Introduction to stochastic Schrödinger equations.

Workshop: The stochastic Schrödinger equations in selected physics problems.

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We consider the focusing nonlinear Schrödinger equations in dimension d and with general power nonlinearity:

$$\begin{cases} i\dfrac{du}{dt} + \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^d. \end{cases}$$

For instance if $d = \sigma = 1$, this is a common model for the propagation in optical fibers (where x is the time and t is the longitudinal coordinate).

Existence, uniqueness of solutions :

$\rightarrow$ global in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ if $\sigma < 2/d$.

$\rightarrow$ local in $H^1(\mathbb{R}^d)$ if $2/d \leq \sigma < 2/(d-2)$ (∞ si $d = 1, 2$).

$\rightarrow$ blow-up occurs if $\sigma \geq 2/d$.

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Conserved quantities:

- Mass: $M(u) = \int_{\mathbb{R}^d} |u|^2 dx$

- Energy: $H(u) = \dfrac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \dfrac{1}{2\sigma+2}\int_{\mathbb{R}^d} |u|^{2\sigma+2} dx$

If $\sigma < 2/d$, $H(u) \geq c_1 \int_{\mathbb{R}^d} |\nabla u|^2 dx - c_2 \left(\int_{\mathbb{R}^d} |u|^2 dx\right)^k$.

$\rightarrow$ global existence.

Dispersion:

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- Inverse Fourier transform:

$$v(x,t) = \frac{1}{(4i\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(i \frac{|x-y|^2}{4t} \right) u_0(y) dy.$$

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$$U(t) = e^{it\Delta} \longrightarrow |U(t)v(\cdot,t)|_{L^{2\sigma+2}(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}}|v|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)}$$

$\rightsquigarrow$ allows to solve the nonlinear equation under the integral form:

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)|u(s)|^{2\sigma}u(s)ds.$$

A first model:

If the dispersion is small and random and nonlinear effect are small:

$$\begin{cases} i\frac{dv^\epsilon}{dt} + \epsilon m(t)\Delta v^\epsilon + \epsilon^2 |v^\epsilon|^{2\sigma} v^\epsilon = 0,\, x \in \mathbb{R}^d,\, t > 0 \\ \\ v^\epsilon(0) = v_0, \end{cases}$$

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If m is sufficiently mixing, formally when $\epsilon \to 0$:

$$\begin{cases} i\frac{du}{dt} + \dot{\beta} o \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ \\ u(0) = u_0. \end{cases}$$

The time white noise. :

β a brownian motion, *i.e.* $\beta(t,\omega)$ satisfies:

- $(\beta(t_1), \beta(t_2), \ldots, \beta(t_n))$ gaussian vector for any $t_1, \ldots, t_n$

- Independent increments: $\mathbb{E}((\beta(t) - \beta(s))\beta(t)) = 0$, $t \geq s$.

- $\mathbb{E}(\beta(t)^2) = t$.

$\rightarrow \mathbb{E}((\beta(t) - \beta(s))^2) = t - s$

$\rightarrow \mathbb{E}\left(\left(\frac{\beta(t)-\beta(s)}{t-s}\right)^2\right) = \frac{1}{t-s}$, β is nowhere differentiable ...

$\rightarrow \mathbb{E}(\beta(t)\beta(s)) = \min\{t, s\}$.

$\rightarrow$ Formally: $\mathbb{E}(\dot\beta(t)\dot\beta(s)) = \delta_{t=s}$

$\rightarrow$ Formally : $\dot\beta(t,\omega) = \sum_{k\in\mathbb{N}} \chi_k(\omega) f_k(t)$ with (f_k) a basis of $L^2([0,T])$ and χ_k independent $\mathcal{N}(0,1)$.

$\rightarrow$ $\dot\beta$ is the time white noise.

The product with $\dot\beta$ is ill-defined. Two possibilities:

• Ito : $f(t)\dot\beta(t) \approx f(t)\dfrac{\beta(t+\delta)-\beta(t)}{\delta}$

$\rightarrow$ good mathematical properties

• Stratonovitch : $f(t)\dot\beta(t) \approx f(t)\dfrac{\beta(t+\delta)-\beta(t-\delta)}{2\delta}$

$\rightarrow$ arise naturally in applications

$$\begin{cases} i\frac{du^\epsilon}{dt} + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2})\Delta u^\epsilon + |u^\epsilon|^{2\sigma} u^\epsilon = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ u^\epsilon(0) = u_0. \end{cases}$$

Set $M(t) = \dfrac{1}{\epsilon}\displaystyle\int_0^t m(\frac{\sigma}{\epsilon^2})d\sigma = \epsilon \int_0^{t/\epsilon^2} m(\sigma)d\sigma$ then $M(t) - M(s) = \epsilon \displaystyle\int_{s/\epsilon^2}^{t/\epsilon^2} m(\sigma)d\sigma$ is more and more independent from $M(s)$.

$$M(t) = \epsilon \sum_0^{t/\epsilon} \frac{1}{\epsilon} \int_{k/\epsilon}^{(k+1)/\epsilon} m(\sigma)d\sigma$$

$\longrightarrow$ M converges in law to a brownian motion as $\epsilon \to 0$.

$\longrightarrow$ $\dfrac{1}{\epsilon} m(\dfrac{t}{\epsilon^2})$ converges to a time white noise.

$$\begin{cases} i\frac{du}{dt} + \dot{\beta} o \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ u(0) = u_0. \end{cases}$$

$\longrightarrow$ It is a Stratonovich product.

$\longrightarrow$ The mass is still a conserved quantity but the Hamiltonian structure is destroyed. The energy has a complicated evolution.

$\longrightarrow$ The aim is to study the limit equation and to justify the limit (Marty, de Bouard and D., Tsutsumi and D.).

$\longrightarrow$ For a periodic m, the scaling is different (Zharnitsky, Grenier, Jones, Turitsyn).

Effect of a random dispersion: $v(t) = S(t,s)v_s$ solution of

$$\begin{cases} i dv + \Delta v \circ d\beta = 0,\ x \in \mathbb{R}^d,\ t \geq s, \\ v(s) = v_s. \end{cases}$$

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$\rightarrow$ $\widehat{v}(t,\xi) = e^{-i|\xi|^2(\beta(t)-\beta(s))}\widehat{v}_s(\xi),\ t \geq s,\ \xi \in \mathbb{R}^d.$

$\rightarrow$ If $v_s \in L^2(\mathbb{R}^d)$, then $v(\cdot) \in C([s,T];L^2(\mathbb{R}^d))$ a.s. and $|v(t)|_{L^2} = |v_s|_{L^2}$, a.s. $t \geq s$.

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$\rightarrow$ $|v(t)|_{L^\infty(\mathbb{R}^d)} \leq c \dfrac{1}{|\beta(t) - \beta(s)|^{d/2}} |v_s|_{L^1(\mathbb{R}^d)}.$

Theorem Let $\sigma < \dfrac{2}{d}$ and $u_0 \in L^2_x$ p.s., $\mathcal{F}_0$-measurable, there exists a unique solution in $L^r_{loc}(0,\infty; L^p(\mathbb{R}^d))$ a.s. with $p = 2\sigma + 2 \le r < \dfrac{4(\sigma+1)}{d\sigma}$; moreover, $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$, a.s.

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Conjecture: the result holds for $\sigma < \frac{4}{d}$.

The limit $\epsilon \to 0$ **:** $d = 1$ and $\sigma \le 2$

$$idu^\epsilon + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}) \Delta u^\epsilon + |u^\epsilon|^{2\sigma} u^\epsilon = 0,\ u^\epsilon(0) = u_0$$

Assume that m is stationary, has zero average and that $t \mapsto \varepsilon \int_0^{t/\varepsilon^2} m(s)ds$ converges in distribution to a brownian motion (for instance if m is markov and sufficiently mixing).

Theorem

For any $\varepsilon > 0$ and $u_0 \in H^1(\mathbb{R})$, there exists a unique u_ε in $H^1(\mathbb{R})$, defined on on $[0, \tau_\varepsilon(u_0))$ and for all $T > 0$

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon(u_0) \le T) = 0.$$

The process $u_\varepsilon \mathbb{1}_{[\tau_\varepsilon > T]}$ converges in distribution to u in $C([0, T]; H^1(\mathbb{R}))$.

Models with spatial noises:

Two types of perturbations :

$$i\frac{du}{dt} + (\Delta u + |u|^{2\sigma} u) = \dot{\eta}, \text{(additive noise)}.$$

$$i\frac{du}{dt} + (\Delta u + |u|^{2\sigma} u) = u \circ \dot{\eta}, \text{(multiplicative noise)}.$$

The space time white noise:

- $W(t,x,\omega) = \sum_{i\in\mathbb{N}} \beta_i(t,\omega)e_i(x)$ where (β_i) are independant brownian motions and (e_i) is a basis of $L^2(\mathbb{R}^d)$.

(note that the series diverges a.s. in $L^2(\mathbb{R}^d)$ ...)

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- alternative definition:

$$\frac{dW}{dt} = \frac{\partial^d B}{\partial x_1 \dots \partial x_d \partial t}$$

where B is a brownian motion with $d+1$ variables (the brownian sheet).

Spatial correlation

Let $k(x, y)$ be a kernel. We define:

$$\tilde{W}(x, t) = \int_{\mathbb{R}^d} k(x, y) W(y, t) dy = \sum_i \Phi e_i \beta_i$$

where $\Phi e_i(x) = \int_{\mathbb{R}^d} k(x, y) e_i(y) dy$.

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- $\mathbb{E}\left(\tilde{W}(t,x)\tilde{W}(s,y)\right) = \min\{t,s\} \int_{\mathbb{R}^d} k(x,z)k(y,z)dz$.

$\rightsquigarrow c(x,y) = \int_{\mathbb{R}^d} k(x,z)k(y,z)dz$: correlation function.

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• If $k(x,y) = k(x-y)$, $c(x,y) = c(x-y)$ $\rightsquigarrow$ the noise is spatially homogeneous.

• The spatial smoothness of $\tilde{W}$ depends on the smoothness of k. If $k = \delta_{x=y}$, we recover the space time white noise.

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• The spatial smoothness of $\tilde{W}$ depends on the smoothness of k. If $k = \delta_{x=y}$, we recover the space time white noise.

• If we require that $\tilde{W}$ is in $L^2(\mathbb{R}^d)$, we need $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ $\rightsquigarrow$ we cannot treat homogeneous noise.

• For the mathematical study $\rightsquigarrow$ at least $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

We rewrite the two stochastic equations as :

$$idu + (\Delta u + |u|^{2\sigma} u)dt = d\tilde{W}, (\text{ additive noise}).$$

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ d\tilde{W}, (\text{ multiplicative noise}).$$

They are the limit of the schrödinger equations with smooth random perturbation:

$$i\frac{du}{dt} + (\Delta u + |u|^{2\sigma} u) = \frac{1}{\epsilon} m\left(\frac{t}{\epsilon^2}, x\right),$$

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The covariance operator is associated to the kernel:

$$k(x,y) = \mathbb{E} \int_{\mathbb{R}^d} m(0)(y) m(t)(x) dt.$$

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$U(t) = e^{it\Delta} \rightsquigarrow$

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$\rightarrow$ If k is sufficiently smooth, the deterministic mathematical analysis can be reproduced and we have existence and uniqueness in $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ under the same conditions on σ as in the deterministic cas (for $d \leq 3$).

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ d\tilde{W}.$$

• With multiplicative noise, the mass is still a conserved quantity.

• The energy is now a non constant quantity. It evolves as:

$$H(u(t)) = H(u_0) - Im \int_{\mathbb{R}^d} \int_0^t \bar{u} \nabla u \cdot \nabla dW dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} |u|^2 f_\Phi^2 dx ds.$$

$$\mathbb{E}(H(u(t))) = \mathbb{E}(H(u_0)) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \mathbb{E}(|u|^2) f_\Phi^2 dx ds.$$

$\rightarrow$ It grows linearly in time.

• Similar formula in the additive case.

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The deterministic theory:

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u(x)|^2 dx, \;\; G(u) = Im \int_{\mathbb{R}^d} u(x) x \nabla \bar{u}(x) dx.$$

$$\frac{dV(u)}{dt} = 4G(u), \;\; \frac{dG(u)}{dt} = 4H(u) + \frac{2-\sigma d}{\sigma+1} \int_0^t \int_{\mathbb{R}^d} |u|^{2\sigma+2} dx ds.$$

Blow-up : $\sigma \geq 2/d$

The deterministic theory:

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u(x)|^2 dx, \quad G(u) = Im \int_{\mathbb{R}^d} u(x) x \nabla \bar{u}(x) dx.$$

$$\frac{dV(u)}{dt} = 4G(u), \quad \frac{dG(u)}{dt} = 4H(u) + \frac{2 - \sigma d}{\sigma + 1} \int_0^t \int_{\mathbb{R}^d} |u|^{2\sigma+2} dx ds.$$

$\rightarrow V(u(t)) \leq V(u_0) + 4tG(u_0) + 8t^2 H(u_0)$

$\rightarrow$ impossible if $H(u_0) < 0$.

Blow-up : $\sigma \geq 2/d$, multiplicative noise:

$$
\begin{aligned}
&V(u(t)) \\
&= V(u_0) + 4G(u_0)t + 8H(u_0)t^2 \\
&+4\frac{2-\sigma d}{\sigma+1}\int_0^t \int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds \\
&+8\int_0^t \int_0^s \int_0^{s_1} \int_{\mathbb{R}^d} |u(s_2,x)|^2 f_\phi^1(x) dx ds_2 ds_1 ds \\
&+4\int_0^t \int_0^s \int_{\mathbb{R}^d} |u(s_1,x)|^2 x \cdot f_\phi^2(x) dx\, dW(s_1) ds \\
&-16\,\mathrm{Im}\int_0^t \int_0^s \int_0^{s_1} \int_{\mathbb{R}^d} \overline{u}(s_2,x)\nabla u(s_2,x)\cdot f_\phi^2 dx\, dW(s_2) ds_1 ds
\end{aligned}
$$

Blow-up : $\sigma \geq 2/d$, multiplicative noise:

$$V(u(t))$$

$$= V(u_0) + 4G(u_0)t + 8H(u_0)t^2$$

$$+4\frac{2-\sigma d}{\sigma+1}\int_0^t\int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds$$

$$+8\int_0^t\int_0^s\int_0^{s_1}\int_{\mathbb{R}^d} |u(s_2,x)|^2 f_\phi^1(x) dx ds_2 ds_1 ds$$

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Blow-up : $\sigma \geq 2/d$, multiplicative noise:

$$\mathbb{E}(V(u(t)))$$

$$= V(u_0) + 4G(u_0)t + 8H(u_0)t^2$$

$$+4\frac{2-\sigma d}{\sigma+1}\mathbb{E}\int_0^t\int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds$$

$$+8\mathbb{E}\int_0^t\int_0^s\int_0^{s_1}\int_{\mathbb{R}^d} |u(s_2,x)|^2 f_\phi^1(x) dx ds_2 ds_1 ds$$

Theorem : If $\sigma \geq \frac{2}{d}$, $u_0 \in L^2(\Omega; \Sigma) \cap L^{2\sigma+2}(\Omega; L^{2\sigma+2}(\mathbb{R}^d))$, k sufficiently smooth. If there exists $\bar{t} > 0$ such that

$$\mathbb{E}(V(u_0)) + 4\mathbb{E}(G(u_0))\bar{t} + 8\mathbb{E}(H(u_0))\bar{t}^2 + \frac{4}{3}\bar{t}^3 m_\phi \mathbb{E}(M(u_0)) < 0$$

then

$$\mathbb{P}(\tau^*(u_0) \leq \bar{t}) > 0.$$

- similar result for an additive noise.

The noise has a stronger effect :

The noise has a stronger effect : $\sigma \geq 2/d$ for an additive noise or $\sigma > 2/d$ for a multiplicative noise.

For an arbitrary initial data u_0, it is possible to construct f such that the solution of

$$\frac{du_f}{dt} + \Delta u_f + |u_f|^{2\sigma} u_f = u_f \frac{df}{dt}, \quad u(0) = u_0,$$

is such that $u(t_0)$ satisfies:

$$\mathbb{E}(V(u(t_0))) + 4\mathbb{E}(G(u(t_0)))\bar{t} + 8\mathbb{E}(H(u(t_0)))\bar{t}^2 + \frac{4}{3}\bar{t}^3 m_\phi \mathbb{E}(M(u(t_0))) < 0$$

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so that a stochastic solution starting from $u(t_0)$ has a positive probability to blow-up before $\bar{t} + t_0$.

If the noise is non degenerate, *i.e.* $ker\Phi = \{0\}$ or that the noise hits all frequencies, then the solution u_f is in the support of the stochastique solution:

$$\mathbb{P}(\sup_{t\in[0,t_0]} \|u - u_f\|_\Sigma \leq \epsilon) > 0, \ \ \forall \epsilon > 0.$$

$\leadsto$ the stochastic solution u blows up before $\bar{t} + t_0$

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$\rightsquigarrow$ the stochastic solution u blows up before $\bar{t} + t_0$

$\rightarrow$ all solutions blow up with positive probability in arbitrary small time.