Discrete scale invariance in trapped ion quantum simulators

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Quantum computing workshop, CEA Saclay

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Outline

Overview of quantum simulators

What is it and why should I care

Trapped ion simulator

Scale invariance- classical vs quantum

Discrete scale invariance for two bosons

Efimov effect

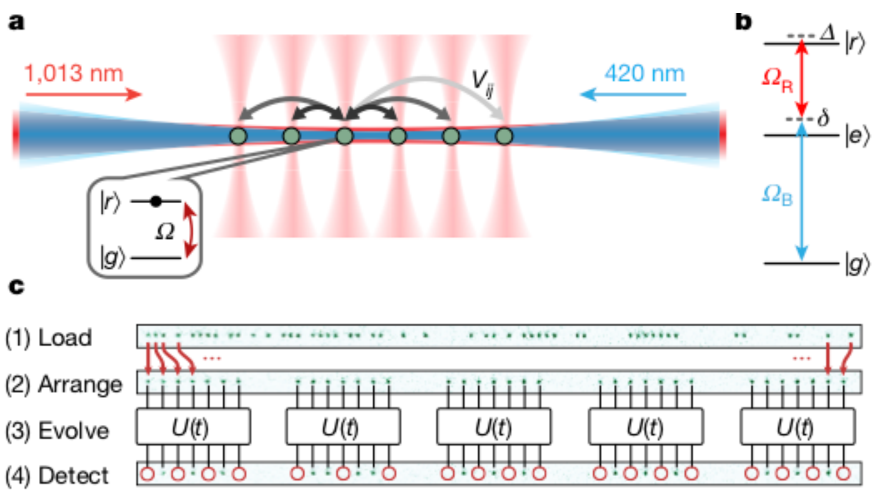
Time fractals

Future work: scattering

What is a quantum simulator?

Probing many-body dynamics on a 51-atom quantum simulator

Hannes Bernien[1], Sylvain Schwartz[1,2], Alexander Keesling[1], Harry Levine[1], Ahmed Omran[1], Hannes Pichler[1,3], Soonwon Choi[1], Alexander S. Zibrov[1], Manuel Endres[4], Markus Greiner[1], Vladan Vuletić[2] & Mikhail D. Lukin[1]

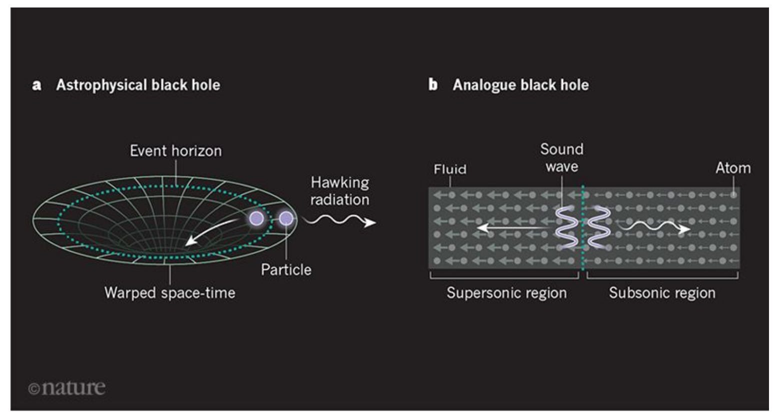


RESEARCH NEWS & VIEWS

EXPERIMENTAL PHYSICS

Quantum simulation of black-hole radiation

It is extremely difficult to observe the radiation that is thought to be emitted by black holes. The properties of this radiation have now been analysed using an analogue black hole comprising a system of ultracold atoms. SEE LETTER P.688



Muñoz de Nova et. al., *Nature* **569**, 688–691 (2019)

What is a quantum simulator?

- A deficient quantum computer?
- Any quantum system which simulates another?
- At what point does simulation become computation?
- Does *trustworthiness* play a role in our definition?
- Why might it be used?
  - Exert greater control over the system
  - Easier observation
  - Problem is computationally intractable

What distinguishes a (quantum) computer from a simulator?

- The short answer: **universality**
  - Computers possess collection of operations (“gates”) which get you all possible operations

- Other aspects: **analog vs digital**
  - computers are thought of as digital, while simulators are (usually) analog

- Theoretically, a quantum computer is does strictly more than a simulator
- *However*, for what they can do, current simulators are currently larger and more easily controlled
  - Harvard’s 51-qubit and Maryland’s 53-qubit simulators were used to study phase transitions in Ising systems
  - Maryland’s can be reconfigured to full QC with small adjustments
  - Only recently have computers of similar size been created by IBM (49 qubits) and Google (72 qubits)

Why bother with quantum simulators?

"Historically, the advance of technology is often hastened by the use of short- to medium-term incentives as a stepping-stone to long-term goals."

- Nielsen and Chuang

Simulators based on trapping ions or neutral atoms have reached sizes which defy simulation by classical computers, suggesting avenues for new discovery.

What types of physical systems may be studied with these technologies?

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Classical scale invariance

Let's consider a classical system with one coordinate *r* and corresponding momentum *p*. Suppose the Hamiltonian has the following form.

$$H(p,r) = p^\gamma h(pr)\ , \quad \gamma \in \mathbb{R}$$

We can show that this classical system is scale invariant. Consider the simultaneous rescaling of p and r by a real parameter λ

$$r \to r' = \lambda r,\ p \to p' = \lambda^{-1} p$$

Then the Hamiltonian transforms as

$$H(p,r) \to H(p',r') = \lambda^{-\gamma} H(p,r)$$

Suppose we also rescale time so that

$$t \to t' = \lambda^{\gamma} t$$

Hamilton's equations for the unprimed coordinates are

$$\frac{dp}{dt} = -\frac{\partial H(p,r)}{\partial r}, \; \frac{dr}{dt} = \frac{\partial H(p,r)}{\partial p}$$

Exactly the same equations of motion hold for the rescaled variables with the same functional form for the Hamiltonian H,

$$\frac{dp'}{dt'} = -\frac{\partial H(p',r')}{\partial r'}, \; \frac{dr'}{dt'} = \frac{\partial H(p',r')}{\partial p'}$$

We conclude that the system is scale invariant.

Quantum scale anomalies

- In QM and QFT, scale invariance can be spoiled
  - A type of *anomalous quantum symmetry breaking*

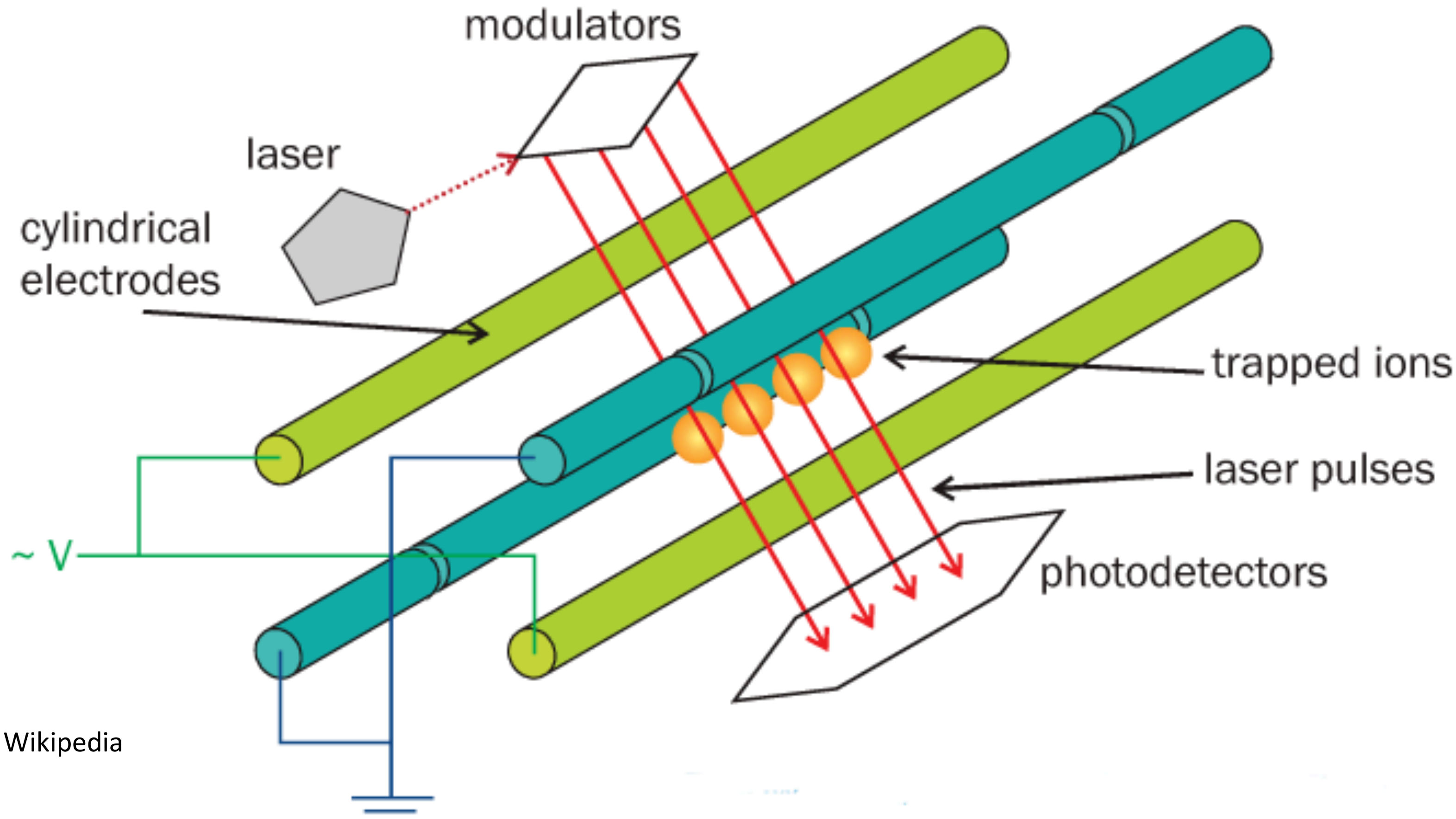
- Dilemma is clear: how can we reconcile continuous symmetry with discrete bound states? What happened?
  - Same issue arises in inverse square potential bound states
  - Formally, such potentials aren't self-adjoint, informally they're too singular

- It may happen that a discrete subgroup of the symmetry is preserved in a subspace of the Hilbert space (e.g. the bound state spectrum)

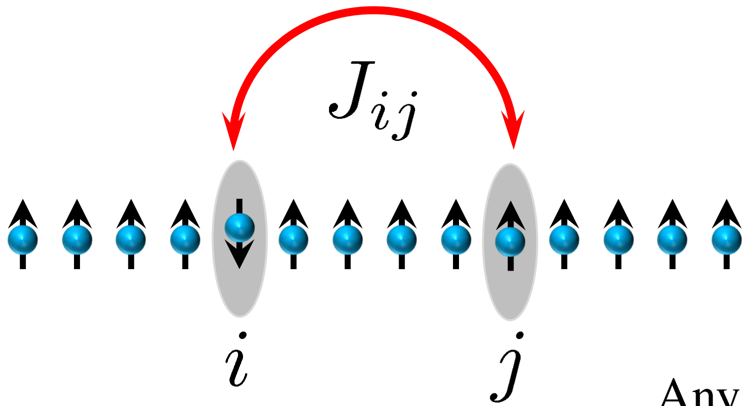
This was observed by Efimov for bound states of three bosons when the two-body interactions are pointlike and the interaction strength is tuned to produce a zero-energy two-body resonance.

Efimov, Sov. J. Nucl. Phys. 12, 589 (1971); Efimov, Phys. Rev. C47 1876 (1993)
Bedaque, Hammer, van Kolck, Phys. Rev. Lett. 82 463 (1999)

Realization with trapped ions

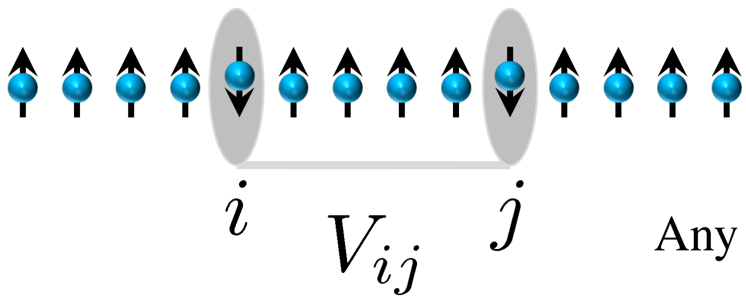


Wikipedia



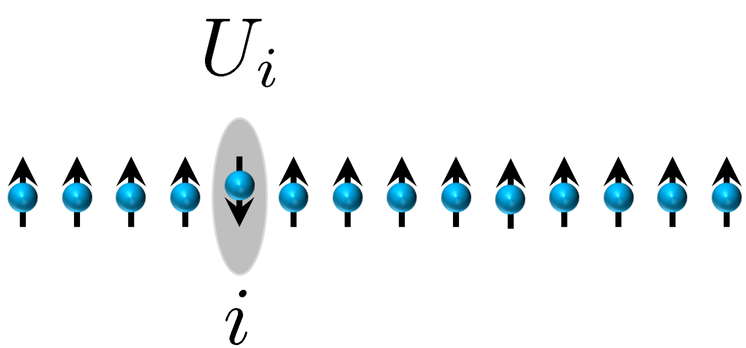
$$J_{ij} = \frac{J_0}{|r_i - r_j|^\alpha}$$

Any $0 < \alpha < 3$, but $0.5 < \alpha < 1.8$ favored.



$$V_{ij} = \frac{V_0}{|r_i - r_j|^\beta}$$

Any $0 < \beta < 3$, but $0.5 < \beta < 1.8$ favored.



Zhang *et al.*, Nature 543, 217 (2017), Zhang *et al.*, Nature 551, 601 (2017)

Our trapped ion Hamiltonian has the form

$$H = T + V_2 + U + C$$

where

$$T = \frac{1}{4} \sum_i \sum_{j \neq i} J_{ij} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)$$

$$V_2 = \frac{1}{8} \sum_i \sum_{j \neq i} V_{ij} (1 - \sigma_i^z)(1 - \sigma_j^z)$$

$$U = \frac{1}{2} \sum_i U_i (1 - \sigma_i^z)$$

and C is a constant.

Note that V_2 terms vanish for spin up states

We can rewrite the Ising-like terms in the Hamiltonian as

$$V_2 + U = \frac{1}{8}\sum_i \sum_{j\neq i} V_{ij}\sigma_i^z\sigma_j^z - \frac{1}{2}\sum_i U_i'\sigma_i^z + C'$$

where

$$U_i' = U_i + \frac{1}{2}\sum_{i\neq j} V_{ij}$$

$$C' = \frac{1}{8}\sum_i \sum_{j\neq i} V_{ij} + \frac{1}{2}\sum_i U_i$$

We will regard the state with all spins pointing in the positive z direction as the vacuum state. Then any spin in the negative z direction can be regarded as a hardcore boson. It is not possible to have more than one hardcore boson on the same site.

In terms of the hardcore boson annihilation and creation operators, our Hamiltonian is now

$$H = \underbrace{\frac{1}{2}\sum_i \sum_{j\neq i} J_{ij}[b_i^\dagger b_j + b_j^\dagger b_i]}_{\text{kinetic/hopping}} + \underbrace{\frac{1}{2}\sum_i \sum_{j\neq i} V_{ij} b_i^\dagger b_i b_j^\dagger b_j}_{\text{interaction}} + \sum_i U_i b_i^\dagger b_i + C$$

Let us now take

$$U_i = -\sum_{j\neq i} J_{ij} = -\sum_{j\neq i} \frac{J_0}{|r_i - r_j|^\alpha}$$

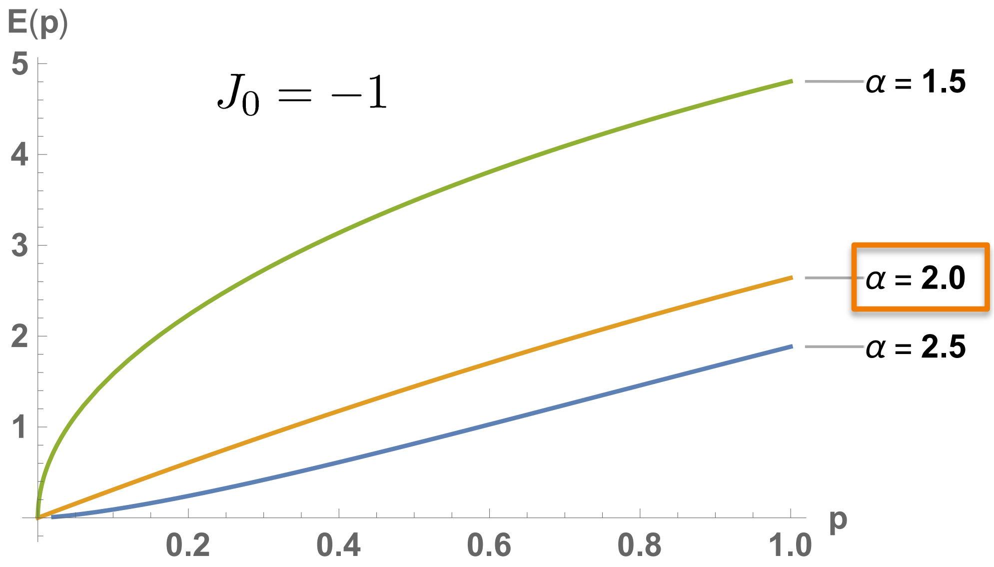
This choice ensures that a zero-momentum boson has zero energy. We now consider the dispersion relation for one boson.

For a boson with momentum p, the energy is

$$E(p) = 2J_0 \sum_{n>0} \frac{\cos(pn) - 1}{n^\alpha} = J_0 \left[\mathrm{Li}_\alpha(e^{ip}) + \mathrm{Li}_\alpha(e^{-ip}) - 2\mathrm{Li}_\alpha(1) \right]$$

At low momenta, this can be simplified as

$$E(p) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + J_0\zeta(\alpha-2)p^2 + O(p^4) \;\; \text{for} \;\; \alpha < 3$$

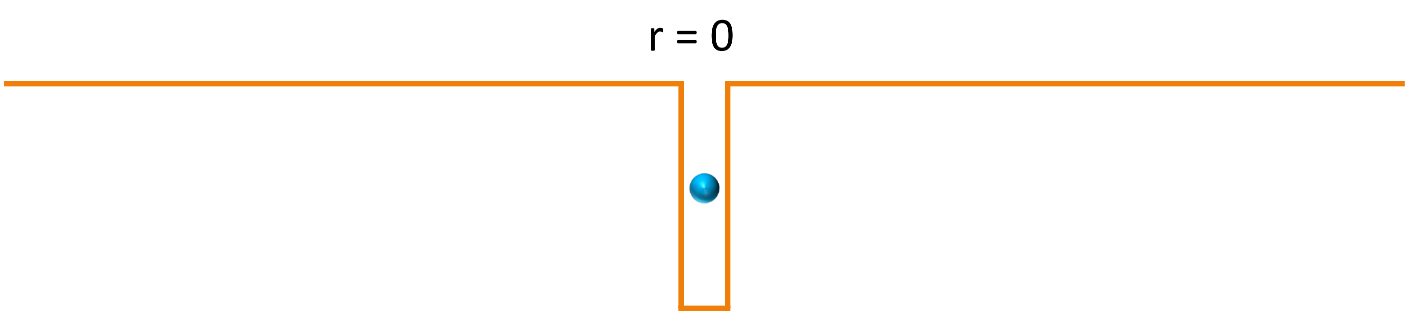


- Linear dispersion
- Connection to graphene & relativistic electrons?

We now introduce a single-site deep trapping potential that traps one boson at some site i_0

$$U_i = -\sum_{j \neq i} \frac{J_0}{|r_i - r_j|^\alpha} - u\delta_{i,i_0} \qquad \text{for } u \gg 1$$

We choose the position of site i_0 to be r = 0. We subtract a constant from the Hamiltonian so that the energy of this state is exactly zero.



Discrete scale invariance for two bosons

We now add one more boson to the system. We regard the immobile boson at r = 0 as a static source.

The low-energy effective Hamiltonian for the mobile boson is

$$H(p,r) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + \frac{V_0}{|r|^{\beta}}$$

where we have dropped terms of $O(p^2)$. We will consider the case where both J_0 and V_0 are negative. In order that the Hamiltonian have classical scale invariance, we take $\beta = \alpha - 1.$

Therefore

$$H(p,r) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + \frac{V_0}{|r|^{\alpha-1}}$$

With this choice, $0 < \alpha - 1 < 2$ theoretically.

In the limit of zero energy, the bound-state wave functions have the following forms for even and odd parity

$$\psi_+(r) = \frac{1}{2}\left(|r|^{i\delta_+} + |r|^{-i\bar{\delta}_+}\right)$$
$$\psi_-(r) = \frac{1}{2}\,\mathrm{sgn}(r)\left(|r|^{i\delta_-} + |r|^{-i\bar{\delta}_-}\right)$$

where

$$2J_0\delta_+\Gamma(1-\alpha)\sin(\alpha\pi/2)\Gamma(i\delta_+)\sinh(\delta_+\pi/2) = V_0\Gamma(2-\alpha+i\delta_+)\cos((\alpha-i\delta_+)\pi/2)$$

$$2J_0\delta_-\Gamma(1-\alpha)\sin(\alpha\pi/2)\Gamma(i\delta_-)\cosh(\delta_-\pi/2) = iV_0\Gamma(2-\alpha+i\delta_-)\sin((\alpha-i\delta_-)\pi/2)$$

The case $\alpha = 2$ corresponds to a Hamiltonian of the form

$$H(p,r) = -\pi J_0|p| + \frac{V_0}{|r|}$$

For the case $\alpha = 2,$

$$\delta_+ = \frac{V_0}{J_0\pi}\coth(\delta_+\pi/2), \quad \delta_- = \frac{V_0}{J_0\pi}\tanh(\delta_-\pi/2)$$

We can rewrite the zero-energy bound-state solutions as

$$\psi_+(r) = \cos[\delta_+\ln(|r|)], \quad \psi_-(r) = \mathrm{sgn}(r)\cos[\delta_-\ln(|r|)]$$

Under the scale transformations $r \to \lambda_\pm r,$

$$\psi_+(r) \to \cos[\delta_+\ln(|r|) + \delta_+\ln(\lambda_+)]$$

$$\psi_-(r) \to \mathrm{sgn}(r)\cos[\delta_-\ln(|r|) + \delta_-\ln(\lambda_-)]$$

The wave functions exhibit discrete scale invariance when the scale factors are

$$\lambda_+ = \exp(\pi/\delta_+), \quad \lambda_- = \exp(\pi/\delta_-)$$

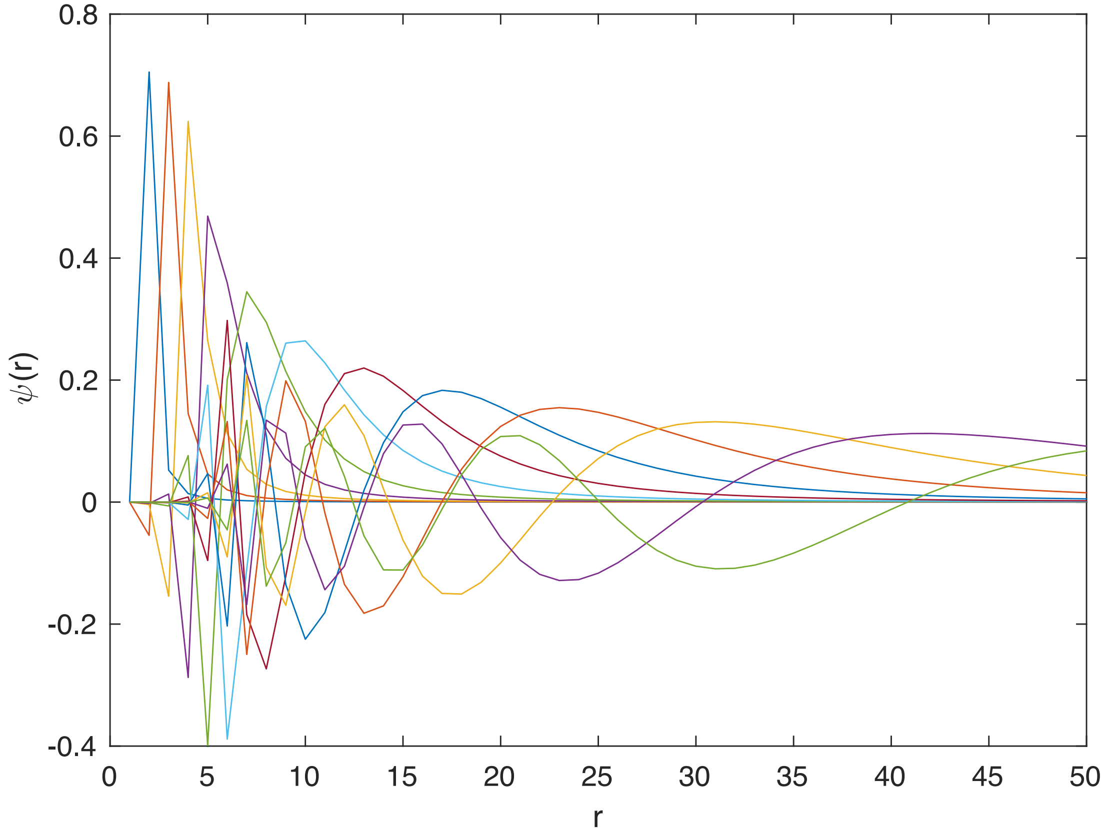
The bound state energies form a geometric progression

$$E_+^{(n)} = \epsilon_+ \lambda_+^{-n}, \quad E_-^{(n)} = \epsilon_- \lambda_-^{-n} \quad (\alpha = 2 \ \text{case})$$

The formulas for general α are

$$\lambda_+ = \exp(\pi/\mathrm{Re}\,\delta_+), \quad \lambda_- = \exp(\pi/\mathrm{Re}\,\delta_-)$$
$$E_+^{(n)} = \epsilon_+ \lambda_+^{-(\alpha-1)n}, \quad E_-^{(n)} = \epsilon_- \lambda_-^{-(\alpha-1)n}$$

The first twelve even-parity bound-state wave functions:



$$\alpha = 2,\ \beta = 1,\ J_0 = -1, \text{ and } V_0 = -30$$

n	$E_+^{(n)}$	$E_+^{(n-1)}/E_+^{(n)}$	$E_-^{(n)}$	$E_-^{(n-1)}/E_+^{(n)}$
0	-27.05304149	–	-26.5188669	–
1	-11.93067205	2.267520336	-11.79861873	2.247624701
2	-6.977774689	1.709810446	-6.919891389	1.705029468
3	-4.553270276	1.5324754	-4.521425357	1.530466798
4	-3.139972298	1.450098869	-3.120231851	1.449067112
5	-2.233327278	1.405961557	-2.220194049	1.405386998
6	-1.617052389	1.381110033	-1.607920414	1.380786033
7	-1.182654461	1.367307563	-1.176124883	1.367134084
8	-0.869406941	1.360300229	-0.864656962	1.360221377
9	-0.640405903	1.357587332	-0.636916042	1.357568195
10	-0.471738446	1.357544438	-0.469161911	1.357561276
11	-0.347112043	1.359037968	-0.345207121	1.359073675
12	-0.254996818	1.361240684	-0.253589633	1.361282464
13	-0.187011843	1.363532996	-0.18597462	1.363571189
theory	–	$\lambda_+ = 1.3895595319$	–	$\lambda_- = 1.3895595319$

Time fractals

We use a phase convention where all of the bound-state wave functions are real valued. Let us construct a coherent superposition of the first N even-parity bound states, where N is large.

$$|S\rangle = \sum_{n=0}^{N-1} |\psi_+^{(n)}\rangle$$

We could have just as easily chosen odd-parity bound states. We now consider the amplitude

$$A(t) = \mathrm{Re}[Z(t)], \quad Z(t) = \langle S|\exp(-iHt)|S\rangle$$

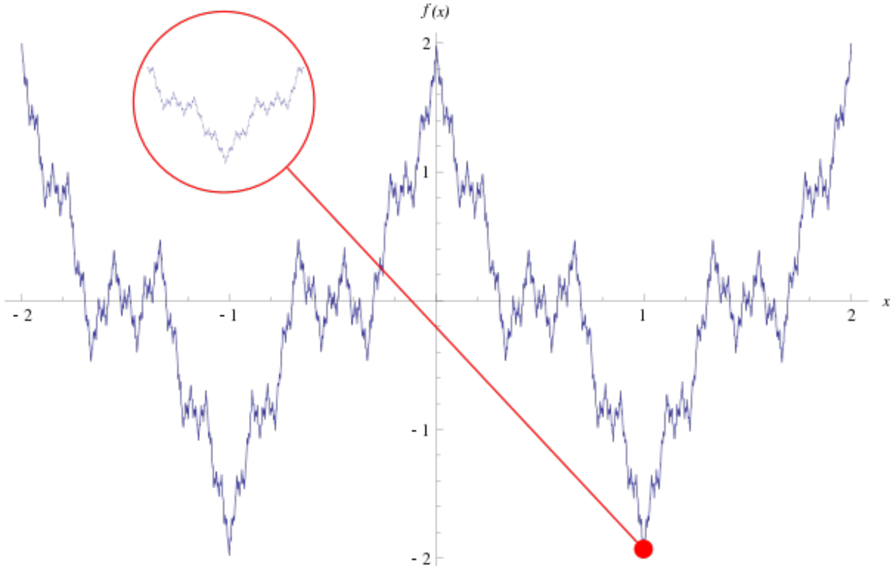
Aside from corrections of relative size 1/N from endpoint terms at n = 0 and n = N – 1, the amplitude is invariant under the discrete rescaling of time.

This is a particular case of the Weierstrass function,

$$w(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t) \qquad 0 < a < 1 < b \text{ with } ab \geq 1$$

In our case we take $a \to 1$ and have truncated after a finite number of terms.
The next slide shows a picture of the Weierstrass function for
a = 0.5, b = 3.

$$a = 0.5, \quad b = 3$$



The Weierstrass function has fractal dimension

$$D = 2 + \frac{\log a}{\log b}$$

Hardy, Trans. Amer. Math. Soc. 17, 301 (1916)
Hunt, Proc. Amer. Math. Soc. 126, 791 (1998)

Conclusion and outlook

- Trapped ion quantum simulators offer a new way to produce quantum scale anomalies and do so quite naturally.

- Several parameters can be varied to accommodate different types of systems with fractal behavior (e.g. relativistic vs nonrel. dispersion)

- Further aspects to explore
  - N-boson system exhibits multi-halo structures
  - scattering and reactions in the trap

- More generally, what can quantum simulators do for us?
  - Perhaps provide a boost to “mainstream” QC

- At the very least, there should be interesting physics to explore

Extra slides

We now show how to produce time fractals directly with the trapped ion quantum simulator. We start with the function S(r) defined as

$$|S\rangle = \sum_{r\neq 0} S(r)|r\rangle$$

Using this function, we define the following product of single-qubit rotations

$$U(\epsilon) = \prod_{r\neq 0} \exp[-i\epsilon\sigma_r^y S(r)]$$

When acting on the state $|o\rangle$ consisting only of the immobilized boson at r = 0, we get

$$U(\epsilon)|o\rangle = \left[1 - \frac{\epsilon^2}{2}\langle S|S\rangle + O(\epsilon^3)\right]|o\rangle + \epsilon|S\rangle + O(\epsilon^2)|X\rangle$$

Recall that we had added a constant to the Hamiltonian so that

$$\langle o| \exp[-iHt] |o\rangle = \langle o|o\rangle = 1$$

We can now use the quantum simulator to determine

$$B(\epsilon, t) = |\langle o|U^\dagger(\epsilon) \exp[-iHt] U(\epsilon)|o\rangle|^2$$

This corresponds to measuring the projection operator $|o\rangle\,\langle o|$ n the state

$$U^\dagger(\epsilon) \exp[-iHt] U(\epsilon)|o\rangle$$

If we expand in ϵ we get

$$B(\epsilon, t) = \left|1 - \epsilon^2 \langle S|S\rangle + \epsilon^2 Z(t) + O(\epsilon^3)\right|^2$$

This can be rewritten as

$$
\begin{aligned}
B(\epsilon, t) &= 1 - 2\epsilon^2 \langle S|S\rangle + \epsilon^2 [Z(t) + Z^*(t)] + O(\epsilon^3) \\
&= 1 - 2\epsilon^2 \langle S|S\rangle + 2\epsilon^2 A(t) + O(\epsilon^3),
\end{aligned}
$$

We can therefore extract the time fractal amplitude A(t).