A brief introduction to group theory

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Symmetry-breaking versus symmetry-preserving many-body schemes

Motivation and outline

Group theory is the natural framework to describe symmetry operations like rotations, translations and other internal symmetries. Most of the symmetries are continuous and therefore require continuous Lie groups with their associated Lie algebras.

- Finite groups
- Representations
- Continuous Lie groups
- Algebras and exponential mapping
- Representations

Group axioms

General definition of group

A group G is a set of elements $G = \{A, B, C, \ldots\}$ and a composition law $\cdot$ such that

- $A \cdot B \in G$
- Identity $A \cdot I = I \cdot A$
- Inverse $\exists A^{-1} \mid A^{-1}A = AA^{-1} = I$
- Associativity $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

Remarks: The composition law is not commutative in general

Group table

	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

Examples:

- $\mathbb{R}$ with $+$
- $\mathbb{R} - \{0\}$ with $\times$
- Permutation group S_n
- Rotations along a symmetry axis

Examples

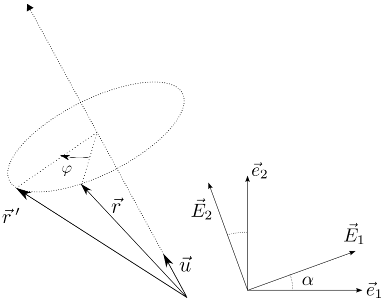
Discrete group S_3

$P_1 = (abc) = (1)(2)(3) = I$,
$P_2 = (bca) = (123)$,
$P_3 = (cab) = (132), P_4 = (acb) = (1)(23)$
$P_5 = (bac) = (3)(12), P_6 = (cba) = (2)(13)$

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_3	P_1	P_6	P_4	P_5
P_3	P_3	P_1	P_2	P_5	P_6	P_4
P_4	P_4	P_5	P_6	P_1	P_2	P_3
P_5	P_5	P_6	P_4	P_3	P_1	P_2
P_6	P_6	P_4	P_5	P_2	P_3	P_1

$\{P_1,\ P_2,\ P_3\}$ subgroup

Continuous group: Rotation along a symmetry axis



$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad R(0) = \mathbb{I}, R(\alpha)R(-\alpha) = \mathbb{I} \quad SO(2)$$

A few results and definition

- **Group order:** number of elements (discrete, $S_3 = 3!$) or parameters (continuous, $SO(2) = 1$)
- **Subgroup** H, Cosets: $X^{-1}Y \in H$. Lagrange: g/h integer
- **Invariant subgroup** $G_i^{-1}HG_i = H$, G/H quotient group $\{\mathbb{R}^3, +\}$, S invariant, $\mathbb{R}^3/S = \mathbb{R}$
- **Caley theorem**: Every finite group is isomorphic to a subgroup of S_n
- **Crystallographic groups:** C_n $(2\pi/n)$, C_{nh}, $C_{n\nu}$, S_n, D_n (diedric), D_{nh}, D_{nd}, T (tetradiedric), T_h, T_d, O (Octaedric), O_n. Crystallographic restriction theorem: $n = 1, 2, 3, 4, 6$

Representations

Representations

Homomorphism of G onto linear applications in a vector space $\mathcal{V}$
$g \to D(g)$. Given a base of $\mathcal{V}$, $D(g)$ is a matrix

- $D(g)D(g') = D(gg')$
- $D(e) = I$
- $(D(g))^{-1} = D(g^{-1})$

Example: $C_4 = \{I, X, X^2, X^3\}$ Cyclic group of order 4 ($X^4 = I$)

	I	X	X^2	X^3	
$C_4^{[1]}$	1	1	1	1	Homo (Triv)
$C_4^{[2]}$	1	i	-1	-i	Iso
$C_4^{[A]}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	Iso

or direct sums of them, like $C_4^{[A]} \oplus C_4^{[2]}$

Representations

- **Regular representation:**
$gh_i = \sum_{j=1}^{n} D^{\mathrm{reg}}(g)_{ji} h_j$. $TrD^{\mathrm{reg}}(g) = 0$ except if $g = I$.
- **Reducible/irreducible:** $D(g)$ is said reducible if

$$D(g) = \left(\begin{array}{c|c} D^{(1)}(g) & 0 \\ \hline 0 & D^{(2)}(g) \end{array} \right) \text{for all } \; g \in G$$

There exists an invariant subspace $\mathcal{V}_1 \subset \mathcal{V}$

- **Unitary:** $D^{\dagger}(g)D(g) = \mathbb{I} \; \forall g \in G$. Every rep of a finite group is equivalent to a unitary rep.
- **Tensor product:** $D^{(1)} \otimes D^{(2)}(v_1 \otimes v_2) = D^{(1)}(v_1) \otimes D^{(2)}(v_2)$.
If $D^{(1)}$ and $D^{(2)}$ are irreps, their tensor product is not. Its decomposition in irreps is the Clebsh-Gordan decomposition.

Irreps

- **Orthogonality:** Given $D^{(\mu)}(g)$ an unitary irrep labeled with μ

$$\sum_{g\in G}\left[D^{(\lambda)}(g)_{ij}\right]^*\left[D^{(\mu)}(g)\right]_{kl}=\frac{g}{n_\lambda}\delta_{\lambda\mu}\delta_{ik}\delta_{jl}$$

  n_λ is the dimension of the irrep λ.

- **Character:** Define $\chi^{(\lambda)}(g)=\mathrm{Tr}D^{(\lambda)}(g)$. Orthogonality:

$$\sum_{g\in G}\chi^{(\lambda)\,*}(g)\chi^{(\nu)}(g)=g\delta_{\lambda\mu}\quad\sum_{\lambda}\chi^{(\lambda)\,*}(g_1)\chi^{(\lambda)}(g_2)=\frac{1}{C_1}\delta_{C_1C_2}$$

- **Theorem:** $\sum_\mu n_\mu^2=g$
- **Schur lemma:** $D(g)T=TD(g)\;\forall g\in G$ implies $T=a\mathbb{I}$

Irreps of S_3

	P_1	P_2	P_3	P_4	P_5	P_6
Trivial	1	1	1	1	1	1
Sign	1	1	1	-1	-1	-1
Matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Unitary	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \eta & 0 \\ 0 & \eta^* \end{pmatrix}$	$\begin{pmatrix} \eta^* & 0 \\ 0 & \eta \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^* \\ \eta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix}$

$\eta = e^{2\pi i/3}$

$$6 = 1 + 1 + 2^2$$

The trivial representation is always an irrep

There are 3 equivalence classes (equal the number of irreps)

Lie groups

A Lie group G is a set of elements $g(x)$ parameterized in terms of variables x belonging to a n dimensional manifold M
$G = \{g(x), x \in M\}$ and such that

- $g(x)g(y) = g(z)$ with $z = \phi(x, y)$ differentiable.
- $g^{-1}(x) = g(y)$ with $y = \Psi(x)$ differentiable.

Group order: Dimension of M (number of parameters)
Matrix Lie groups: Defined in terms of finite dimensional matrices
Compact Lie groups: Closed and bounded
Connected Lie groups: Every pair of matrices A and B are connected by a $M(t)$ in the group.
Simply connected: Every loop in the group can be collapsed continuously to a point.

Examples:

Examples: $GL(n,\mathbb{F})$, $SL(n,\mathbb{F})$ ($\det = 1$)

- Metric preserving groups $M^\dagger G M = G$ ($G = \mathbb{I}$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$). Compact
- Pseudo-metric preserving groups $M^\dagger \mathbb{I}_{pq} M = \mathbb{I}_{pq}$ $O(p,q)$, $U(p,q)$, Lorentz $O(3,1)$
- Simplectic $M^T S M = S$ with $S = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ denoted $Sp(n,\mathbb{F})$
- Combinations of the above: Poincare, Galilei, etc
- $UT(n,\mathbb{F})$ (Upper triangular matrices). Heisenberg $UT(3,\mathbb{R})$

Lie algebra

Lie algebra

The Lie algebra of G (denoted g) is the set of matrices X such that $e^{tX} \in G \ \forall t \in \mathbb{R}$

Matrix exponential

Defined through series $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ Convergent for every X

Properties:

- $e^0 = \mathbb{I}$, $(e^X)^{-1} = e^{-X}$, $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$
- $e^{X+Y} = e^X e^Y$ if $[X, Y] = 0$
- $e^{CXC^{-1}} = Ce^X C^{-1}$, $(e^{tX})' = Xe^{tX} = e^{tX} X$

Matrix exponential

Evaluation: Convert X to Jordan form $X = C(D+N)C^{-1}$

$$e^X = Ce^De^NC^{-1}$$

D and N commute and N is nilpotent (finite series expansion).

$$\exp\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

- **Logarithm:** $\log Z = \sum_{m=1}^{\infty}(-1)^{(m+1)}\frac{(z-1)^m}{m}$ (Around $Z = 1$)
- $e^{\log Z} = Z$ and $\log(e^Z) = Z$
- $\lim_{m\to\infty}(\mathbb{I} + X/m + C_m)^m = e^X \qquad (||c_m|| < c_0/m^2)$
- **Trotter formula:** $e^{X+Y} = \lim_{m\to\infty}(e^{X/m}e^{Y/m})^m$
- $\det(e^X) = e^{\mathrm{tr} X}$

Lie algebras, Examples

1. $GL(n,\mathbb{F})$ Any matrix X of dimension n in $\mathbb{F}$
2. $SL(n,\mathbb{F})$, $\mathrm{Tr}X = 0$
3. $U(n)$ $X^\dagger = -X$ Anti-hermitian
4. $SU(n)$ X anti-hermitian and $\mathrm{Tr}X = 0$
5. $Sp(n,\mathbb{C})$ $X = \left(\begin{array}{cc} A & B \\ C & -A^T \end{array} \right)$ B and C symmetric

Lie algebras

Theorem: The set of matrices X of a Lie algebra form a real vector space and $[X, Y] \in g$

$\{X_i, i = 1, \ldots, n\}$ is a basis of the vector space

Structure constants: $[X_i, X_j] = \sum_k C_{ij}^k X_k$

The Lie algebra of a Lie group is an algebra with the commutator as the "product"

Theorem: $X_i = \frac{\partial G(x_1,\ldots,x_n)}{\partial x_i}_{x_i=0}$ if $G(0) = \mathbb{I}$

Adjoint representation: $[A, X_i] = \sum_j ad(A)_{ji} X_j$

Group adjoint representation: $AX_iA^{-1} = \sum_j Ad(A)_{ji} X_j$
$e^{tad(A)} = Ad(exp(tA))$ and $ad(A) = [Ad(exp(tA)]'_{t=0}$

Operator algebra

Let $\hat{X}_{ij}$ be a set of operators satisfying $[\hat{X}_{ij}, \hat{X}_{rs}] = \hat{X}_{is}\delta_{jr} - \hat{X}_{rj}\delta_{si}$

For example $f_i^\dagger f_j$ and $b_i^\dagger b_j$ (f and b fermion and boson operators, respectively) Also $\hat{X}_{ij} = x_i \partial_j$ satisfy the requirement

Define $\mathcal{A} = \sum_{ij} A_{ij}\hat{X}_{ij}$, $\mathcal{B}$, etc then

$$[\mathcal{A}, \mathcal{B}] = \mathcal{C}$$

with $C = [A, B]$ i.e. there is a one-to-one correspondence between A and $\mathcal{A}$ satisfying commutation relations. For instance, in $so(3)$

$$X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \hat{L}_3 = x_2\partial_1 - x_1\partial_2$$

Cartan-Killing form

Define a pseudo-metric in the algebra

$$g_{ij} = \mathrm{Tr} ad(X_i) ad(X_j)$$

According to its eingenvalues divides the algebra in three subspaces

- V_0 invariant and nilpotent algebra (eigenvalue 0)
- V_- Compact subalgebra (negative eigenvalues)
- V_+ No special structure (positive eigenvalues)

If V_0 is empty the algebra is semi-simple or simple

From the algebra to the group

The exponential of the algebra is not able in general to recreate the whole group.

Cartan: But a wise choice of products of exponentials do.

The problem is associated to the compact subalgebras

Example: $sl(2,\mathbb{R})$ can be generated using the product of two exponentials, one associated to the compact subalgebra $so(2)$ and the other to the algebra of $SL(2,\mathbb{R})/SO(2)$

$$\begin{pmatrix} a & b+c \\ b-c & -a \end{pmatrix} = \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}\begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix}$$

From the algebra to the group

Isomorphic algebras do not lead to isomorphic Lie groups. But there exists a unique simply connected group that can be put in correspondence with the others.

Example: $su(2)$ and $so(3)$ are isomorphic but the corresponding groups are not. $SU(2)$ is "twice as large" as $SO(3)$ and "covers" it.

In $SU(2)$, a rotation along the z axis needs 4π to come back to the identity, whereas in $SO(3)$ you only need 2π

The correspondence between $SU(2)$ and $SO(3)$ is

$$R_{ij} = \frac{1}{2}\mathrm{Tr}[\sigma_i u \sigma_j u^{-1}]$$

u and $-u$ lead to the same R

Baker-Campbell-Hausdorff

There are many different arrangement of products of exponentials of the algebra that lead to the group.

They are connected through the BCH formula

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots$$

Every term in the formula is a commutator of elements of the algebra and therefore belongs to it

The formula is valid also for homomorphisms of the algebra and can be used to generate homomorphisms in the group (remember ad and Ad).

Structure of Lie algebra

The study in depth the structure of Lie algebras it is convenient to analyze the "secular" equation $[Z, X] = \lambda X$ for a given Z. It is equivalent to a secular equation in the adjoint representation

$$\det(ad(Z) - \lambda\mathbb{I})) = 0$$

The coefficients of the characteristic polynomial

$$\det(ad(Z) - \lambda\mathbb{I})) = \sum_{j=0}^{n} (-\lambda)^{N-j} \phi_j(z_1, \ldots, z_n)$$

fulfill $\phi_0 = 1$ $\phi_n = 0$ and are functions of combinations of the parameters of the algebra

Rank and Invariant

Rank: Is the number of independent coefficients in the secular equation of the adjoint representation

Invariant operators: The element of the algebra $\phi_j(X_1, \ldots, X_n)$ commute with all elements of the algebra

$$[\phi_j(X_1, \ldots, X_n), X_j] = 0$$

The number of independent invariant operators (Casimir) equals the rank of the algebra.

Example: In $su(2)$

$$\det(ad(Z) - \lambda\mathbb{I})) = (-\lambda)(\lambda^2 + \phi_2(a_1^2 + a_2^2 + a_3^3))$$

Rank 1 algebra. One Casimir $X_1^2 + X_2^2 + X_3^2$

Cartan subalgebra

The subspace of 0 eigenvalue of the secular equation defines a subalgebra (Cartan) with a basis H_i such that $[H_i, H_j] = 0$

In the subspace of the other eigenvalues α the basis is denoted E_α and they have canonical commutation relations

$$[H_i, E_\alpha] = \alpha_i E_\alpha \quad [E_\alpha, E_\beta] = E_{\alpha+\beta}$$

Theorem: For any semi-simple algebra of rank l there are

- l invariant operators ϕ_i
- l matrices H_i basis of the Cartan subalgebra

Eigenvalues of H_i and ϕ_i are essential to characterize irreducible representations of the algebra (and the group)

(*) An algebra is semi-simple if Cartan-Killing is not degenerated

Representations of the algebra

Homomorphisms of the elements of the algebra into linear applications defined in some vector space $\mathcal{V}$

The most interesting are the irreps (no invariant subspaces)

Every irrep of a **compact** Lie groups is equivalent to a unitary one

Compact Lie groups have infinite irreps but is number is countable

Non-compact Lie groups do not have finite dimensional unitary irreps

Building irreps of **compact** Lie algebras is similar to the construction of invariant subspaces of given angular momentum in QM (J_3 is the basis of the Cartan subalgebra and J^2 is the Casimir)

Integration in compact groups, Haar measure

Compact Lie groups have a finite invariant measure (Haar)

$$\int_G f(g)dg = \int_{a_1}^{b_1} dy_1 \ldots \int_{a_n}^{b_n} dy_n f(y_1, \ldots, y_n)\sigma(y_1, \ldots, y_n)$$

Let Γ^p and Γ^q be unitary irreps of a **compact** Lie group

$$\int_G dg(\Gamma^p(g))^*_{jk}(\Gamma^q(g))_{st} = \frac{1}{d_p}\delta_{pq}\delta_{js}\delta_{kt}$$

and also for the characters $\chi^p(q)$

$$\int_G dg\chi^p(g)^*\chi^q(g) = \delta_{pq}$$

$O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact Lie groups

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