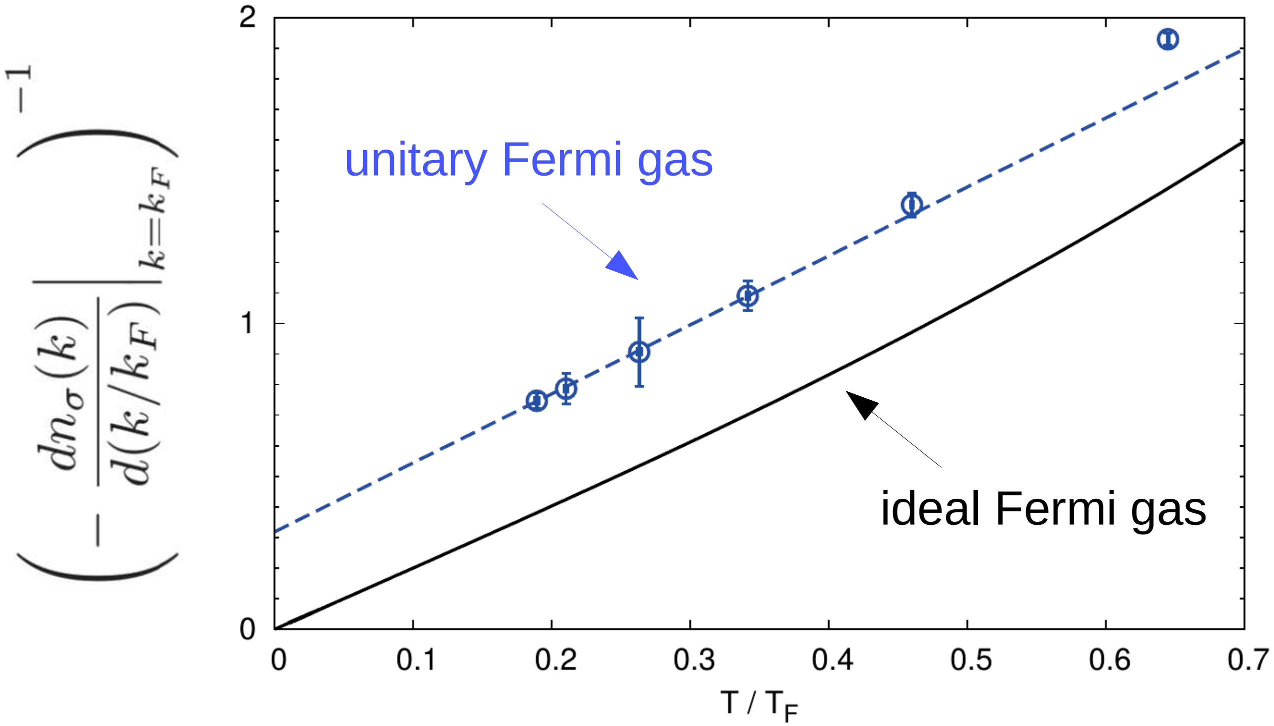
$$\mathcal{C} = -\Gamma(\mathbf{r}=\mathbf{0}, \tau = 0^-)$$



[1] **Carcy, Hoinka, Lingham, Dyke, Kuhn, Hu, Vale, PRL 2019**
[2] **Mukherjee, Patel, Yan, Fletcher, Struck, Zwierlein, PRL 2019**

Momentum distribution





non Fermi liquid behavior

High-order diagrammatic expansion around BCS Hamiltonians

Polarized superfluid phase of the attractive Hubbard model

[Spada *et al.*, *arXiv* 2021]

Hubbard model – 3D cubic lattice

$$H = H_{\rm kin} - \sum_\sigma \mu_\sigma N_\sigma + H_{\rm int}$$

$$H_{\rm kin} = -t \sum_{\langle \mathbf{i},\mathbf{j}\rangle\,\sigma} (c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} + h.c.)$$

$$H_{\rm int} = U \sum_{\mathbf{i}} n_{\mathbf{i}\uparrow}\, n_{\mathbf{i}\downarrow}$$

$$U < 0$$

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Diag. expansion in superfluid (superconducting) phase $\mathcal{O} := \langle c_{\mathbf{0}\uparrow} c_{\mathbf{0}\downarrow} \rangle$

unperturbed quadratic Hamiltonian:

$$H_0 = H_{\text{kin}} - \sum_\sigma \mu_{0,\sigma} N_\sigma + H^{(\Delta_0)}_{\text{pair}}$$

breaks U(1) symmetry

$$H^{(\Delta_0)}_{\text{pair}} := \Delta_0 \sum_{\mathbf{i}} c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{i}\downarrow} + h.c.$$

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$$H_\xi := (1-\xi)\, H_0 + \xi\, H$$

$$Q(\xi) := \langle \hat{Q} \rangle_{H_\xi} \doteq \sum_{N=0}^{\infty} Q_N\, \xi^N$$

$$Q = Q(\xi = 1) = \sum_{N=0}^{\infty} Q_N \quad \text{if series converges}$$

[Abel's theorem]

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[Abel's theorem]

pressure: $P = -\Omega / L^3, \quad \Omega = -T \ln \operatorname{Tr} \exp(-\beta H)$

$$P(\xi) := \frac{T}{L^3} \ln \operatorname{Tr} \exp(-\beta H_\xi) \mathrel{\hat{=}} \sum_{N=0}^{\infty} P_N\, \xi^N$$

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$$(1-\xi)\,\Delta_0 \Leftrightarrow$$ **symmetry breaking field**

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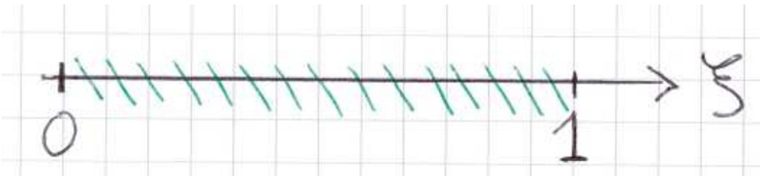
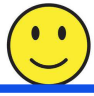
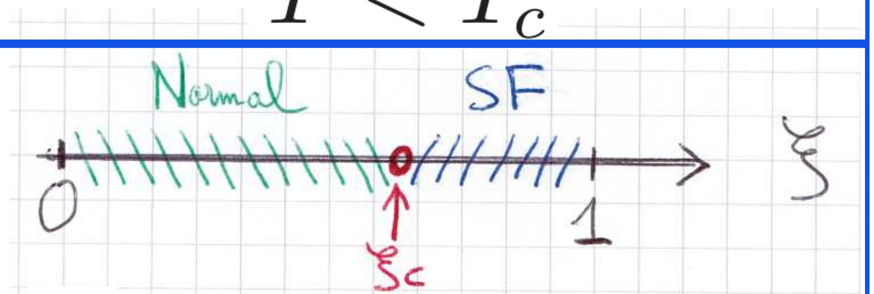
	$T > T_c$	$T < T_c$
$\Delta_0 = 0$	 	 
$\Delta_0 > 0$		

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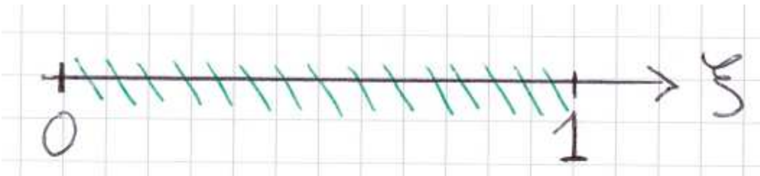
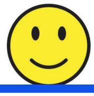
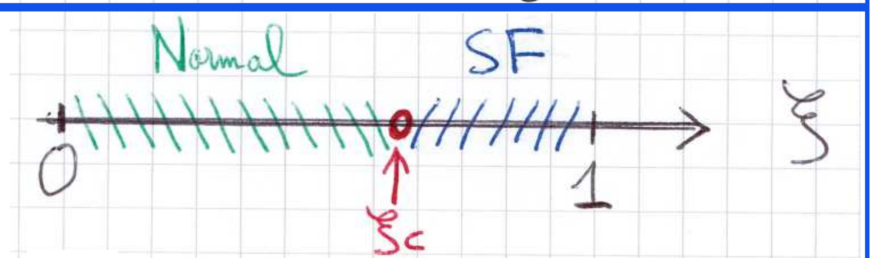
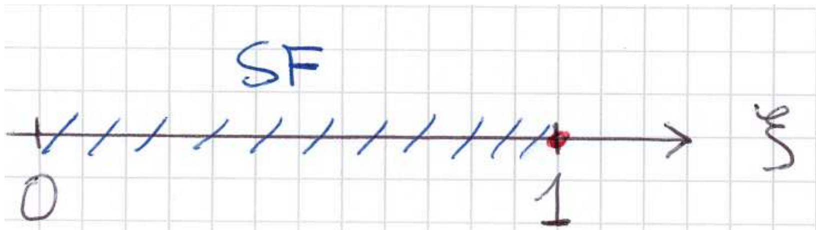
$\Delta_0 = 0$	 	 
$\Delta_0 > 0$		

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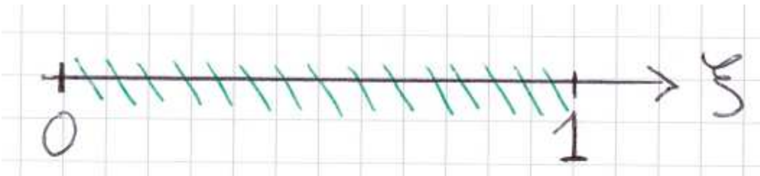
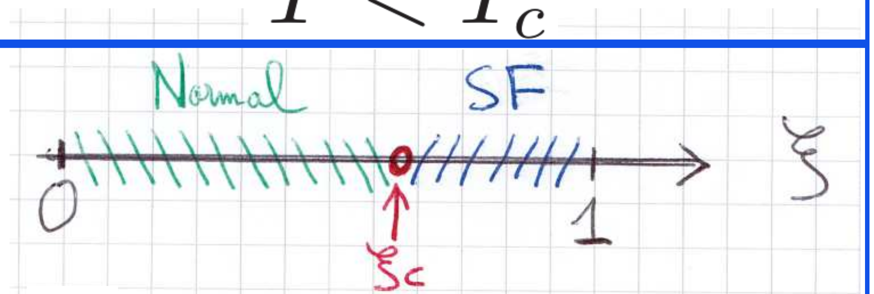
<tr><td>$\Delta_0 = 0$</td><td>

</td><td>

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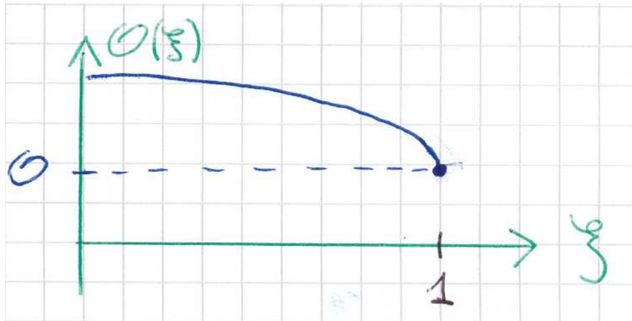
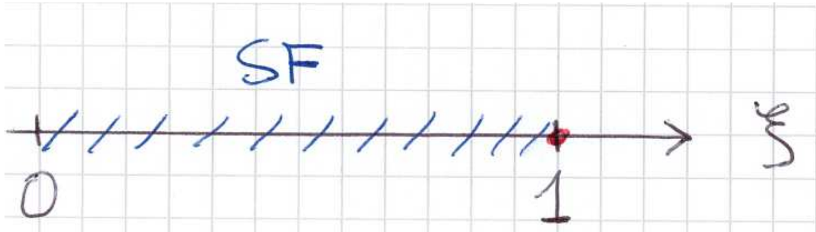
<tr><td>$\Delta_0 > 0$</td><td></td><td>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unperturbed quadratic Hamiltonian:

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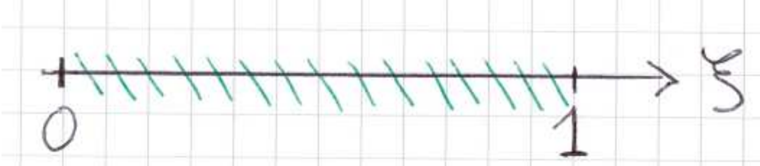
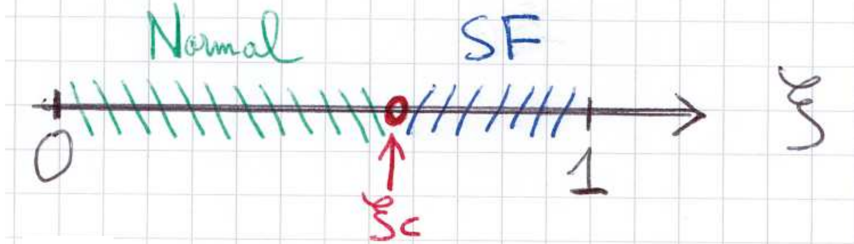
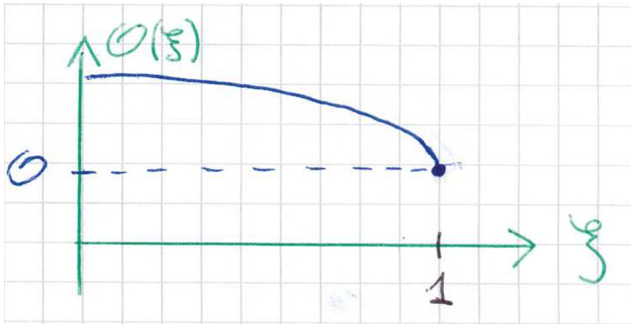
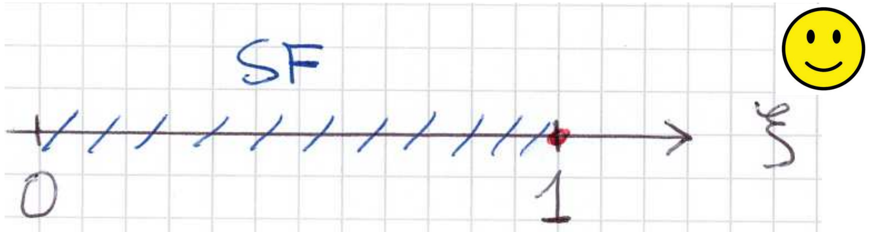
$(1-\xi)\,\Delta_0 \Leftrightarrow$ **symmetry breaking field**

	$T > T_c$	$T < T_c$
$\Delta_0 = 0$		
$\Delta_0 > 0$		 

__unperturbed quadratic Hamiltonian:__

$$H_0 = H_{\text{kin}} - \sum_{\sigma} \mu_{0,\sigma} N_\sigma + H^{(\Delta_0)}_{\text{pair}}$$

breaks U(1) symmetry

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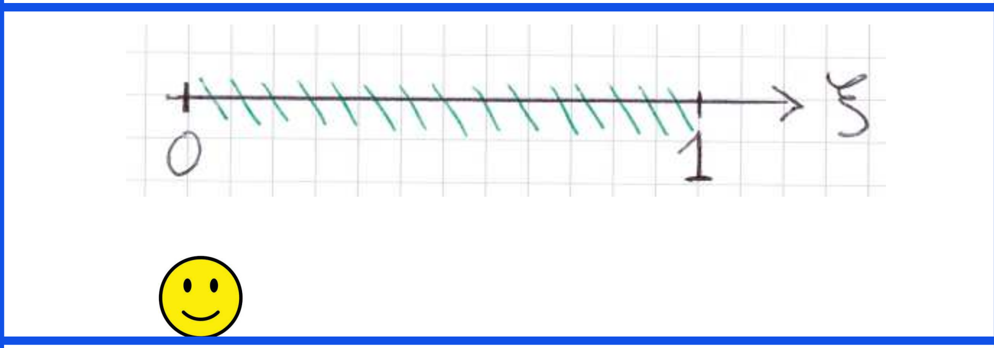
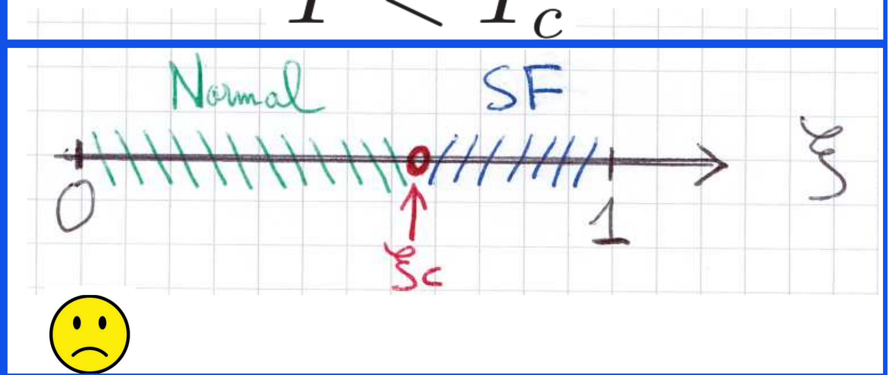
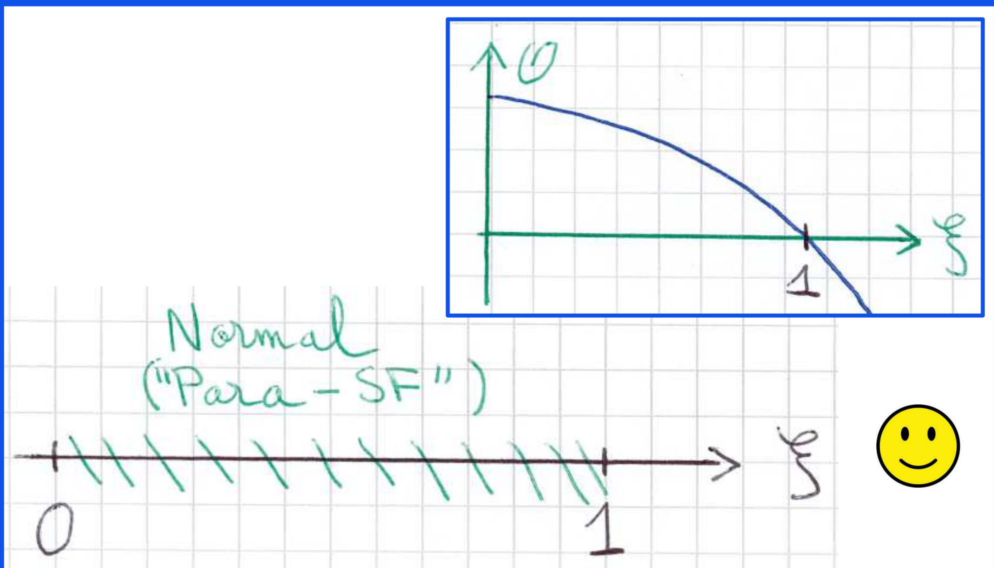
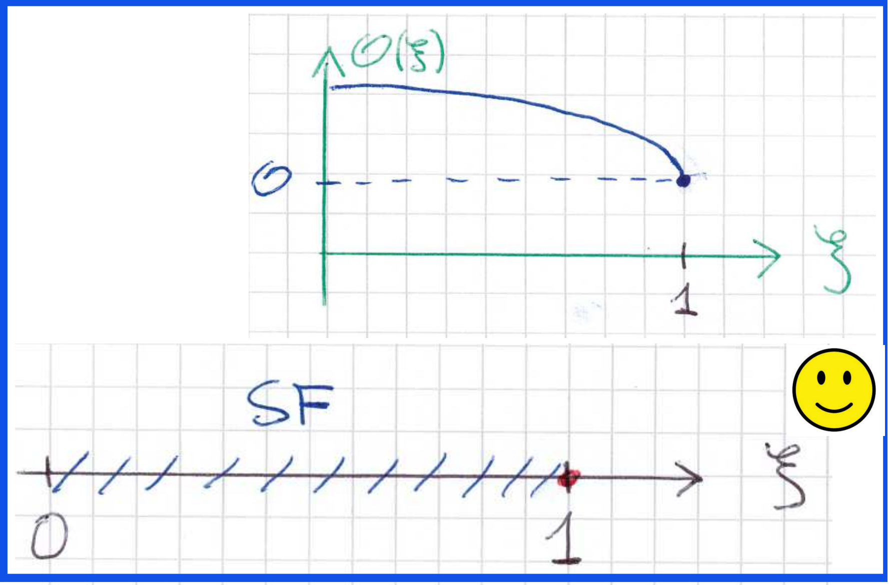
$\Delta_0 = 0$		
$\Delta_0 > 0$		

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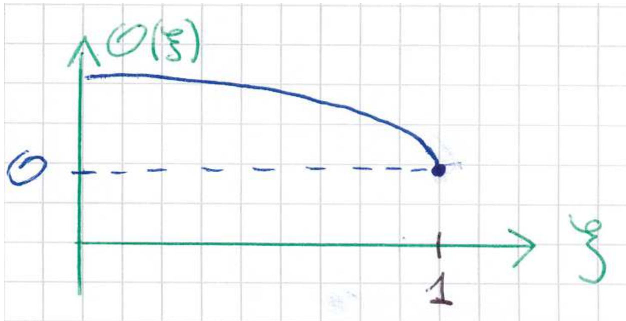

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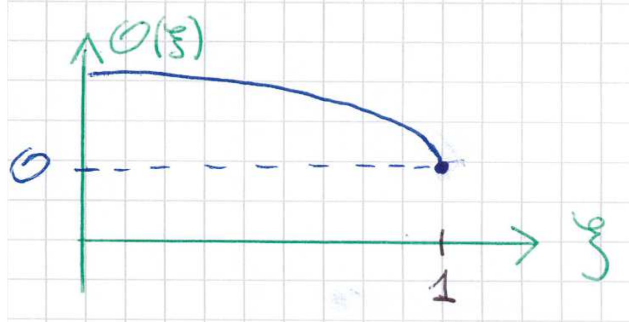

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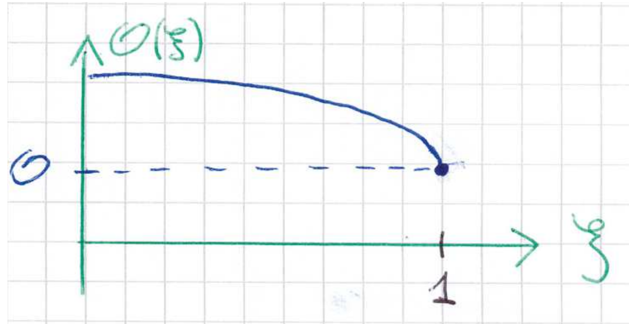

spontaneous **symmetry breaking** – thermodynamic limit $L \to \infty$ before $\xi \to 1^-$

$$\mathcal{O}(\xi) = \sum_{N=0}^{\infty} \mathcal{O}_N\, \xi^N$$

$$\mathcal{O}_N = \lim_{L\to\infty} \mathcal{O}_N^{(L)}$$

$$\mathcal{O} = \mathcal{O}(\xi \to 1^-) = \sum_{N=0}^{\infty} \mathcal{O}_N$$

unperturbed quadratic Hamiltonian: breaks U(1) symmetry

$$H_0 = H_{\mathrm{kin}} - \sum_{\sigma} \mu_{0,\sigma} N_\sigma + H_{\mathrm{pair}}^{(\Delta_0)}$$

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natural choice:
BCS mean-field theory

$$\mu_{0,\sigma} = \mu_\sigma - U \langle n_{\mathbf{0},-\sigma} \rangle_{H_0}$$

$$\Delta_0 = \Delta_{\mathrm{MF}} := -U \langle c_{\mathbf{0}\uparrow} c_{\mathbf{0}\downarrow} \rangle_{H_0}$$

also $\Delta_0 \neq \Delta_{\mathrm{MF}}$

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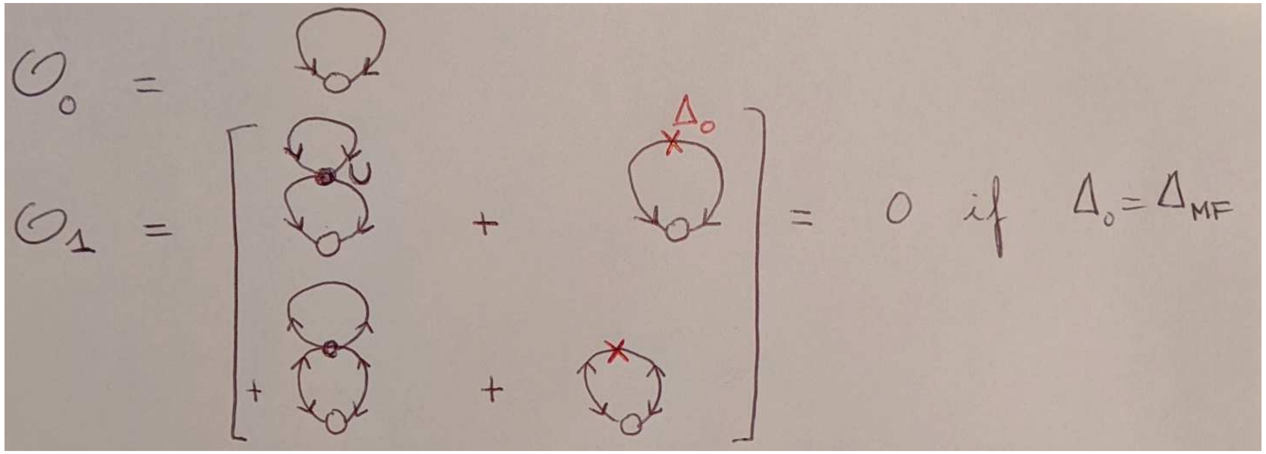

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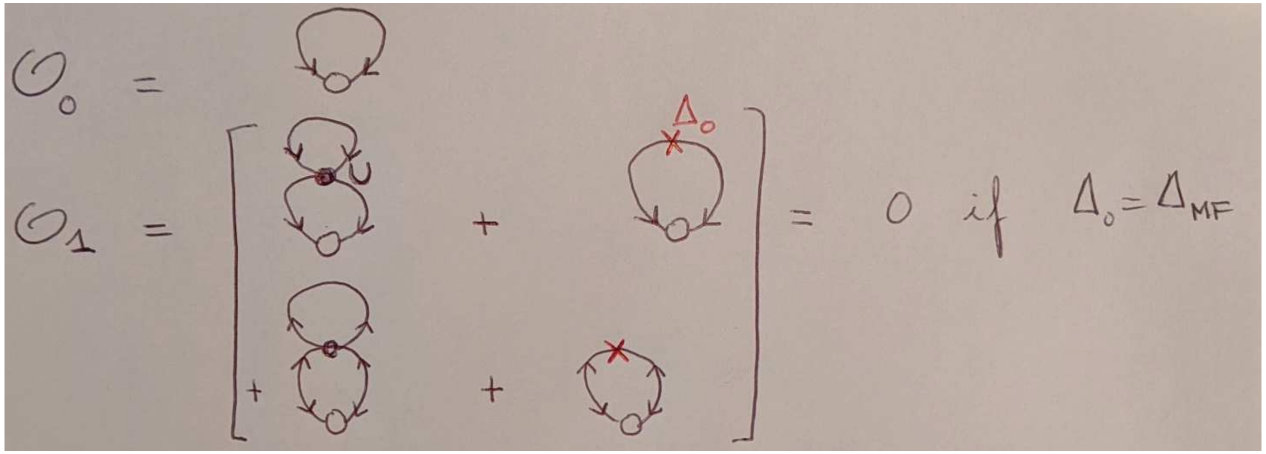

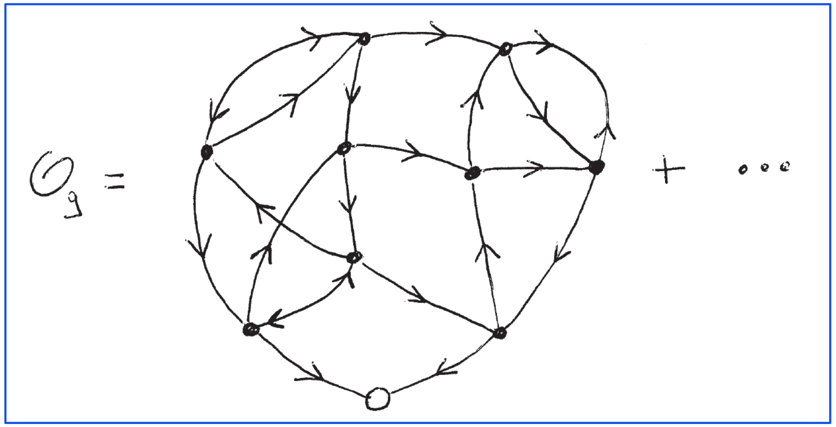

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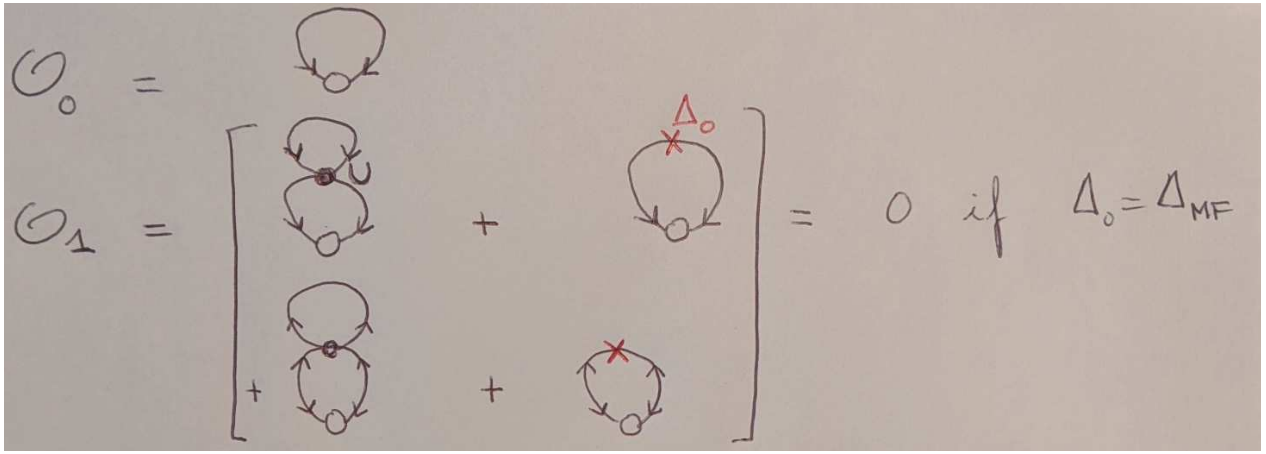

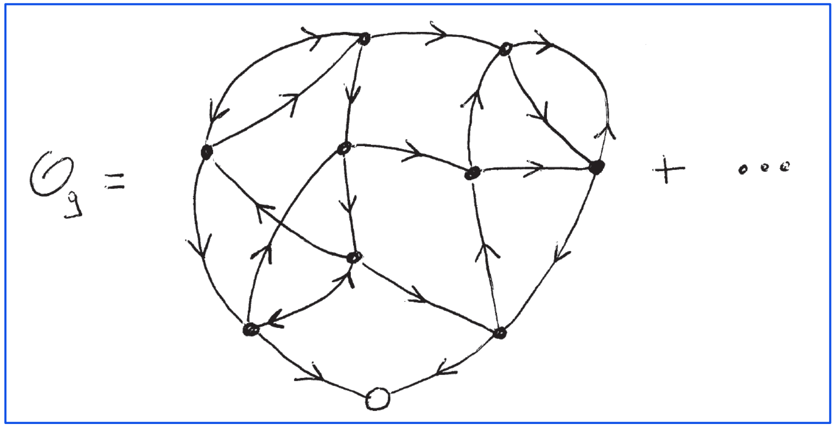

large distances : small contribution

broken symmetry

algorithm: ***CDet*** *[Rossi 2017]* *with* ***Nambu propagators*** algorithm: ***CDet*** *[Rossi 2017]* *with* ***Nambu propagators***

$$\begin{pmatrix} \mathcal{G}_{00}(X-X') & \mathcal{G}_{01}(X-X') \\ \mathcal{G}_{10}(X-X') & \mathcal{G}_{11}(X-X') \end{pmatrix} := -\begin{pmatrix} \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c^\dagger_\uparrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \\ \left\langle \mathrm{T}\, c_\downarrow(X) c_\uparrow(X') \right\rangle_{H_0} & \left\langle \mathrm{T}\, c_\downarrow(X) c^\dagger_\downarrow(X') \right\rangle_{H_0} \end{pmatrix}$$

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$$\mathcal{O}_N = -\frac{(-U)^N}{N!} \int dX_1 \dots dX_N \; \mathrm{cdet}(A)$$

$$\mathrm{cdet}(A) = \det(A) - \sum (\text{disconnected diagrams})$$

$$X = (\mathbf{i}, \tau)$$

$$\begin{pmatrix} \textit{recursively} \\ 3^N \;\; \textit{operations} \end{pmatrix}$$

$$A := \begin{pmatrix} 0 & \delta_{\rm sh} & \dots & \mathcal{G}_{00}(X_1-X_N) & \mathcal{G}_{01}(X_1-X_N) & \mathcal{G}_{0\alpha}(X_1) \\ \delta_{\rm sh} & 0 & \dots & \mathcal{G}_{10}(X_1-X_N) & \mathcal{G}_{11}(X_1-X_N) & \mathcal{G}_{1\alpha}(X_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{G}_{00}(X_N-X_1) & \mathcal{G}_{01}(X_N-X_1) & \dots & 0 & \delta_{\rm sh} & \mathcal{G}_{0\alpha}(X_N) \\ \mathcal{G}_{10}(X_N-X_1) & \mathcal{G}_{11}(X_N-X_1) & \dots & \delta_{\rm sh} & 0 & \mathcal{G}_{1\alpha}(X_N) \\ \mathcal{G}_{\alpha'0}(-X_1) & \mathcal{G}_{\alpha'1}(-X_1) & \dots & \mathcal{G}_{\alpha'0}(-X_N) & \mathcal{G}_{\alpha'1}(-X_N) & \mathcal{G}_{\alpha'\alpha}(0) \end{pmatrix}$$

$$\delta_{\rm sh} = 0 \text{ if } \Delta_0 = \Delta_{\rm MF}$$

$$(\alpha = 1,\; \alpha' = 0)$$

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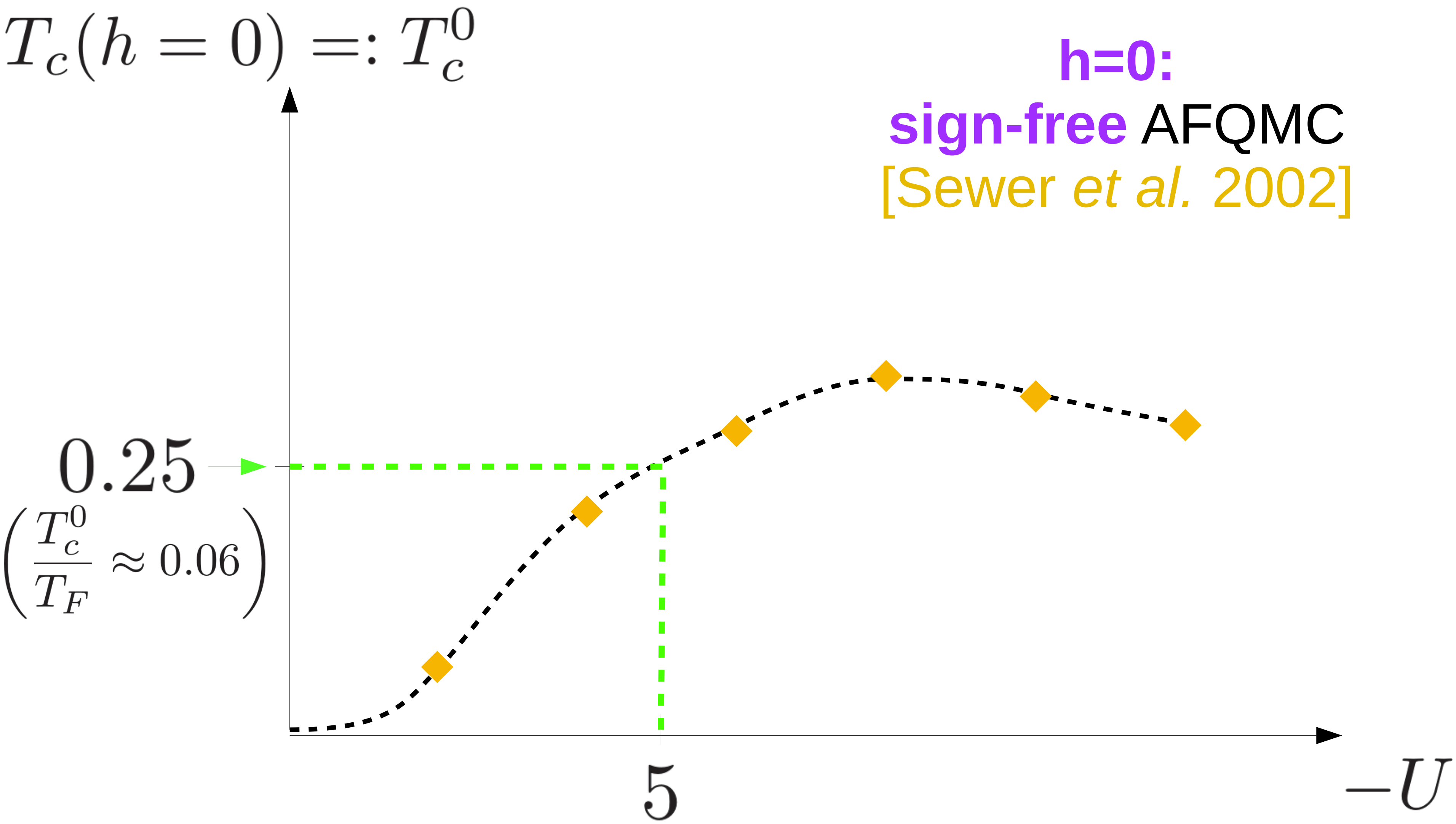
implementation: ***Fast Feynman Diagrammatics*** *library* *[Rossi & Simkovic]*
with ***Many Configuration MC*** *[Simkovic & Rossi, arXiv 2021]*

RESULTS

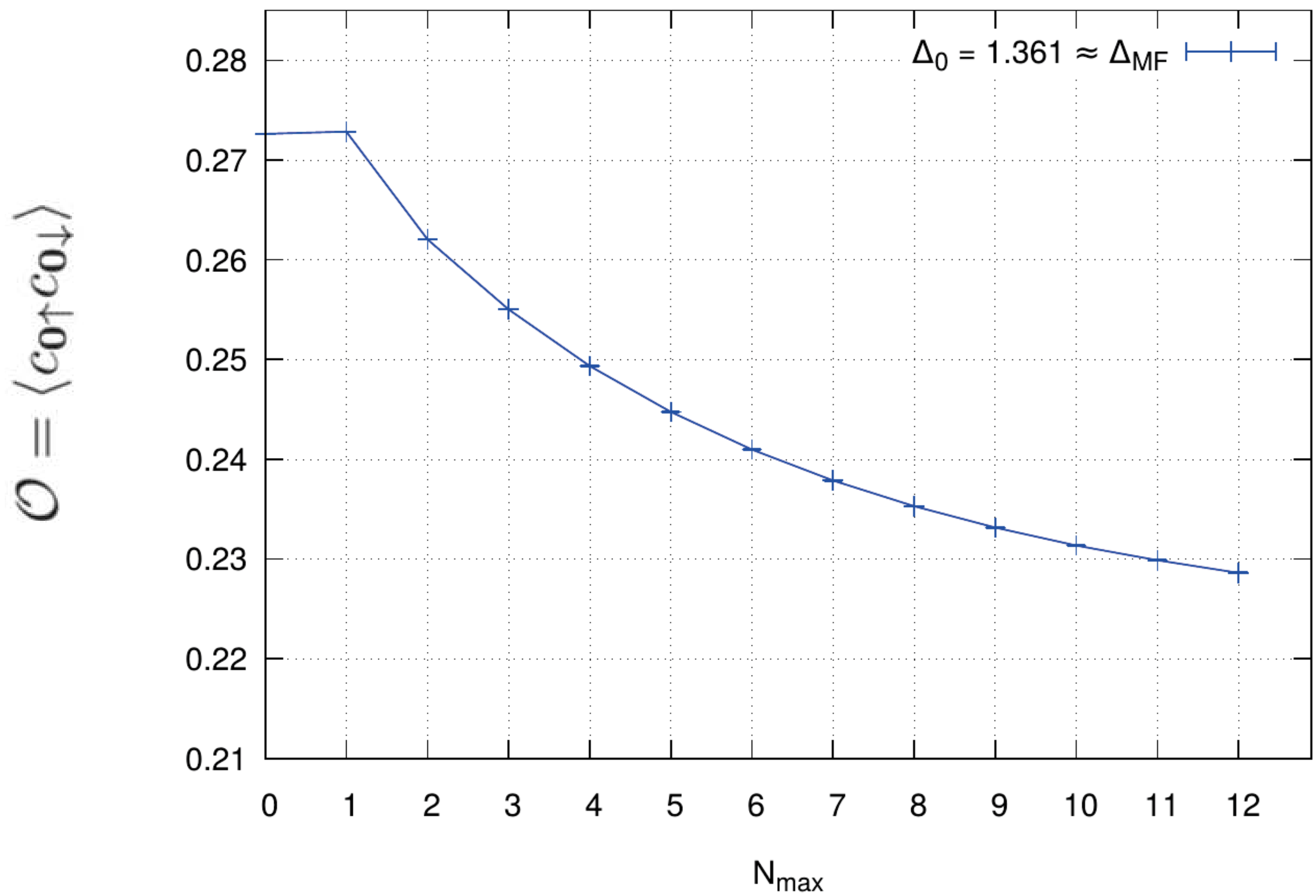
$$\mu_\uparrow = \mu + h, \quad \mu_\downarrow = \mu - h$$

$$t \equiv 1, \boxed{U = -5}$$

$$\mu = -3.38 \Rightarrow \langle n_\uparrow + n_\downarrow \rangle \simeq 0.5 \,(\text{quarter filling})$$



$$T = 1/8 \approx T_c^0/2, \quad h = 0$$



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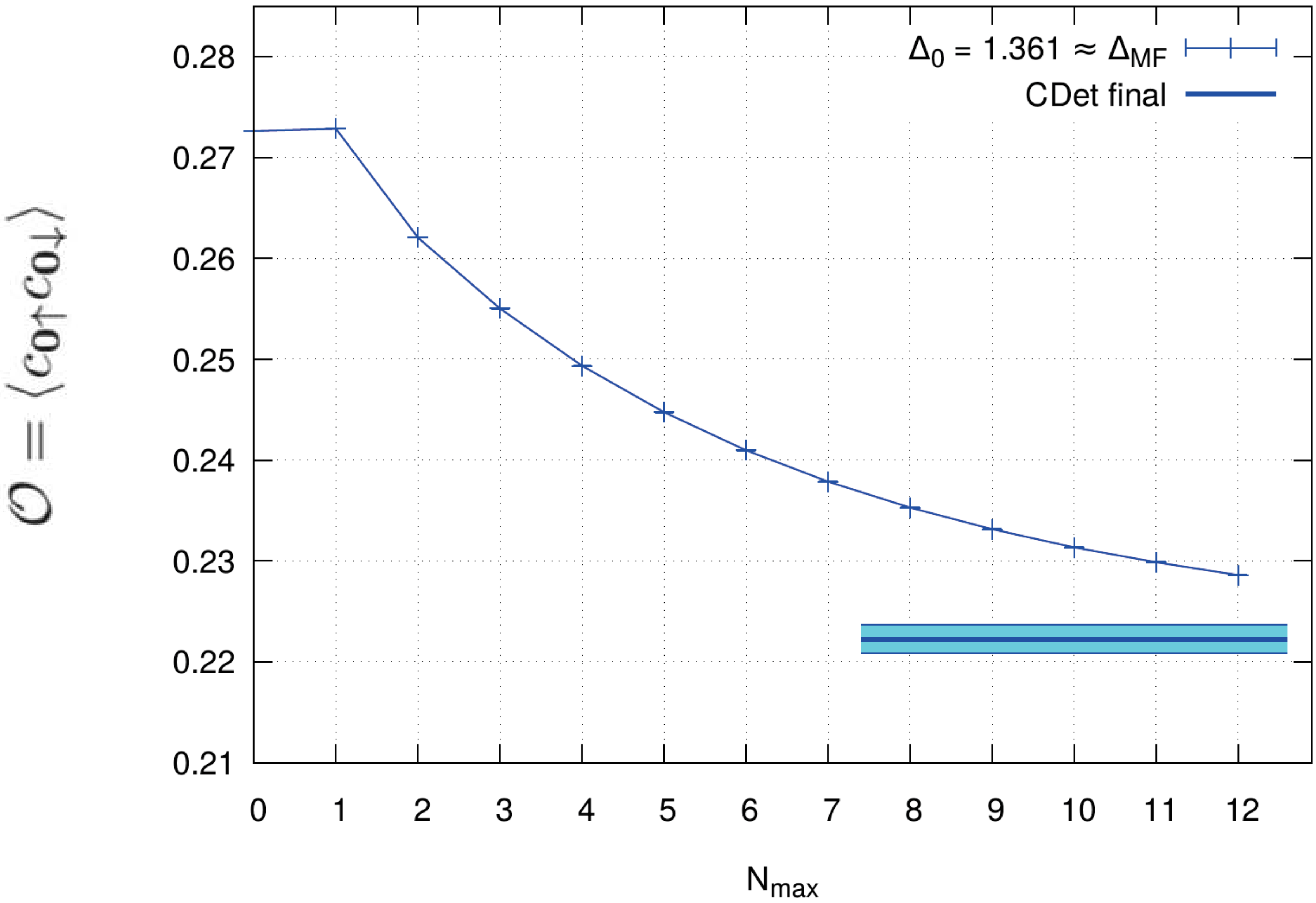

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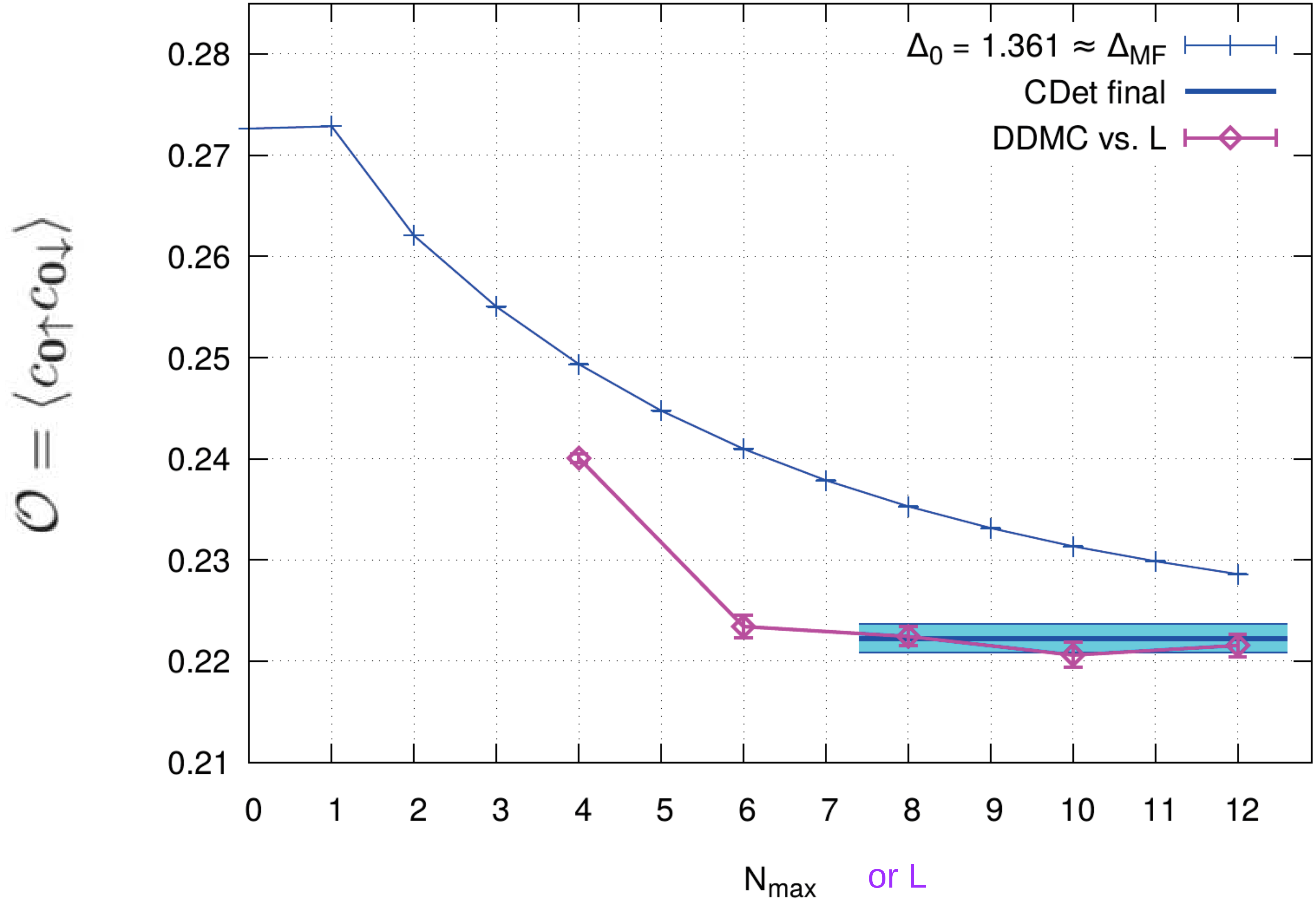
benchmark vs.
Determinant **D**iagrammatic **MC**
[Burovski's code]



Polarized regime

$h \neq 0 \quad \left(h \equiv \frac{\mu_\uparrow - \mu_\downarrow}{2} \right)$

no unbiased results available (sign problem)

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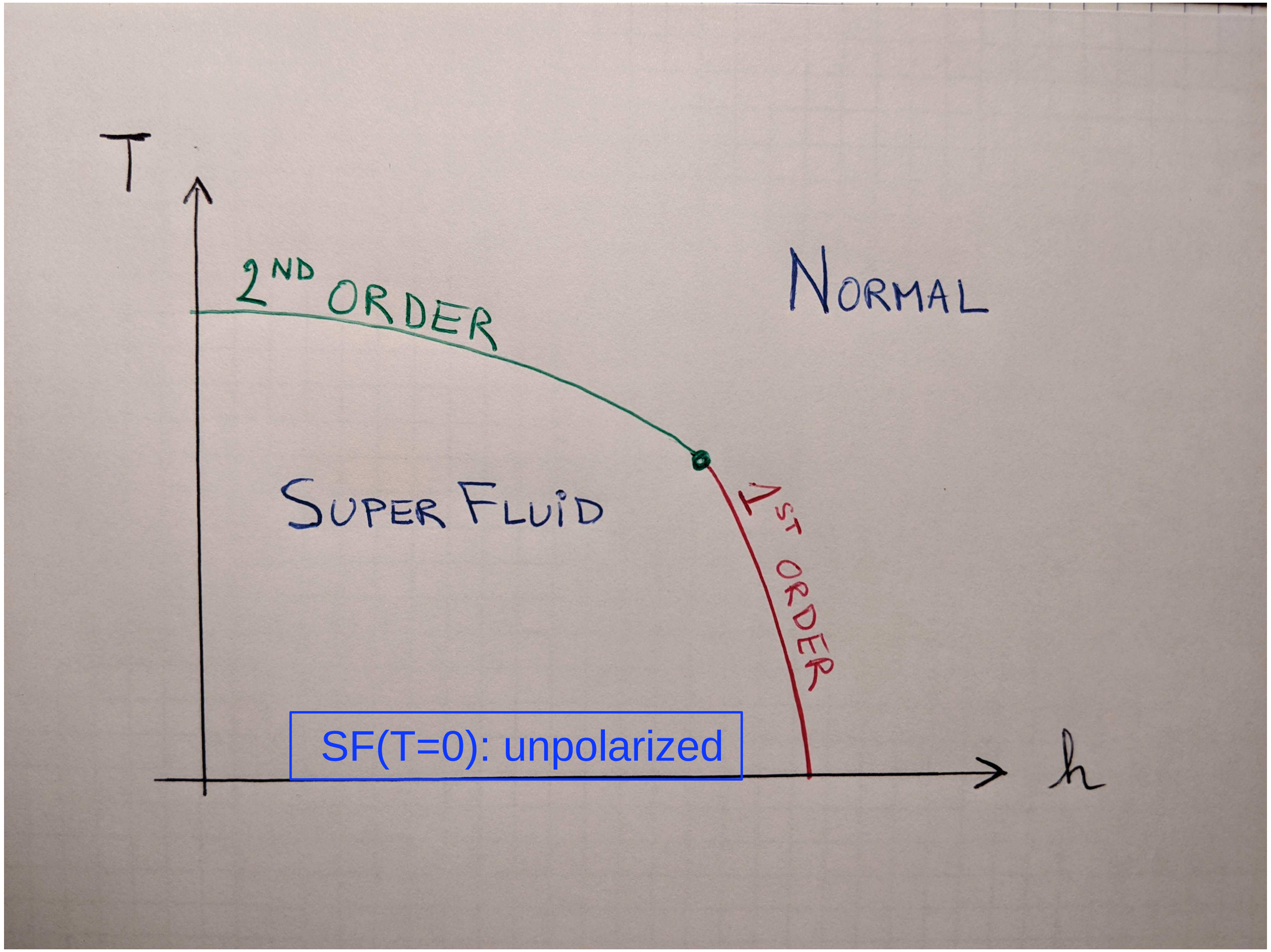
<u>expected phase-diagram topology</u> (discarding FFLO)

from BCS-MF; DMFT [Dao *et al.* 2008; Koga & Werner 2010]

[cold-atom experiments in c° space, ENS & MIT, 2008-2010]



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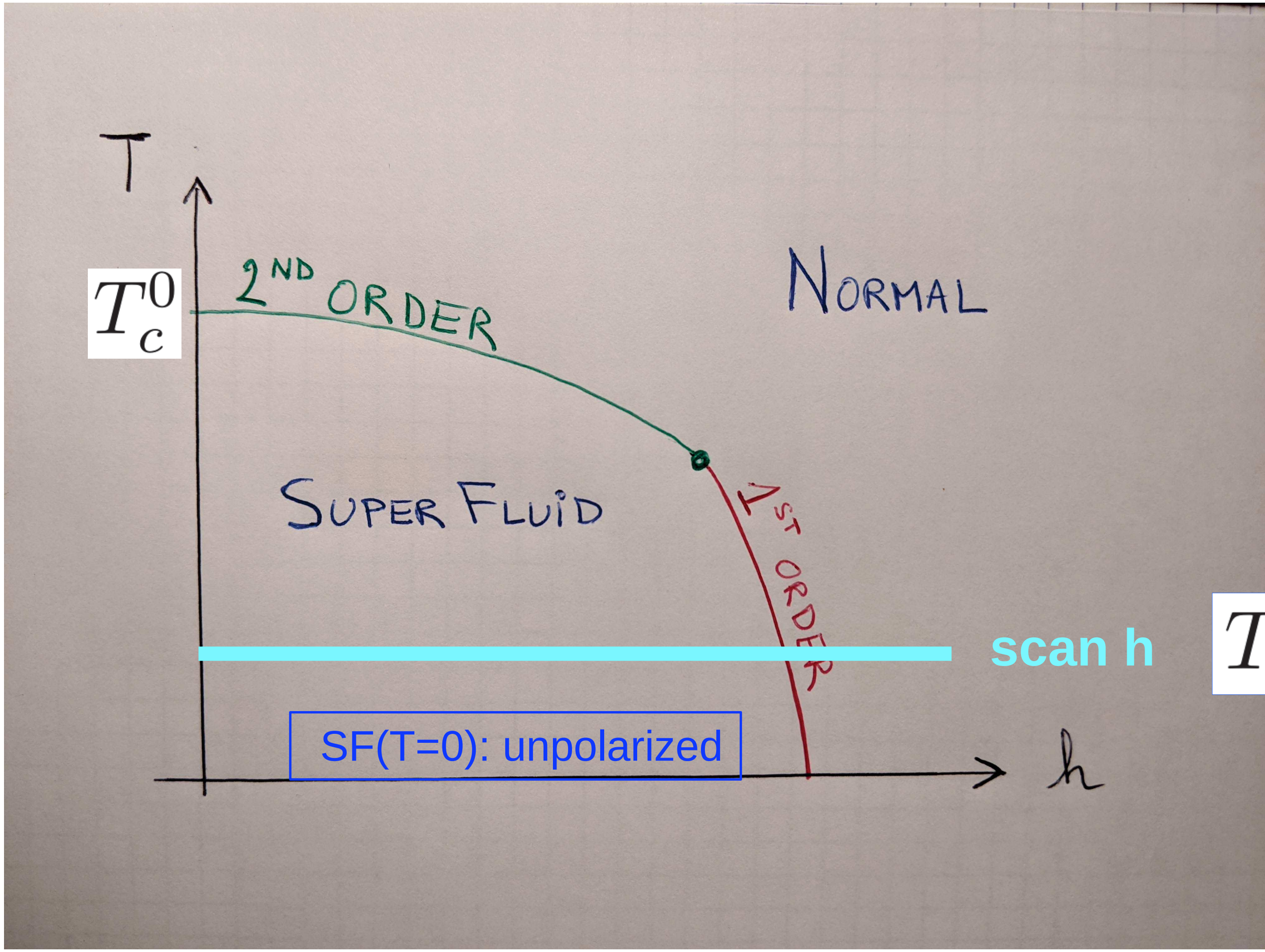
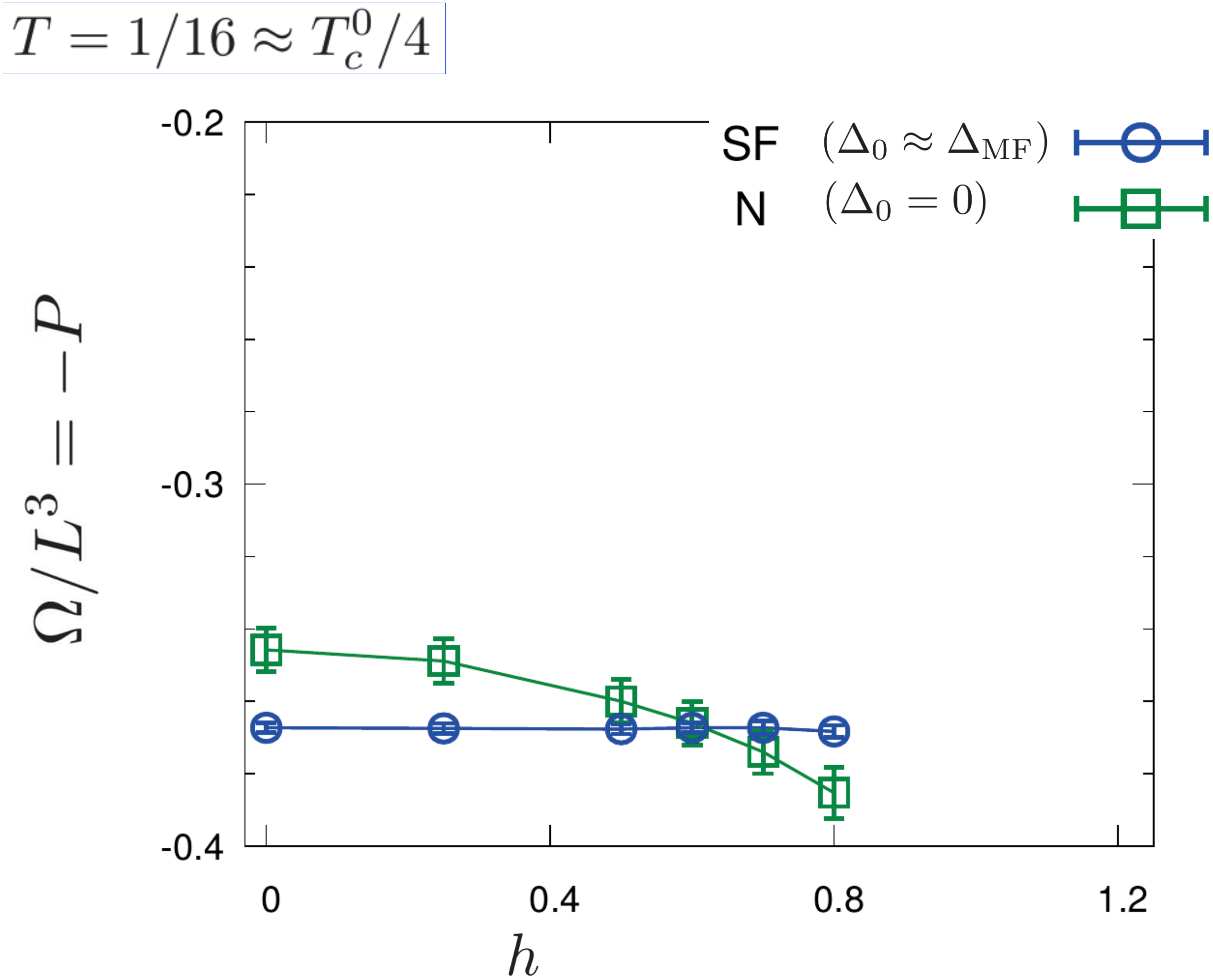
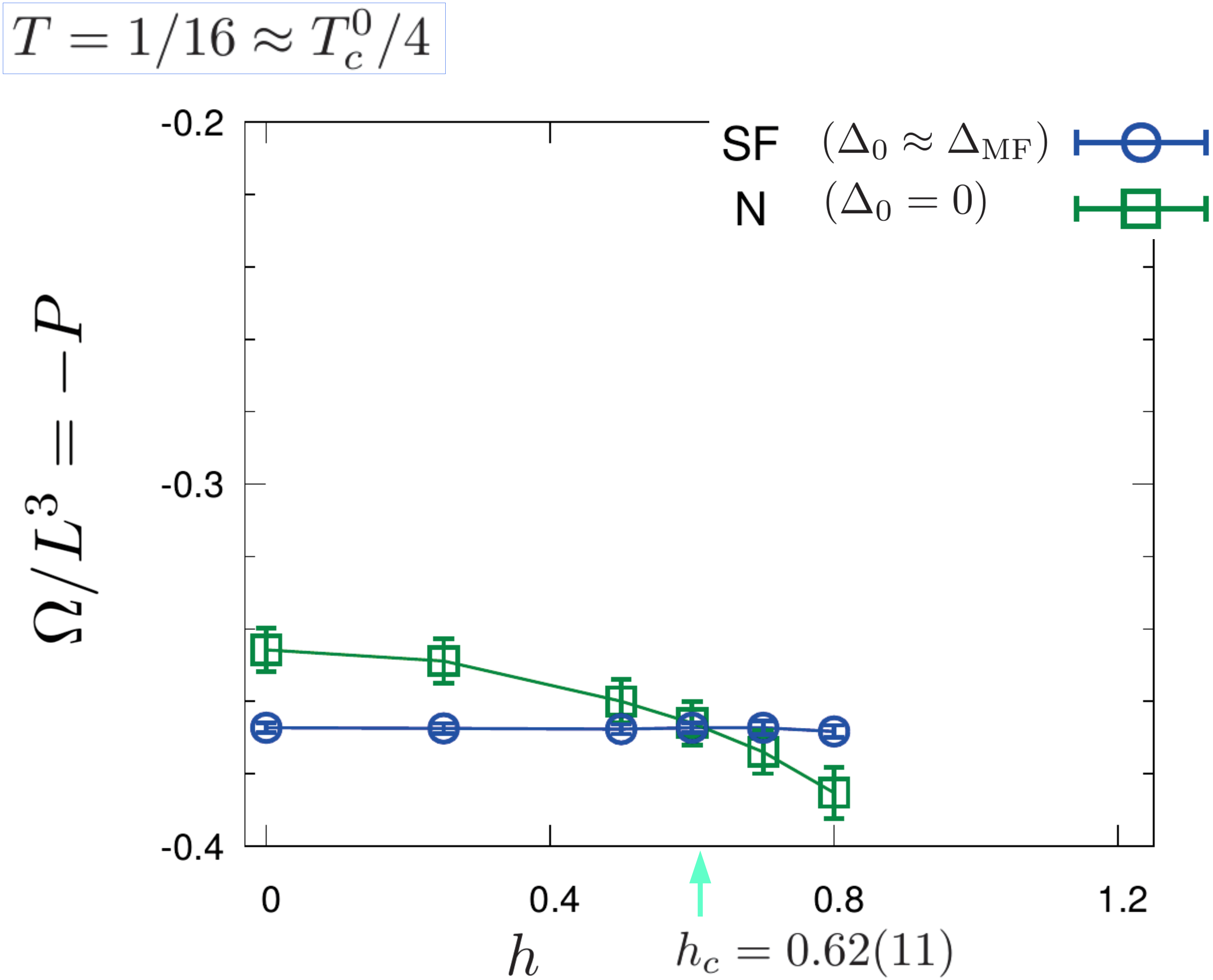


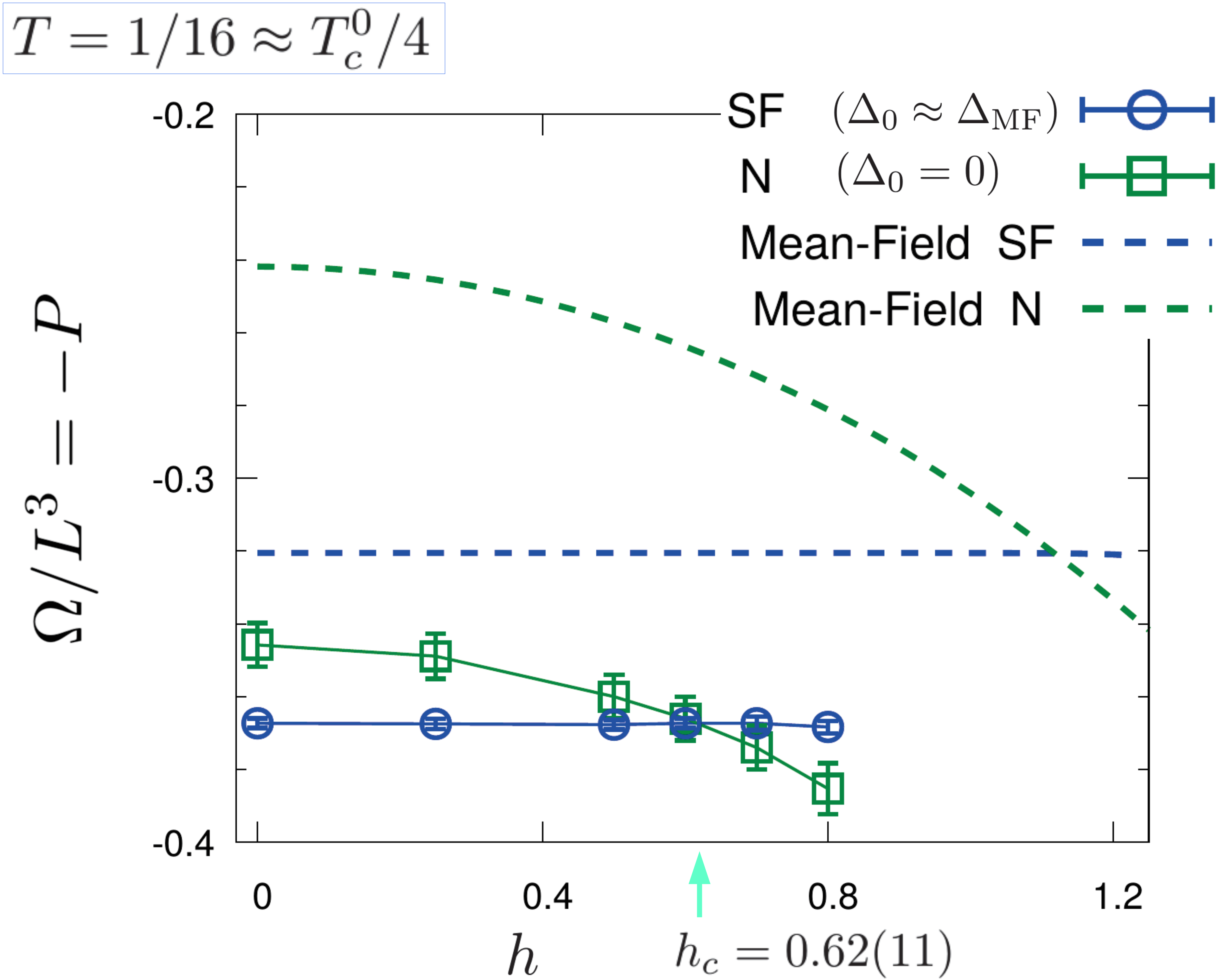
$T \approx T_c^0/4$



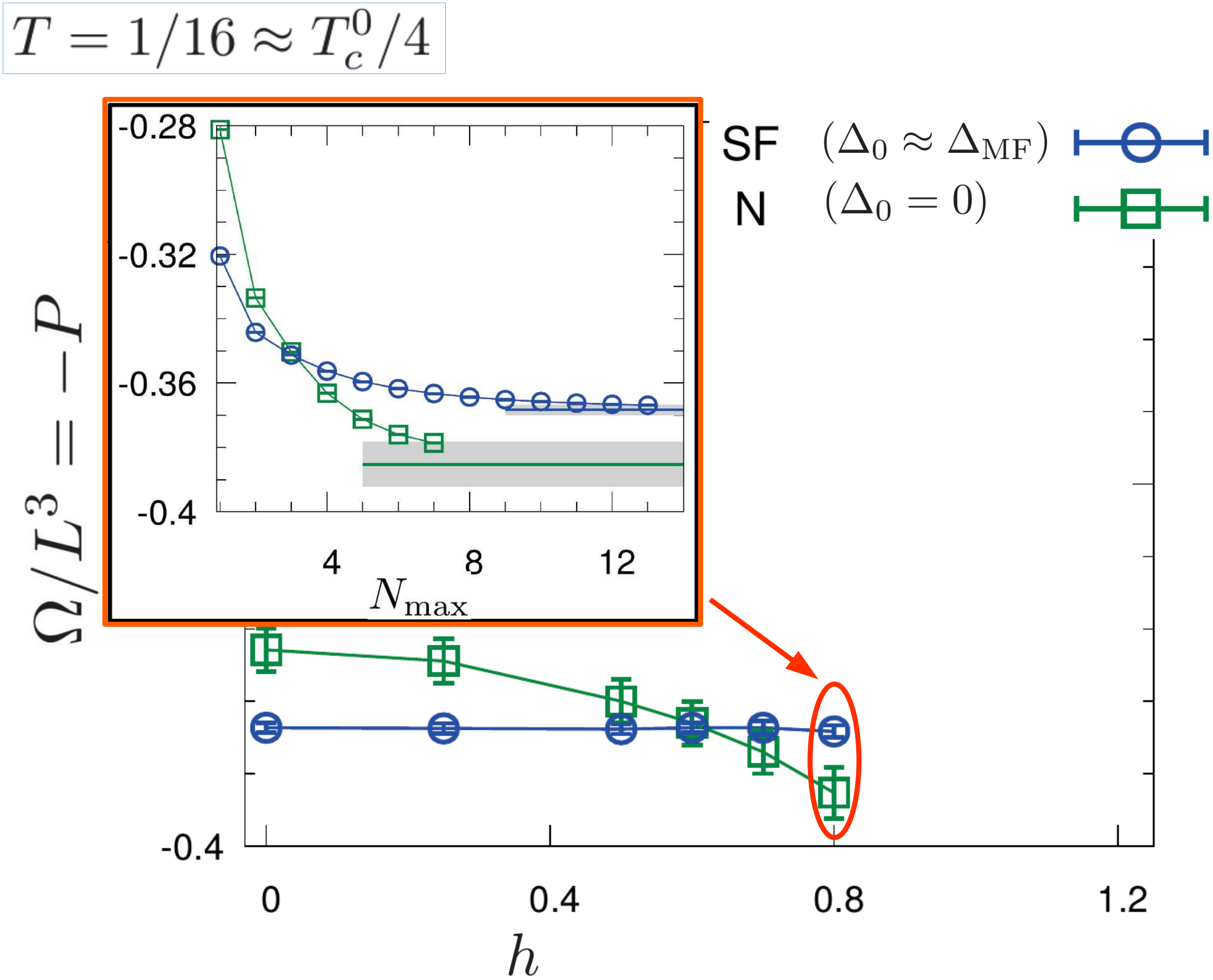
$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\rm MF})$
N $(\Delta_0 = 0)$
$\Omega/L^3 = -P$
-0.2
-0.3
-0.4
0
0.4
0.8
1.2
h



$T = 1/16 \approx T_c^0/4$
$\Omega/L^3 = -P$
-0.2
-0.3
-0.4
0
0.4
0.8
1.2
h
$h_c = 0.62(11)$
SF $(\Delta_0 \approx \Delta_{\mathrm{MF}})$
N $(\Delta_0 = 0)$

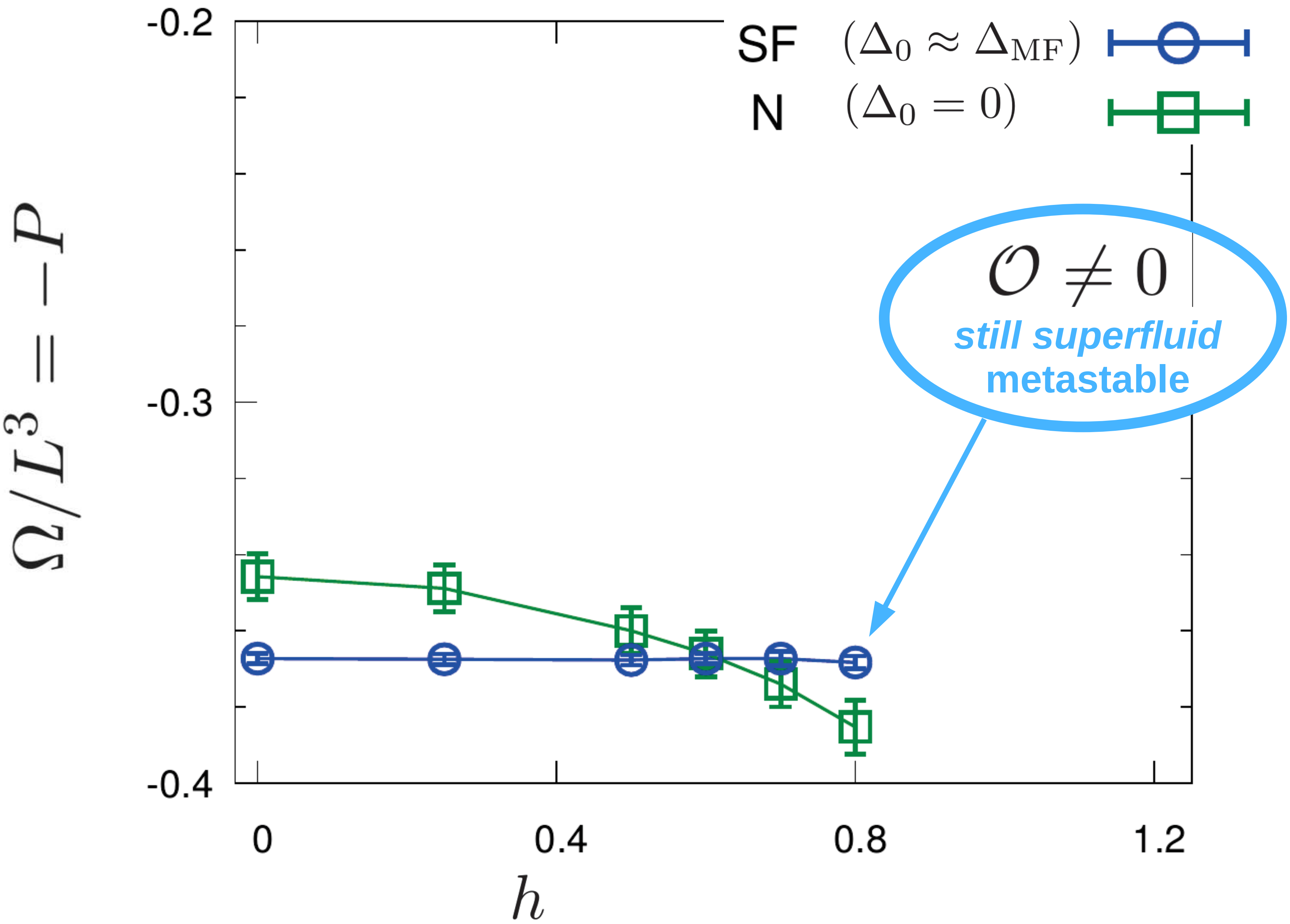


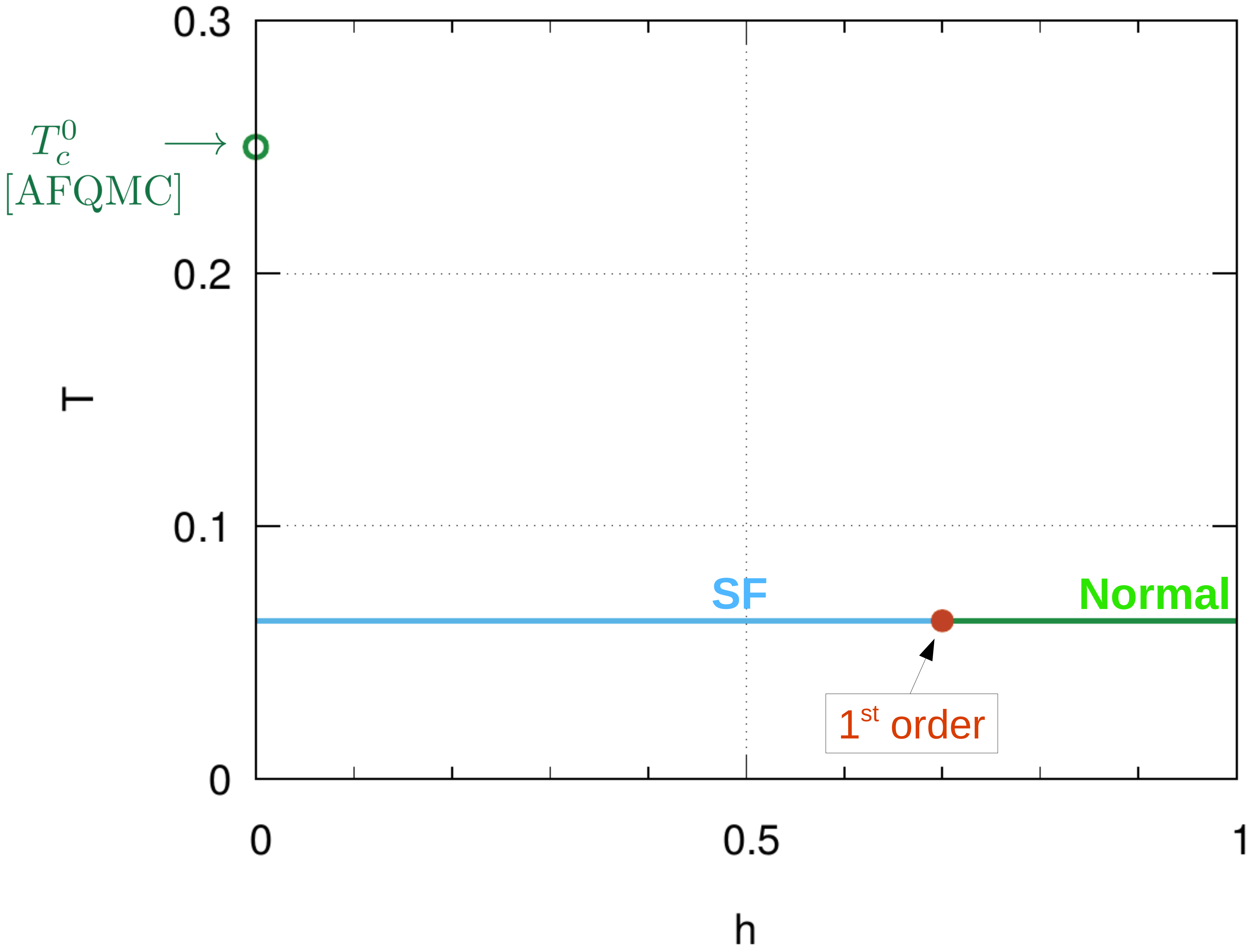
$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\mathrm{MF}})$
N $(\Delta_0 = 0)$
Mean-Field SF
Mean-Field N
-0.2
-0.3
-0.4
0
0.4
0.8
1.2
$\Omega/L^3 = -P$
h
$h_c = 0.62(11)$



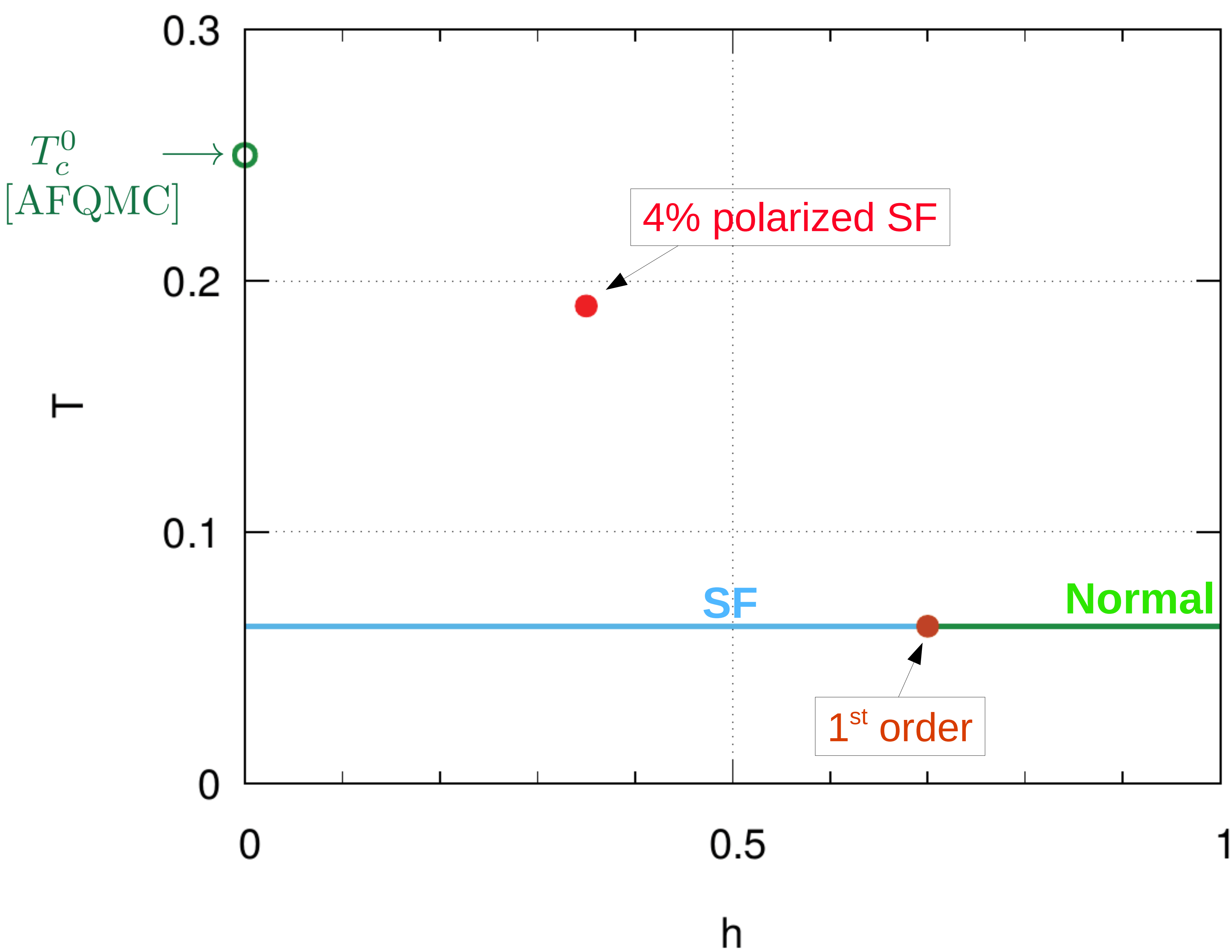
$T = 1/16 \approx T_c^0/4$
SF $(\Delta_0 \approx \Delta_{\rm MF})$
N $(\Delta_0 = 0)$
$\Omega/L^3 = -P$
-0.4
0
0.4
0.8
1.2
h
-0.28
-0.32
-0.36
-0.4
4
8
12
$N_{\rm max}$

$T = 1/16 \approx T_c^0/4$





T_c^0 [AFQMC]
0.3
0.2
0.1
0
T
SF
Normal
1st order
0
0.5
1
h



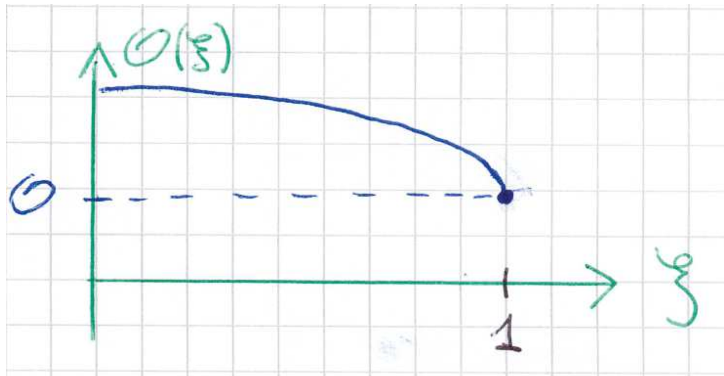
0.3
0.2
0.1
0
T
T_c^0 [AFQMC]
4% polarized SF
SF
Normal
1st order
0
0.5
1
h

Large-order behavior of SF expansion

$(1-\xi) \Leftrightarrow$ **symmetry breaking field**

Goldstone singularity [Patashinskii-Pokrovskii / Brézin-Wallace, 1973]

$$\mathcal{O}(\xi) \underset{\xi\to 1^-}{=} \mathcal{O} + \text{cst}\ \sqrt{1-\xi} + \dots \qquad (T < T_c)$$



Large-order behavior of SF expansion

$$(1-\xi) \Leftrightarrow$$ **symmetry breaking field**

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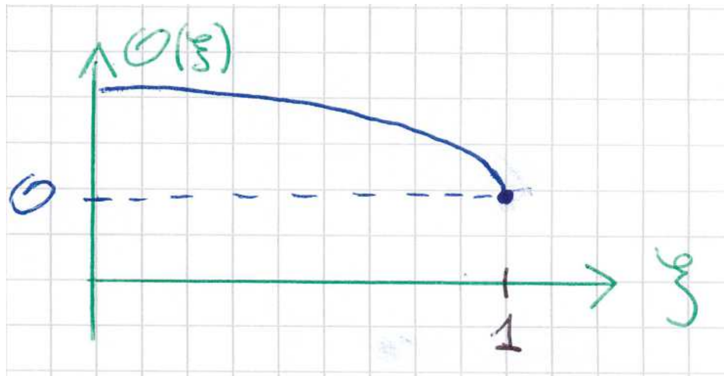
$$\frac{T}{\pi}\left(\frac{\mathcal{O}}{2\,D_s}\right)^{3/2}\sqrt{\Delta_0}$$

SF stiffness



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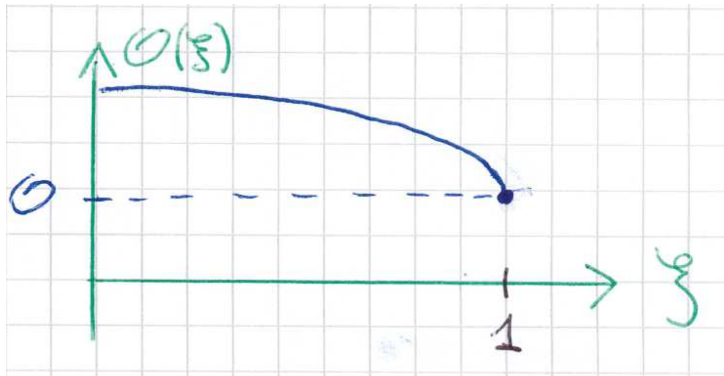
D_s: SF stiffness



$$\Rightarrow \quad \mathcal{O}_N \underset{N\to\infty}{\sim} \frac{\text{cst}}{N^{3/2}}$$

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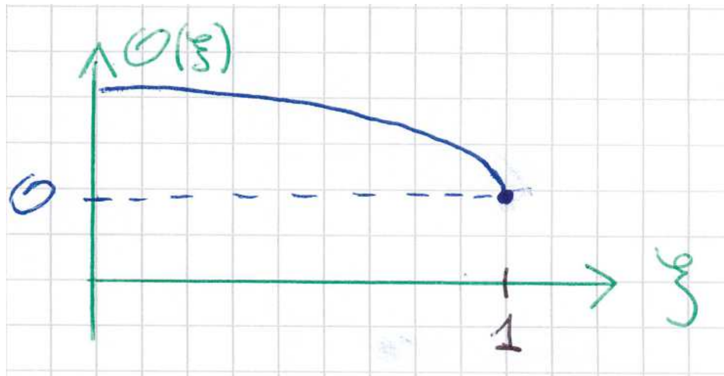

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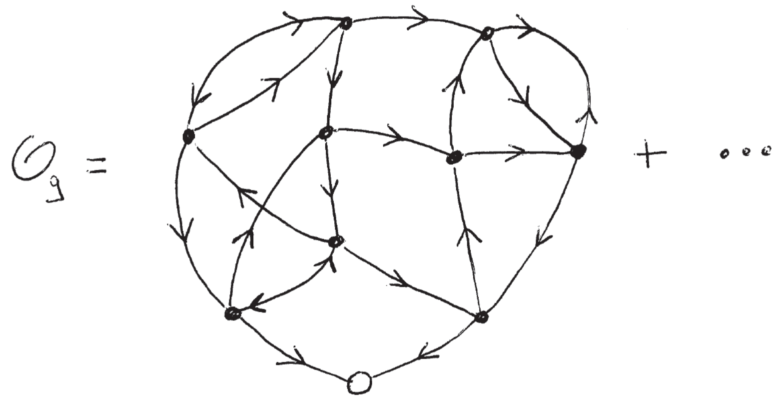
Large N ⟷ Large distances



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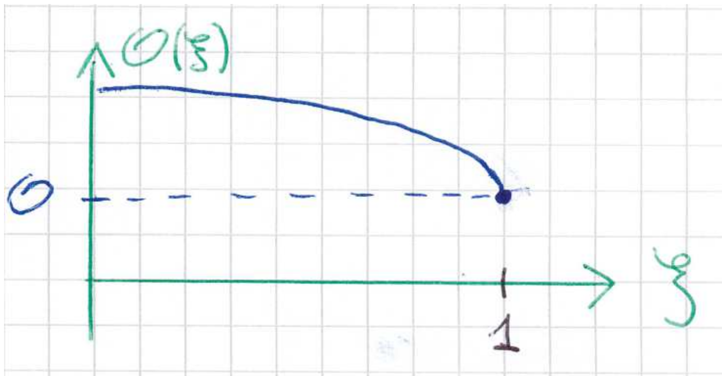

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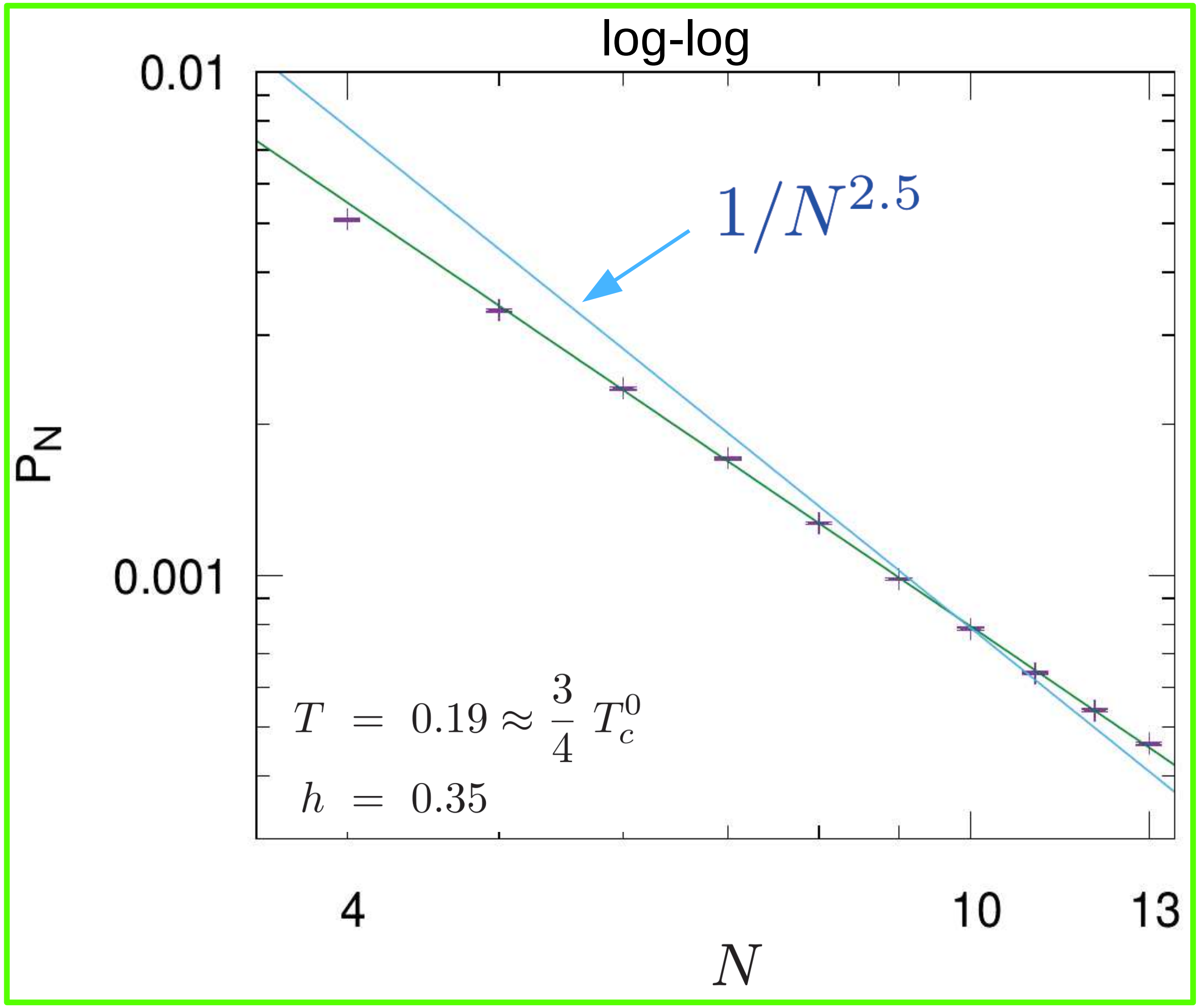
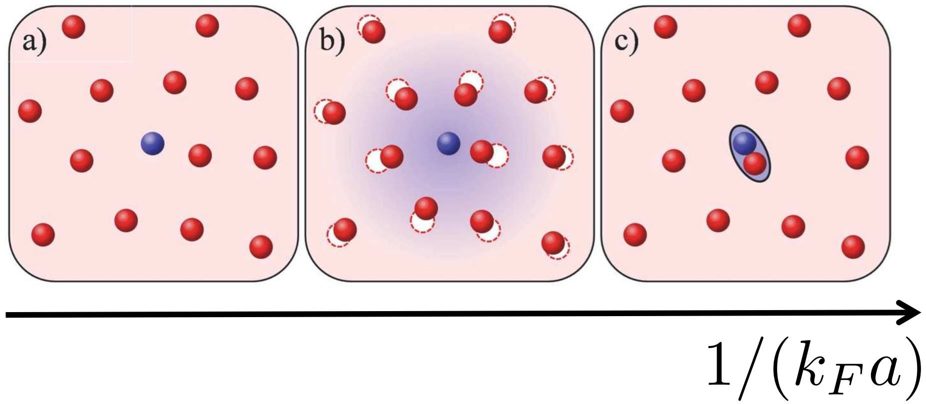
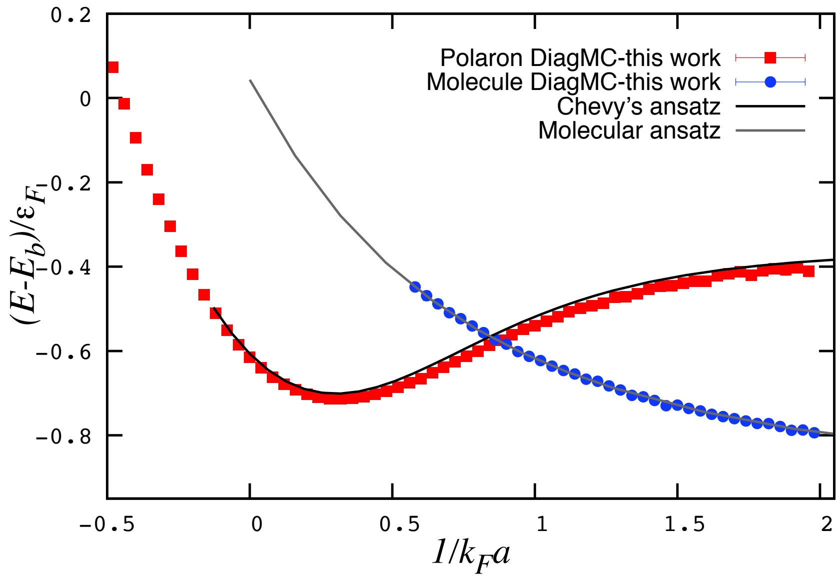


Fermi Polaron (polarized Fermi gas) = particle immersed in a Fermi sea



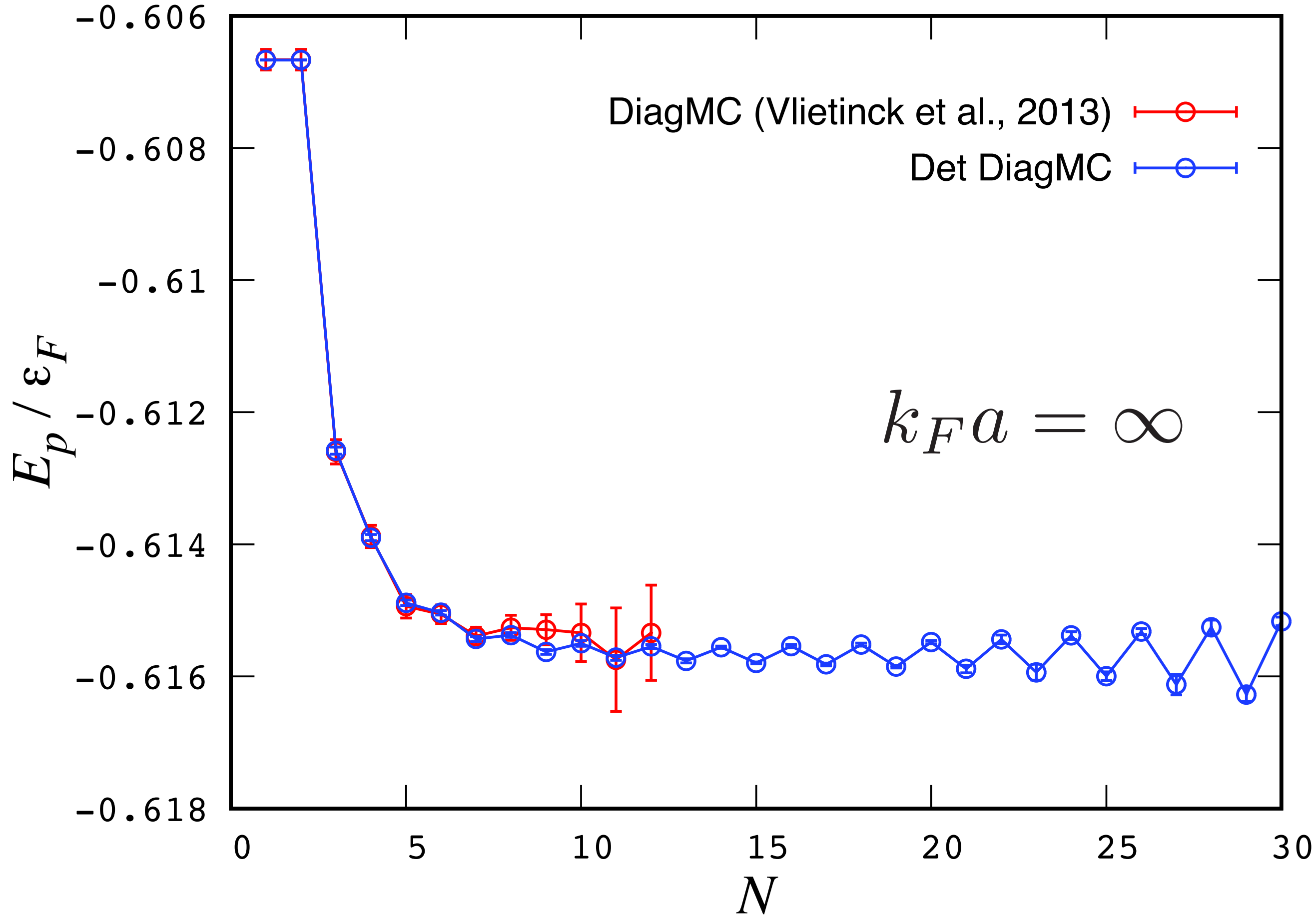
Schirotzek, Wu, Sommer, Zwierlein (2009)

Prokof'ev & Svistunov (PRB 2008)
Vlietinck, Ryckebusch, Van Houcke (PRB 2013)
Kroiss, Pollet (PRB 2015)
Goulko, Mishchenko, Prokof'ev, Svistunov (PRA 2016)



PDet algorithm: KVH, F. Werner, and R. Rossi, Phys. Rev. B **101**, 045134 (2020).

Polaron energy from self-energy: $E_p = \Sigma(\mathbf{p} = 0, \omega = 0, \mu = E_p)$



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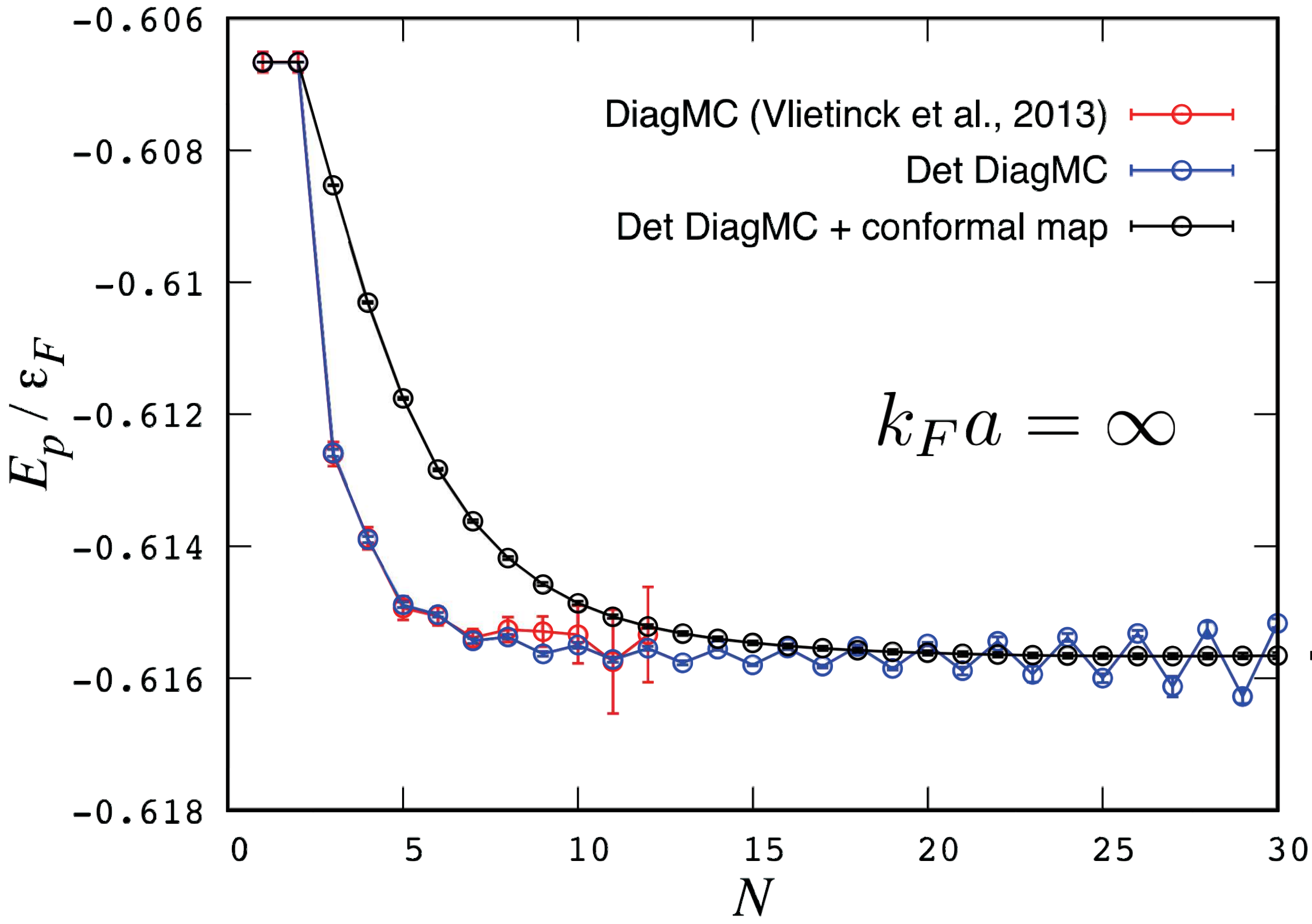

$\rightarrow E_p/\epsilon_F = -0.61565(4)$

Recent experiment (Zwierlein, MIT):

$E_p/\epsilon_F = -0.60(5)$

PDet algorithm: KVH, F. Werner, and R. Rossi, Phys. Rev. B **101**, 045134 (2020).

Conclusions:

Controlled summation of diagrammatic series with zero convergence radius for a strongly correlated fermionic field theory

Rossi, Ohgoe, Van Houcke, Werner, PRL 121, 130405 (2018) and PRL 121, 130406 (2018)

High-order expansion around BCS theory for the attractive 3D Hubbard model

G. Spada, R. Rossi, F. Simkovic, R. Garioud, M. Ferrero, K. Van Houcke, F. Werner, arXiv:2103.12038

Calculation of Fermi polaron properties with unprecedented precision + Large-order asymptotics

Van Houcke, Werner, Rossi, Phys. Rev. B 101, 045134 (2020).

OUTLOOK

HUBBARD MODEL

- Bare vertex → ladders [CDET (normal phase): Simkovic et al. 2020]
  → strong coupling
  Polarized SF at T=0? (« breached-pair»)

- FFLO? (MF:yes)

  [U>0 : stripes]

- 2D: BKT, algebraic order at T>0

- d-wave superconding phase for U>0

UNITARY GAS:

- normal phase: polarized gas; spectral function ..

- SF phase: Borel resummable?