Introduction to stochastic processes

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A quick introduction to probability theory

Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

- Goal: Model a random experiment.

Example 1: toss a coin.

Example 2: pick a number between 0 and 1 (matlab function `rand`).

- Ω: fundamental set (set of all possible realisations).

$\hookrightarrow$ A realization ω is an element of Ω.

Example 1: $\Omega = \{H, T\}$.

Example 2: $\Omega = [0, 1]$.

- $\mathcal{A}$: σ-algebra on Ω (set of events).

Example 1: $\mathcal{A} = \mathcal{P}(\Omega)$;

an event: the result is head, $A = \{H\}$.

Example 2: $\mathcal{A} = \mathcal{B}([0, 1])$;

an event: the number is larger than 1/2, $A = [1/2, 1]$.

- $\mathbb{P}$: probability (gives the probability of events): function from $\mathcal{A}$ to $[0, 1]$ such that:

- $\mathbb{P}(\Omega) = 1$,
- if $(A_j)_{j \geq 1}$ is a numerable family of disjoint sets of $\mathcal{A}$ then $\mathbb{P}(\cup_j A_j) = \sum_j \mathbb{P}(A_j)$.

Example 1: $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}$.

Example 2: $\mathbb{P}([a, b]) = b - a$ for all $0 \leq a \leq b \leq 1$.

Random variable

- Random variable X = random number.

Application $X : \Omega \to \mathbb{R}$.

A realization $X(\omega)$ of a random variable is a real number.

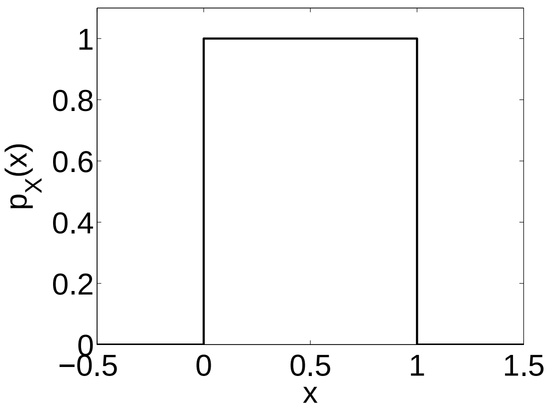
- The distribution of a random variable is characterized by moments of the form $\mathbb{E}[\phi(X)]$ for $\phi \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$:

$$\mathbb{E}[\phi(X)] = \int_\Omega \phi(X(\omega))\mathbb{P}(d\omega)$$

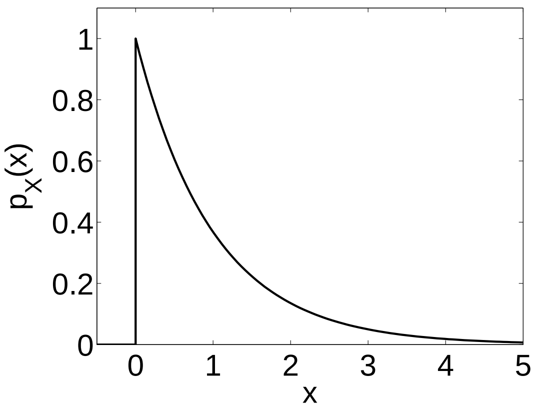
- The distribution of a (continuous) random variable is characterized by the probability density function (pdf) p_X:

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) p_X(x) dx$$

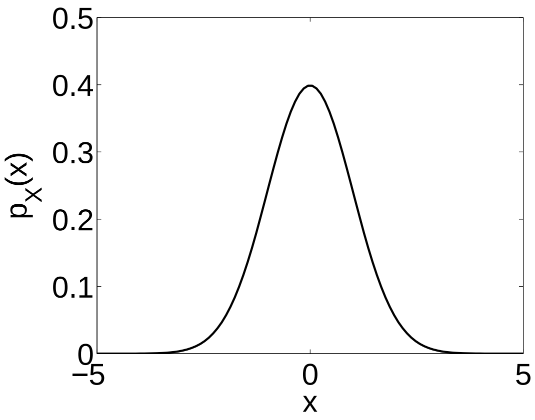
- Usual pdfs:



uniform



exponential



Gaussian

- The mean (expectation) of a random variable X with pdf $p_X(x)$ is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x p_X(x) dx$$

The variance of a random variable X with pdf $p(x)$ is

$$\mathrm{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_{-\infty}^{\infty} x^2 p_X(x) dx - \Big(\int_{-\infty}^{\infty} x p_X(x) dx\Big)^2$$

The variance measures the dispersion of the random variable (around its mean).

- A standard Gaussian random variable X has the pdf

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)$$

Its mean is $\mathbb{E}[X] = 0$ and its variance is $\mathrm{Var}(X) = 1$.
We write $X \sim \mathcal{N}(0, 1)$.

- A Gaussian random variable X with mean μ and variance σ^2 has the pdf

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$$

We write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Random vector

- n-dimensional random vector $\boldsymbol{X}$ = collection of n random variables $(X_1, \ldots, X_n)$.

Application $\boldsymbol{X} : \Omega \to \mathbb{R}^n$.

A realization $\boldsymbol{X}(\omega)$ of a random vector is a vector in $\mathbb{R}^n$.

The distribution of a (continuous) random vector is characterized by the pdf $p_{\boldsymbol{X}}$:

$$\mathbb{E}[\phi(\boldsymbol{X})] = \int_{\mathbb{R}^n} \phi(\boldsymbol{x}) p_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}, \qquad \forall \phi \in \mathcal{C}_b(\mathbb{R}^n, \mathbb{R})$$

The vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is independent if

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{j=1}^{n} p_{X_j}(x_j)$$

or equivalently

$$\mathbb{E}\big[\phi_1(X_1) \cdots \phi_n(X_n)\big] = \mathbb{E}\big[\phi_1(X_1)\big] \cdots \mathbb{E}\big[\phi_n(X_n)\big], \qquad \forall \phi_1, \ldots, \phi_n \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$$

Example: a normalized Gaussian random vector $\boldsymbol{X}$ has the Gaussian pdf

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n}} \exp\Big(-\frac{|\boldsymbol{x}|^2}{2}\Big)$$

It is a vector of independent random normalized Gaussian variables.

Limit theorems

- **Law of Large Numbers.**

Let $(X_n)_{n\geq 0}$ be independent and identically distributed (i.i.d.) random variables. If $\mathbb{E}[|X_1|] < \infty$, then

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + ... + X_n) \overset{n\to\infty}{\longrightarrow} m \text{ almost surely, with } m = \mathbb{E}[X_1]$$

"The empirical mean converges to the statistical mean".

- **Central Limit Theorem.** Fluctuations theory.

Let $(X_n)_{n\geq 0}$ be i.i.d. random variables. If $\mathbb{E}[X_1^2] < \infty$, then

$$\sqrt{n}\left(\bar{X}_n - m\right) = \sqrt{n}\left(\frac{1}{n}(X_1 + X_2 + ... + X_n) - m\right) \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2) \text{ in law}$$

$$\text{where } \begin{cases} m = \mathbb{E}[X_1] \\ \sigma^2 = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] \end{cases}$$

"For large n, the error $\bar{X}_n - m$ obeys the Gaussian distribution $\mathcal{N}(0,\sigma^2/n)$."

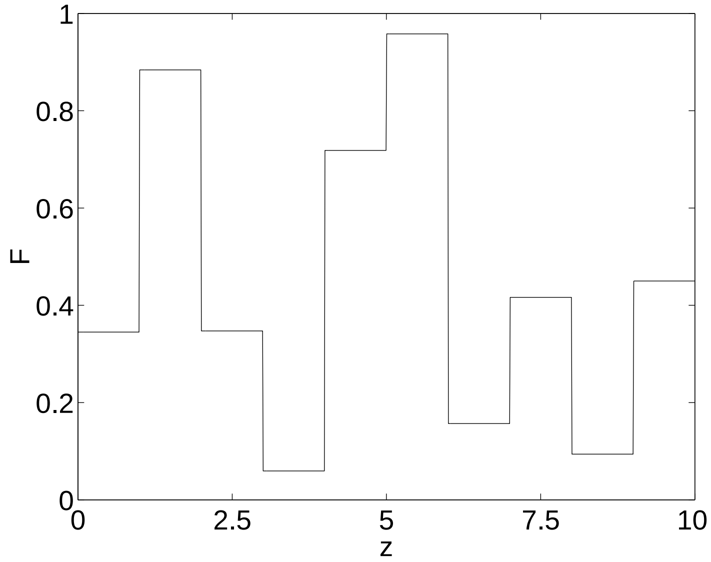
Stochastic processes

Toy model

Let $F(z) \in \mathbb{R}$ be the stepwise constant random process

$$F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$$

where F_i independent random variables $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.



Here F_i are independent with uniform distribution over $(0, 1)$.

Random process

- Random variable X = random number.

A realization of the random variable = a real number.

Distribution of X characterized by moments of the form $\mathbb{E}[\phi(X)]$ where $\phi \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$.

- Stochastic process $(\mu(\boldsymbol{x}))_{\boldsymbol{x} \in \mathbb{R}^d}$ = random function.

A realization of the process = a function from $\mathbb{R}^d$ to $\mathbb{R}$.

Distribution of $(\mu(\boldsymbol{x}))_{\boldsymbol{x} \in \mathbb{R}^d}$ characterized by moments of the form $\mathbb{E}[\phi(\mu(\boldsymbol{x}_1), \dots, \mu(\boldsymbol{x}_n))]$, for any $n \in \mathbb{N}^*$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$, $\phi \in \mathcal{C}_b(\mathbb{R}^n, \mathbb{R})$.

Example: Gaussian process.

Gaussian process

- Gaussian process $(\mu(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ characterized by its first two moments $m(\boldsymbol{x}_1) = \mathbb{E}[\mu(\boldsymbol{x}_1)]$ and $R(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{E}[\mu(\boldsymbol{x}_1)\mu(\boldsymbol{x}_2)]$.

Any linear combination $\mu_\lambda = \sum_{i=1}^n \lambda_i \mu(\boldsymbol{x}_i)$ has Gaussian distribution $\mathcal{N}(m_\lambda, \sigma_\lambda^2)$ with

$$m_\lambda = \sum_{i=1}^n \lambda_i \mathbb{E}[\mu(\boldsymbol{x}_i)] \quad \text{and} \quad \sigma_\lambda^2 = \sum_{i,j=1}^n \lambda_i\lambda_j \mathbb{E}[\mu(\boldsymbol{x}_i)\mu(\boldsymbol{x}_j)] - m_\lambda^2$$

- Simulation: in order to simulate $(\mu(\boldsymbol{x}_1), \dots, \mu(\boldsymbol{x}_n))$:

- evaluate the mean vector $M_i = \mathbb{E}[\mu(\boldsymbol{x}_i)]$ and the covariance matrix $C_{ij} = \mathbb{E}[\mu(\boldsymbol{x}_i)\mu(\boldsymbol{x}_j)] - \mathbb{E}[\mu(\boldsymbol{x}_i)]\mathbb{E}[\mu(\boldsymbol{x}_j)]$.
- generate a random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $\boldsymbol{Y} = \boldsymbol{M} + \mathbf{C}^{1/2}\boldsymbol{X}$. The vector $\boldsymbol{Y}$ has the distribution of $(\mu(\boldsymbol{x}_1), \dots, \mu(\boldsymbol{x}_n))$.

Note: the computation of the square root is expensive (use Cholesky method).

Brownian motion

- Brownian motion $(W_z)_{z\geq 0}$ (starting from 0)= real Gaussian process with mean 0 and covariance function

$$\mathbb{E}[W_z W_{z'}] = z \wedge z'$$

The realizations of the Brownian motion are continuous but not differentiable.

The increments of the Brownian motion are independent:
if $z_n \geq z_{n-1} \geq \cdots \geq z_1 \geq z_0 = 0$, then $(W_{z_n} - W_{z_{n-1}}, \dots, W_{z_2} - W_{z_1}, W_{z_1})$ are independent Gaussian random variables with mean 0 and variance

$$\mathbb{E}\big[(W_{z_j} - W_{z_{j-1}})^2\big] = z_j - z_{j-1}$$

- Simulation: in order to simulate $(W_h, W_{2h}, ..., W_{nh})$:

- evaluate the covaraince matrix $\mathbf{C} = (C_{jl})_{j=,l=1,\dots,n}$ with $C_{jl} = (j \wedge l)h$.
- generate a random vector $\boldsymbol{X} = (X_1, ..., X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $\boldsymbol{Y} = \mathbf{C}^{1/2}\boldsymbol{X}$.

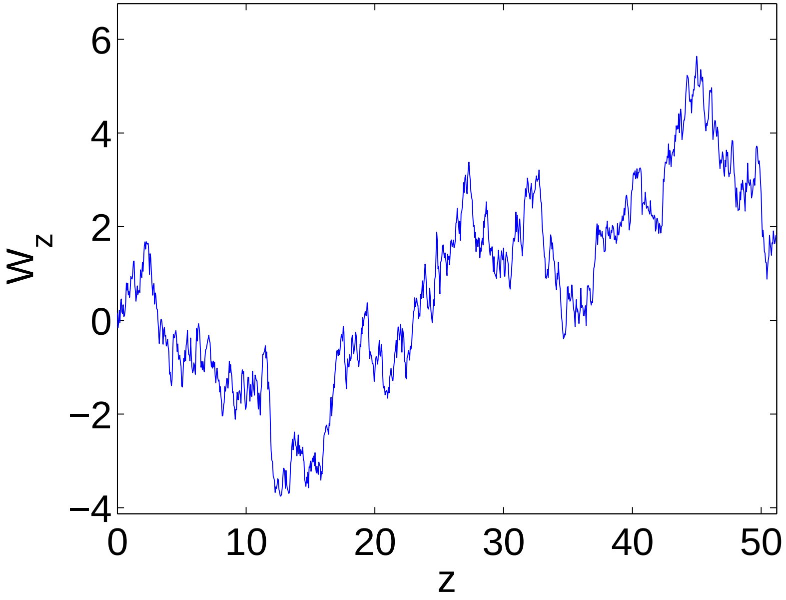
The vector $\boldsymbol{Y}$ has the distribution of $(W_h, W_{2h}, ..., W_{nh})$.

- Simulation: in order to simulate $(W_h, W_{2h}, \ldots, W_{nh})$:

- generate a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)^T$ of n independent Gaussian random variables with mean 0 and variance 1.

- compute $Y_j = \sqrt{h} \sum_{i=1}^{j} X_i$.

The vector $\boldsymbol{Y}$ has the distribution of $(W_h, W_{2h}, \ldots, W_{nh})^T$.



Stationary random process

- $(\mu(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ is stationary if $(\mu(\boldsymbol{x}+\boldsymbol{x}_0))_{\boldsymbol{x}\in\mathbb{R}^d}$ has the same distribution as $(\mu(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ for any $\boldsymbol{x}_0 \in \mathbb{R}^d$.

Sufficient and necessary condition:

$$\mathbb{E}\big[\phi(\mu(\boldsymbol{x}_1),\dots,\mu(\boldsymbol{x}_n))\big] = \mathbb{E}\big[\phi(\mu(\boldsymbol{x}_0+\boldsymbol{x}_1),\dots,\mu(\boldsymbol{x}_0+\boldsymbol{x}_n))\big]$$

for any n, $\boldsymbol{x}_0,\dots,\boldsymbol{x}_n \in \mathbb{R}^d$, $\phi \in \mathcal{C}_b(\mathbb{R}^n,\mathbb{R})$.

- Example: Gaussian process $\mu(\boldsymbol{x})$ with mean zero $\mathbb{E}[\mu(\boldsymbol{x})] = 0 \ \forall \boldsymbol{x}$ and covariance function $\mathbb{E}[\mu(\boldsymbol{x}')\mu(\boldsymbol{x}'+\boldsymbol{x})] = c(\boldsymbol{x})$.

- Bochner's theorem: a function $c(\boldsymbol{x})$ is a covariance function of a stationary process if and only if its Fourier transform $\hat{c}(\boldsymbol{k})$ is nonnegative.

$$\hat{c}(\boldsymbol{k}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} c(\boldsymbol{x}) d\boldsymbol{x}$$

- Spectral representation (of real-valued stationary Gaussian process):

$$\mu(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sqrt{\hat{c}(\boldsymbol{k})}\hat{n}_{\boldsymbol{k}} d\boldsymbol{k}$$

with $\hat{n}_{\boldsymbol{k}}$ complex white noise, i.e.:
$\hat{n}_{\boldsymbol{k}}$ complex-valued, Gaussian, $\hat{n}_{-\boldsymbol{k}} = \overline{\hat{n}_{\boldsymbol{k}}}$, $\mathbb{E}\left[\hat{n}_{\boldsymbol{k}}\right] = 0$, and $\mathbb{E}\left[\hat{n}_{\boldsymbol{k}}\overline{\hat{n}_{\boldsymbol{k}'}}\right] = (2\pi)^d \delta(\boldsymbol{k} - \boldsymbol{k}')$.
(the representation is formal, one should use stochastic integrals $d\hat{W}_{\boldsymbol{k}} = \hat{n}_{\boldsymbol{k}} d\boldsymbol{k}$).

We have $\hat{n}_{\boldsymbol{k}} = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} n(\boldsymbol{x}) d\boldsymbol{x}$ where $n(\boldsymbol{x})$ is a real white noise, i.e.:
$n(\boldsymbol{x})$ real-valued, Gaussian, $\mathbb{E}\left[n(\boldsymbol{x})\right] = 0$, and $\mathbb{E}\left[n(\boldsymbol{x})n(\boldsymbol{x}')\right] = \delta(\boldsymbol{x} - \boldsymbol{x}')$.
(in 1D, formally, $n(x) = dW_x/dx$).

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(in 1D, formally, $n(x) = dW_x/dx$).

- Simulation ($d = 1$): in order to simulate $(\mu(x_1), \ldots, \mu(x_n))$, $x_j = (j-1)h$:

- compute the covariance vector $\boldsymbol{C} = (c(x_1), \ldots, c(x_n))$.
- generate a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- filter with the square root of the Fourier transform of $\boldsymbol{C}$:

$$\boldsymbol{Y} = \mathrm{IDFT}\big(\sqrt{\mathrm{DFT}(\boldsymbol{C})} \times \mathrm{DFT}(\boldsymbol{X})\big)$$

$\hookrightarrow$ $\boldsymbol{Y}$ is a realization of $(\mu(x_1), \ldots, \mu(x_n))$ (in practice, use FFT and IFFT).

Random differential equations and ordinary differential equations

Goal: determine the limit $\lim_{\varepsilon \to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

for a fairly general random process $(F(z))_{z \geq 0}$.

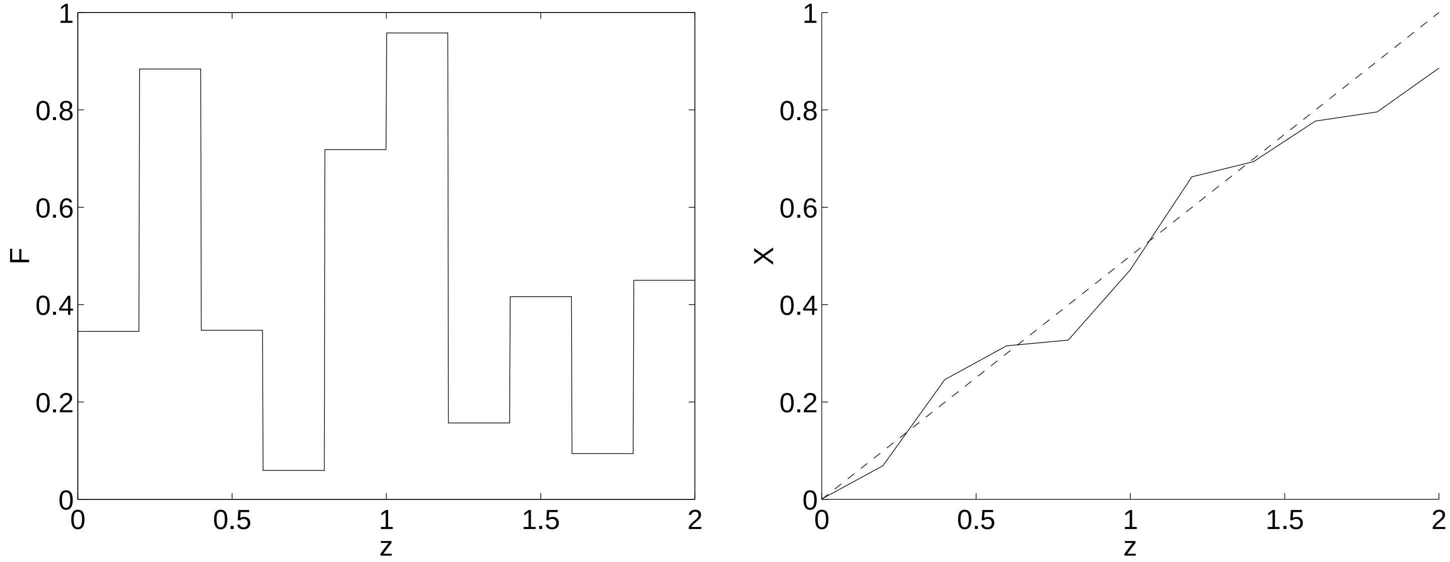
Method of averaging: Toy model

Let $X^\varepsilon(z) \in \mathbb{R}$ be the solution of

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon})$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.

($z \mapsto t$, particle in a random velocity field)

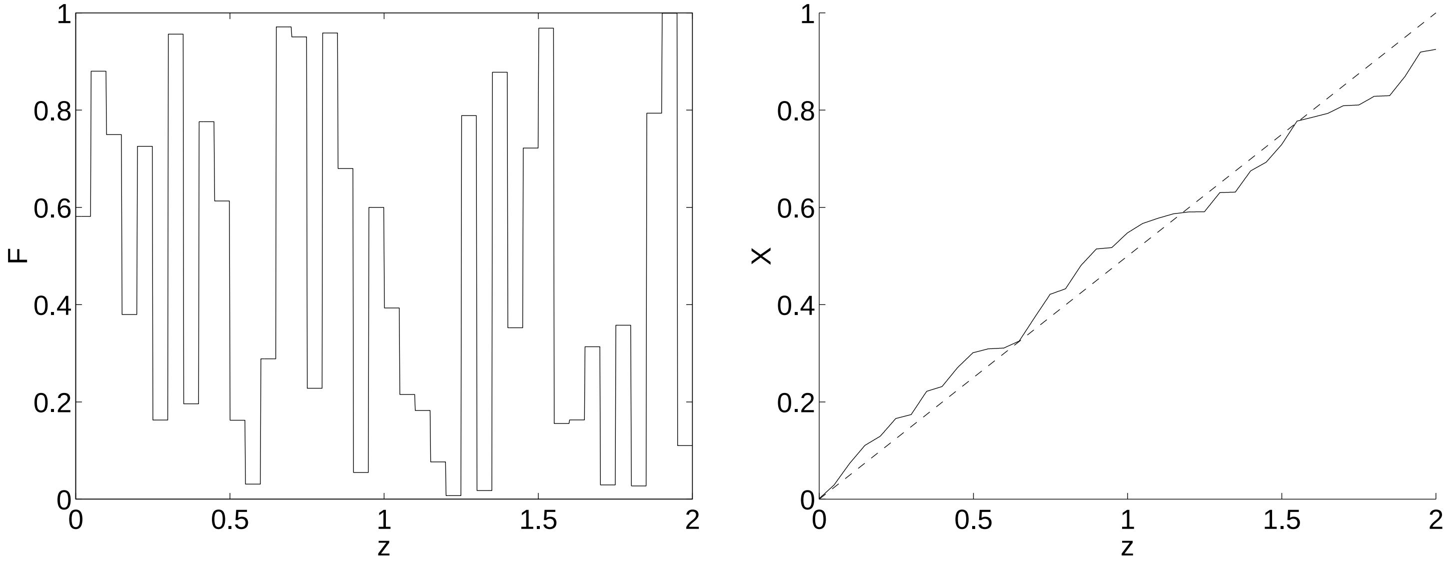


$\varepsilon = 0.2$

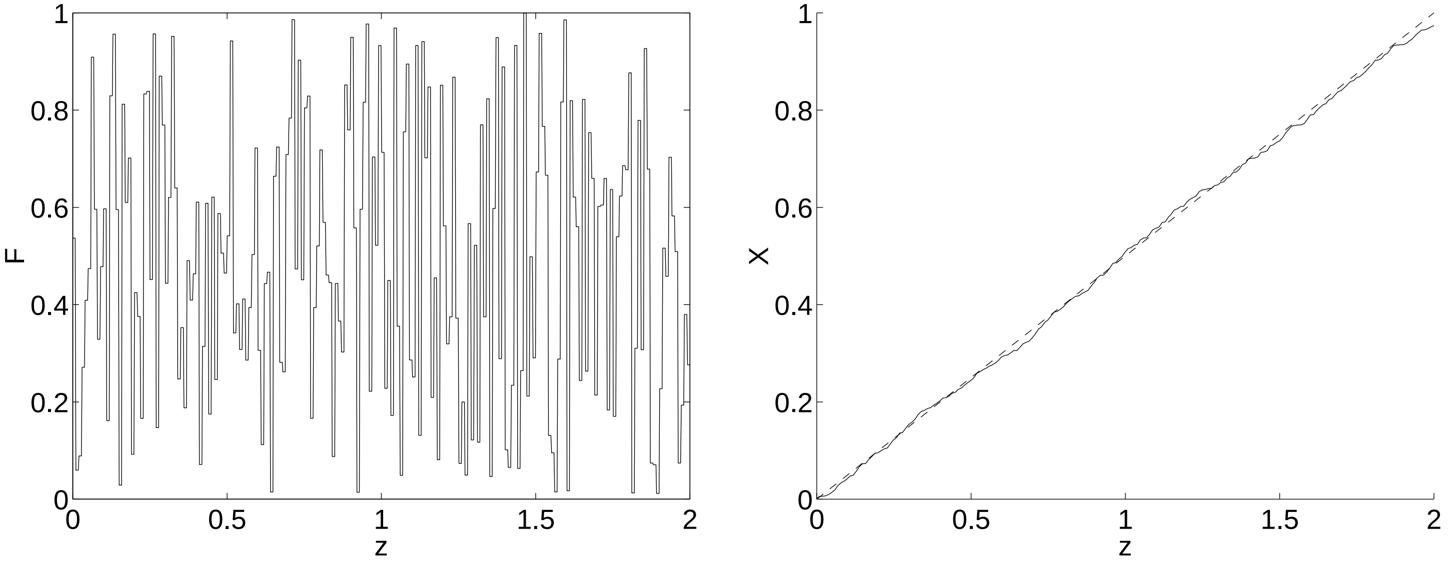
$$\begin{aligned} X^{\varepsilon}(z) &= \int_0^z F\left(\frac{s}{\varepsilon}\right) ds = \varepsilon \int_0^{\frac{z}{\varepsilon}} F(s) ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon}\right]}^{\frac{z}{\varepsilon}} F(s) ds \\ &= \underset{\begin{array}{c} {\color{blue}\varepsilon \to 0} \downarrow \\ z \end{array}}{\varepsilon \left[\frac{z}{\varepsilon}\right]} \quad \times \quad \underset{\begin{array}{c} \text{a.s.} \downarrow (LLN) \\ \mathbb{E}[F_1] = \bar{F} \end{array}}{\frac{1}{\left[\frac{z}{\varepsilon}\right]} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right)} \quad + \quad \underset{\begin{array}{c} \text{a.s.} \downarrow \\ 0 \end{array}}{\varepsilon \left(\frac{z}{\varepsilon} - \left[\frac{z}{\varepsilon}\right] \right) F_{\left[\frac{z}{\varepsilon}\right]}} \end{aligned}$$

Thus:

$$X^{\varepsilon}(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad {\color{magenta}\frac{d\bar{X}}{dz} = \bar{F}}.$$



$$\varepsilon = 0.05$$



$$\varepsilon = 0.01$$

Goal: determine the limit $\lim_{\varepsilon \to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

for a stationary random process $(F(z))_{z \geq 0}$.

The previous analysis can be extended provided

$$\frac{1}{Z} \int_0^Z F(z) dz \overset{Z \to \infty}{\longrightarrow} \bar{F}$$

(i.e. F is ergodic)

Next goal: determine the limit $\lim_{\varepsilon \to 0} \boldsymbol{X}^{\varepsilon}(z)$ where

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \boldsymbol{F}\left(\frac{z}{\varepsilon}, \boldsymbol{X}^{\varepsilon}(z)\right)$$

Method of averaging: Khasminskii theorem

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \boldsymbol{F}(\frac{z}{\varepsilon}, \boldsymbol{X}^{\varepsilon}), \quad \boldsymbol{X}^{\varepsilon}(0) = \boldsymbol{x}_0$$

Assume:

$\boldsymbol{x} \mapsto \boldsymbol{F}(z, \boldsymbol{x})$ is Lipschitz,

$z \mapsto \boldsymbol{F}(z, \boldsymbol{x})$ is stationary and ergodic.

Define:

$$\bar{\boldsymbol{F}}(\boldsymbol{x}) = \mathbb{E}[\boldsymbol{F}(z, \boldsymbol{x})]$$

Let $\bar{\boldsymbol{X}}$ be the solution of

$$\frac{d\bar{\boldsymbol{X}}}{dz} = \bar{\boldsymbol{F}}(\bar{\boldsymbol{X}}), \quad \bar{\boldsymbol{X}}(0) = \boldsymbol{x}_0$$

Theorem: for any $Z > 0$,

$$\sup_{z \in [0,Z]} \mathbb{E}\left[|\boldsymbol{X}^{\varepsilon}(z) - \bar{\boldsymbol{X}}(z)|\right] \xrightarrow{\varepsilon \to 0} 0$$

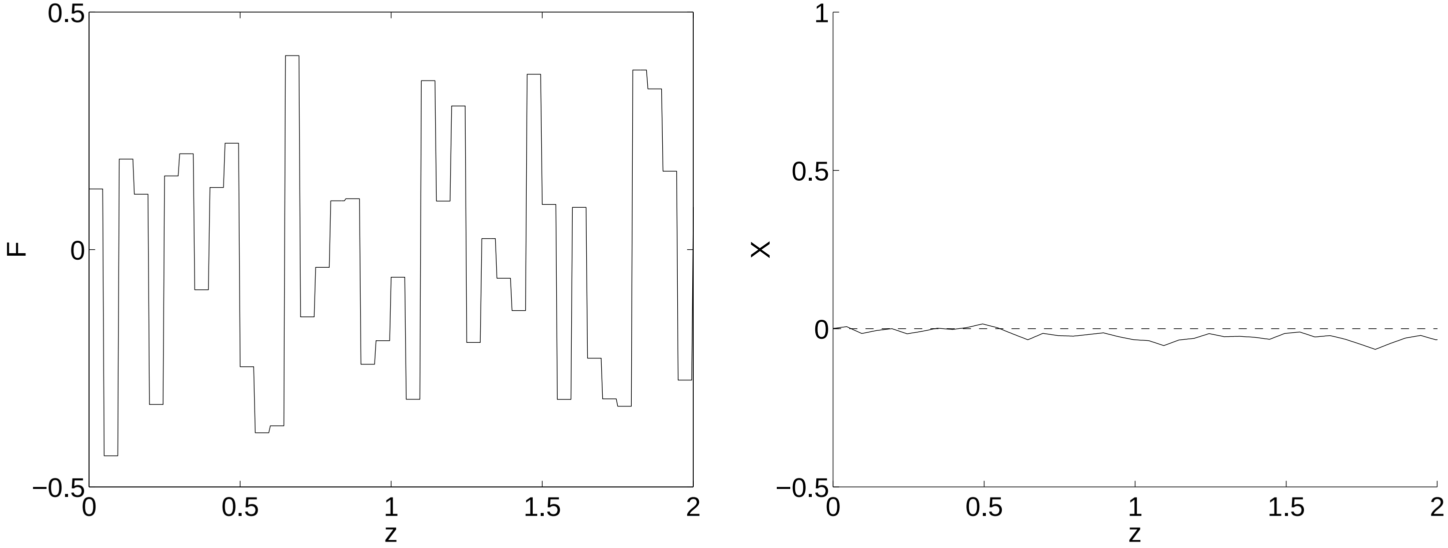
[1] R. Z. Khasminskii, Theory Probab. Appl. **11** (1966), 211-228.

Random differential equations and Brownian motion

Toy model

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon})$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F} = 0$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.



$\varepsilon = 0.05$

For any $z \in [0, Z]$, we have

$$X^\varepsilon(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F} = 0.$$

No macroscopic evolution is noticeable.

→ it is necessary to look at larger z to get an effective behavior

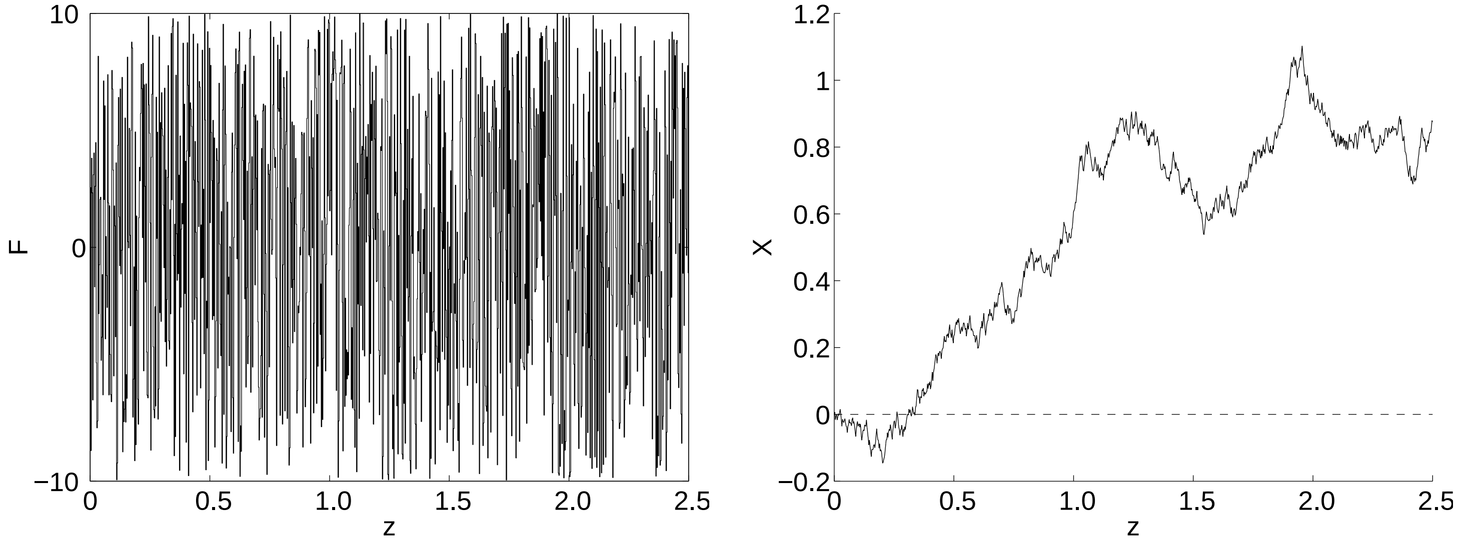
$$z \mapsto \frac{z}{\varepsilon}, \qquad \tilde{X}^\varepsilon(z) = X^\varepsilon(\frac{z}{\varepsilon})$$

$$\frac{d\tilde{X}^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

Diffusion-approximation: Toy model

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = 0$ and $\mathbb{E}[F_i^2] = \sigma^2$.



$\varepsilon = 0.05$

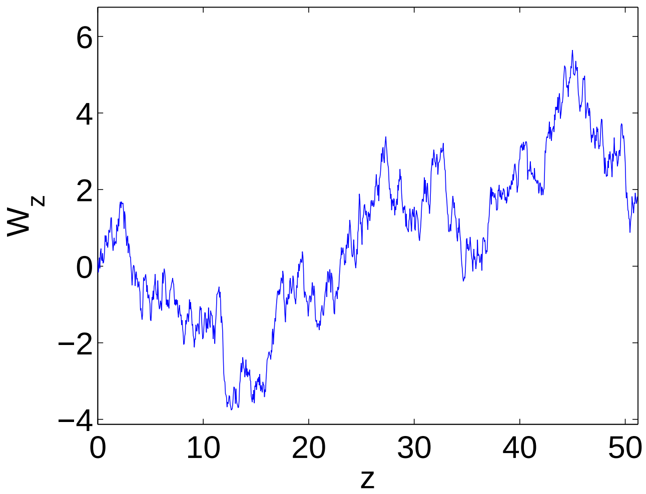
$$\begin{aligned} X^{\varepsilon}(z) &= \int_0^z \frac{1}{\varepsilon} F(\frac{s}{\varepsilon^2}) ds = \varepsilon \int_0^{\frac{z}{\varepsilon^2}} F(s) ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon^2}\right]}^{\frac{z}{\varepsilon^2}} F(s) ds \\ &= \underset{\substack{\varepsilon \to 0 \downarrow \\ \sqrt{z}}}{\varepsilon \sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \times \underset{\substack{\text{law} \downarrow (CLT) \\ \mathcal{N}(0,\sigma^2)}}{\frac{1}{\sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right)} + \underset{\substack{\text{a.s.} \downarrow \\ 0}}{\varepsilon \left(\frac{z}{\varepsilon^2} - \left[\frac{z}{\varepsilon^2}\right] \right) F_{\left[\frac{z}{\varepsilon^2}\right]}} \end{aligned}$$

Thus: $X^{\varepsilon}(z)$ converges in distribution as $\varepsilon \to 0$ to $\bar{X}(z)$ whose distribution is $\mathcal{N}(0, \sigma^2 z)$.

With some more work: The process $(X^{\varepsilon}(z))_{z \geq 0}$ converges in distribution to a Brownian motion $(\sigma W_z)_{z \geq 0}$.

Diffusion processes and stochastic differential equations

Example: Brownian motion



W_z (issued from 0): zero-mean Gaussian process with covariance $\mathbb{E}[W_z W_{z'}] = z \wedge z'$

Its increments are independent and: $\mathbb{E}[(W_{z+h} - W_z)^2] = h$

Let ϕ be a bounded real function:

$$\begin{aligned} u(z,x) \quad &:= \quad \mathbb{E}[\phi(x+W_z)] = \int \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{w^2}{2z}\right) \phi(x+w) dw \\ &= \quad \int \underbrace{\frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right)}_{p_z(x,y)} \phi(y) dy \end{aligned}$$

For x, z fixed, $y \mapsto p_z(x,y)$ is the pdf of the random variable $x + W_z$.
$p_z(x,y)$ is the kernel of the heat operator.

Example: Brownian motion

- The moment

$$u(z,x) := \mathbb{E}[\phi(x+W_z)] = \int p_z(x,y)\phi(y)dy, \qquad p_z(x,y) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right)$$

satisfies the (backward) Kolmogorov equation

$$\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u(z=0,x) = \phi(x)$$

Reciprocal: The partial differential equation

$$\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u(z=0,x) = \phi(x)$$

has the probabilistic representation: $u(z,x) = \mathbb{E}[\phi(x+W_z)]$.
The reciprocal is useful for Monte Carlo simulation techniques for solving PDEs.

- The pdf $y \mapsto p_z(x,y)$ of $x + W_z$ satisfies the Fokker-Planck (Kolmogorov forward) equation as a function of z and y:

$$\frac{\partial p_z}{\partial z} = \frac{1}{2}\frac{\partial^2 p_z}{\partial y^2}, \qquad p_{z=0}(x,y) = \delta(y-x)$$

Stochastic differential equations

Let $X(z)$ be the solution of the one-dimensional stochastic differential equation

$$X(z) = x + \int_0^z \sigma(X(s))dW_s + \int_0^z b(X(s))ds$$

Existence and uniqueness ensured provided b and σ are $\mathcal{C}^1$ with bounded derivatives.
$X(z)$ is continuous a.s., is squared integrable, is adapted (depends on $(W_s)_{0\leq s\leq z}$).

- Itô integral

$$\int_0^z \phi(X(s))dW_s = \lim_{\Delta z\to 0} \sum_k \phi(X(z_k))(W_{z_{k+1}} - W_{z_k}), \qquad 0 = z_0 < z_1 < \cdots < z_n = z$$

Good properties: martingale (mean zero), Itô's isometry:

$$\mathbb{E}\Big[\Big(\int_0^z \phi(X(s))dW_s\Big)^2\Big] = \int_0^z \mathbb{E}[\phi(X(s))^2]ds$$

- Stratonovich integral:

$$\int_0^z \phi(X(s)) \circ dW_s = \lim_{\Delta z\to 0} \sum_k \frac{\phi(X(z_{k+1})) + \phi(X(z_k))}{2}(W_{z_{k+1}} - W_{z_k})$$

- Relation:

$$\int_0^z \phi(X(s)) \circ dW_s = \int_0^z \phi(X(s))dW_s + \frac{1}{2}\int_0^z \sigma(X(s))\phi'(X(s))ds$$

$$X(z) = x + \int_0^z \sigma(X(s))dW_s + \int_0^z b(X(s))ds$$

- Itô's formula:

$$\begin{aligned}\phi(X(z)) &= \phi(x) + \int_0^z \sigma(X(s))\phi'(X(s))dW_s + \int_0^z b(X(s))\phi'(X(s))ds \\ &\quad + \frac{1}{2}\int_0^z \sigma(X(s))^2\phi''(X(s))ds\end{aligned}$$

- $X(z)$ is a diffusion process with the generator $Q = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$
- The moment $u(z,x) := \mathbb{E}\big[\phi(X(z))|X(0) = x\big]$ satisfies the Kolmogorov equation:

$$\frac{\partial u}{\partial z} = Qu, \qquad u(z = 0, x) = \phi(x)$$

- The pdf $y \mapsto p_z(x,y)$ of $X(z)$ (starting from $X(0) = x$) satisfies the Fokker-Planck equation as a function of z and y:

$$\frac{\partial p_z}{\partial z} = Q^* p_z, \qquad p_{z=0}(x,y) = \delta(x-y)$$

where Q^* is the adjoint operator of Q:

$$Q^* p(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(y)p(y)\big) - \frac{\partial}{\partial y}\big(b(y)p(y)\big)$$

Diffusion processes

- Let σ and b be $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions with bounded derivatives.

Let W_z be a Brownian motion.

The solution $X(z)$ of the 1D stochastic differential equation:

$$dX(z) = \sigma(X(z))dW_z + b(X(z))dz$$

is a diffusion process with the generator

$$Q = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

- Let $\boldsymbol{\sigma} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n\times m})$ and $\boldsymbol{b} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ with bounded derivatives.

Let $\boldsymbol{W}_z$ be a m-dimensional Brownian motion.

The solution $\boldsymbol{X}(z)$ of the stochastic differential equation:

$$d\boldsymbol{X}(z) = \boldsymbol{\sigma}(\boldsymbol{X}(z))d\boldsymbol{W}_z + \boldsymbol{b}(\boldsymbol{X}(z))dz$$

is a diffusion process with the generator

$$Q = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x})\frac{\partial}{\partial x_i}$$

with $\mathbf{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$.

Random differential equations and stochastic differential equations

Next goal: determine the limit $\lim_{\varepsilon \to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\frac{z}{\varepsilon^2}\right)$$

for a fairly general process $(F(z))_{z \geq 0}$ with $\mathbb{E}[F(z)] = 0$.

Write

$$X^\varepsilon(z) = \frac{1}{\varepsilon}\int_0^z F\left(\frac{s}{\varepsilon^2}\right)ds = \varepsilon\sqrt{\frac{z}{\varepsilon^2}} \times \frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds$$

We know that

$$\begin{aligned}
\mathbb{E}\Big[\frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds\Big] &= 0, \\
\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big(\frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds\Big)^2\Big] &= \lim_{Z\to\infty} Z\mathbb{E}\left[\Big(\frac{1}{Z}\int_0^Z F(s)ds\Big)^2\right] \\
&= 2\int_0^\infty \mathbb{E}[F(0)F(u)]du,
\end{aligned}$$

The Gaussian property of the limit of $X^\varepsilon(z)$ is ensured by an invariance principle.

Conclusion:

$$\frac{1}{\varepsilon}\int_0^z F\big(\frac{s}{\varepsilon^2}\big)ds \xrightarrow{\varepsilon\to 0} \sqrt{2}\sigma W_z$$

in distribution, where $(W_z)_{z\geq 0}$ is a Brownian motion and

$$\sigma^2 = \int_0^\infty \mathbb{E}[F(0)F(u)]du$$

Next goal: determine the limit $\lim_{\varepsilon \to 0} \boldsymbol{X}^{\varepsilon}(z)$ where

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \frac{1}{\varepsilon} \boldsymbol{F}\left(\frac{z}{\varepsilon^2}, \boldsymbol{X}^{\varepsilon}(z)\right)$$

when

$$\mathbb{E}[\boldsymbol{F}(z, \boldsymbol{x})] = 0 \quad \forall \boldsymbol{x}$$

Diffusion-approximation

$$\frac{d\boldsymbol{X}^\varepsilon}{dz}(z) = \frac{1}{\varepsilon}\boldsymbol{F}\left(\frac{z}{\varepsilon^2}, \boldsymbol{X}^\varepsilon(z)\right), \qquad \boldsymbol{X}^\varepsilon(0) = \boldsymbol{x}_0 \in \mathbb{R}^d.$$

$\boldsymbol{F}$ stationary, centered, and ergodic (in z): $\mathbb{E}[\boldsymbol{F}(z,\boldsymbol{x})] = \boldsymbol{0}$.

Theorem: *The processes* $(\boldsymbol{X}^\varepsilon(z))_{z\geq 0}$ *converge in distribution in* $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ *to the diffusion process* $\boldsymbol{X}$ *with the generator* $\mathcal{L}$*:*

$$\mathcal{L} = \sum_{i,j=1}^d a_{ij}(\boldsymbol{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(\boldsymbol{x})\frac{\partial}{\partial x_j}$$

with

$$\begin{aligned} a_{ij}(\boldsymbol{x}) &= \int_0^\infty \mathbb{E}\left[F_i(0,\boldsymbol{x})F_j(u,\boldsymbol{x})\right] du \\ b_j(\boldsymbol{x}) &= \sum_{i=1}^d \int_0^\infty \mathbb{E}\left[F_i(0,\boldsymbol{x})\partial_{x_i}F_j(u,\boldsymbol{x})\right] du \end{aligned}$$

It means that the pdf $p_z(\boldsymbol{x})$ of $\boldsymbol{X}(z)$ satisfies:

$$\frac{\partial p_z}{\partial z} = \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}\big(a_{ij}(\boldsymbol{x})p_z\big) - \sum_{j=1}^d \frac{\partial}{\partial x_j}\big(b_j(\boldsymbol{x})p_z\big), \qquad p_{z=0}(\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)$$

Diffusion-approximation - the one-dimensional case

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon} f\left(\frac{z}{\varepsilon^2}\right) h\left(X^\varepsilon(z)\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}.$$

f stationary, centered, and ergodic: $\mathbb{E}[f(z)] = 0$.

Theorem: *The processes* $(X^\varepsilon(z))_{z\geq 0}$ *converge in distribution in* $\mathcal{C}^0([0,\infty),\mathbb{R})$ *to the diffusion process* X *with the generator* $\mathcal{L}$*:*

$$\mathcal{L} = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

with

$$a(x) = \frac{1}{2}\sigma^2 h^2(x), \qquad b(x) = \frac{1}{2}\sigma^2 h(x)h'(x), \qquad \sigma^2 = 2\int_0^\infty \mathbb{E}\left[f(0)f(u)\right] du$$

It means that $X(z)$ is the solution of the SDE

$$dX(z) = \sigma h(X(z))dW_z + b(X(z))dz$$

or

$$dX(z) = \sigma h(X(z)) \circ dW_z$$

Remember that $\int_0^z \frac{1}{\varepsilon} f\left(\frac{s}{\varepsilon^2}\right) ds$ converges in distribution to σW_z.
↪ the "natural" integral is Stratonovich.