# Numerical aspects of resonances

R. Lazauskas

## Overview

Methods for non-relativistic QM problem

I will try to give quite general overview of the most popular techniques to determine resonance positions, except for time dependent approaches

## Contents

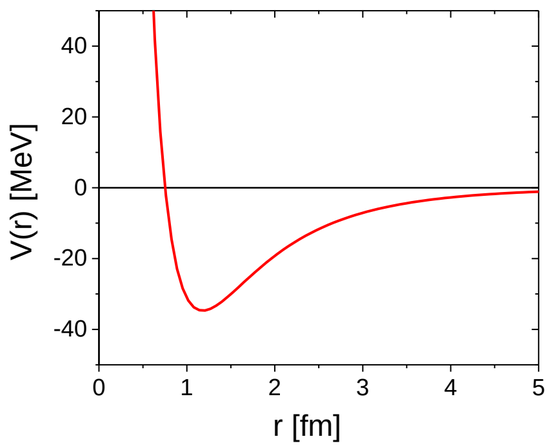
- Direct methods
  - Boundary condition approach: zeroes of Jost function
  - Complex scaling method & its variants
- Extrapolation & co
  - Effective range formulae, Padé expansion
  - ACCC
  - Stabilization technique

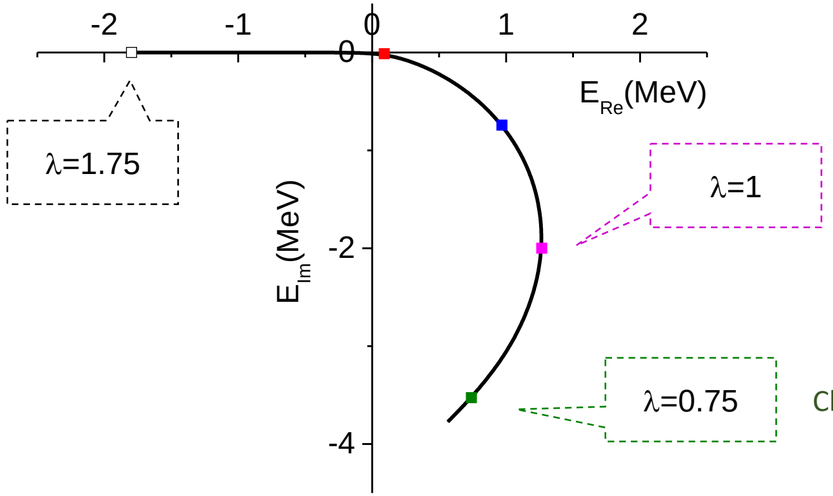
## Example

Local 2-body potential, describing n-$^3$H scattering in P-waves

$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$

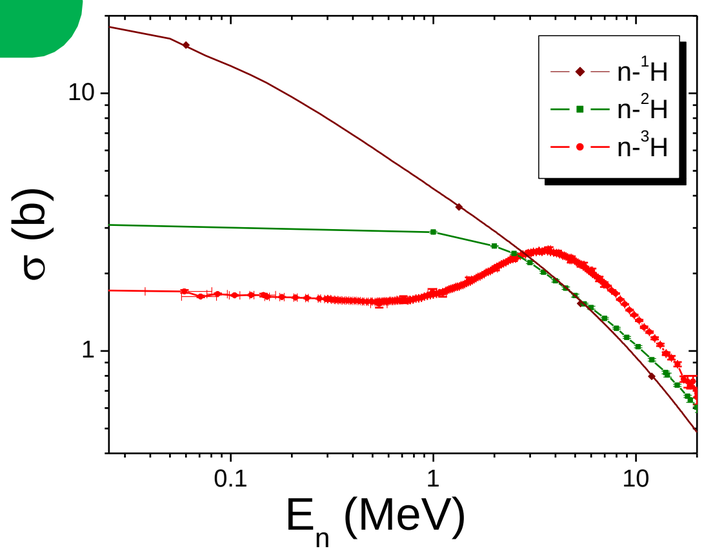


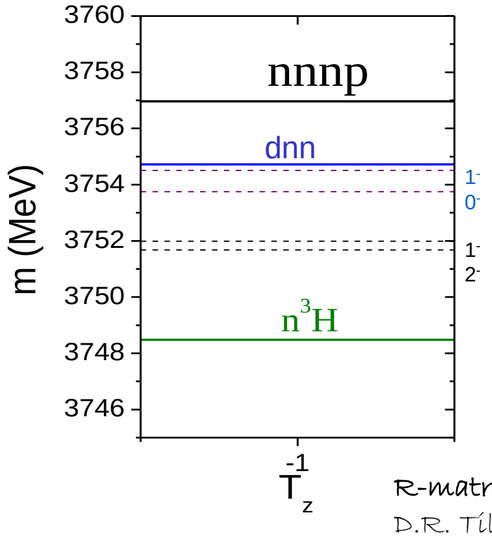

Closely reproduces n-$^3$H $J^\pi = 2^-$ phaseshifts

Closely reproduces n-$^3$H $J^\pi = 0^-$ phaseshifts

## Inspired by




n-$^3$H system contains four broad resonant states in P-waves




| $J^\pi$ | $E(^4H)-E(^3H)$ |
|---|---|
| $2^-$ | 3.19-2.71i |
| $1^-$ | 3.50-3.38i |
| $1^-$ | 6.02-6.5i |
| $0^-$ | 5.27-4.46i |

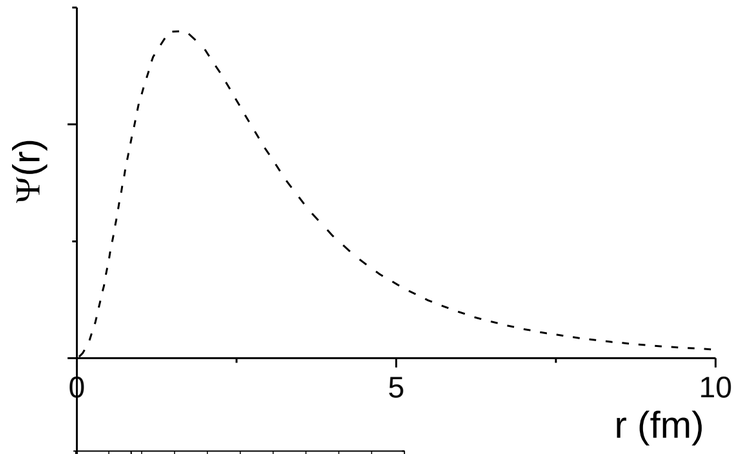
R-matrix analysis:
D.R. Tilley et al., Nucl. Phys. **A 541** (1992) 1

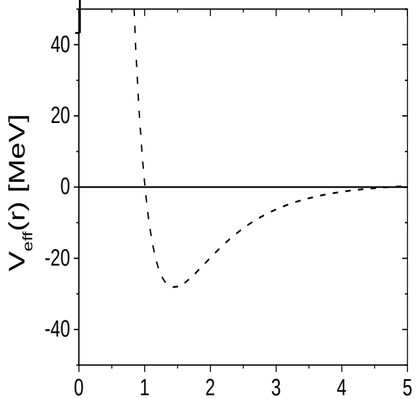
## Example

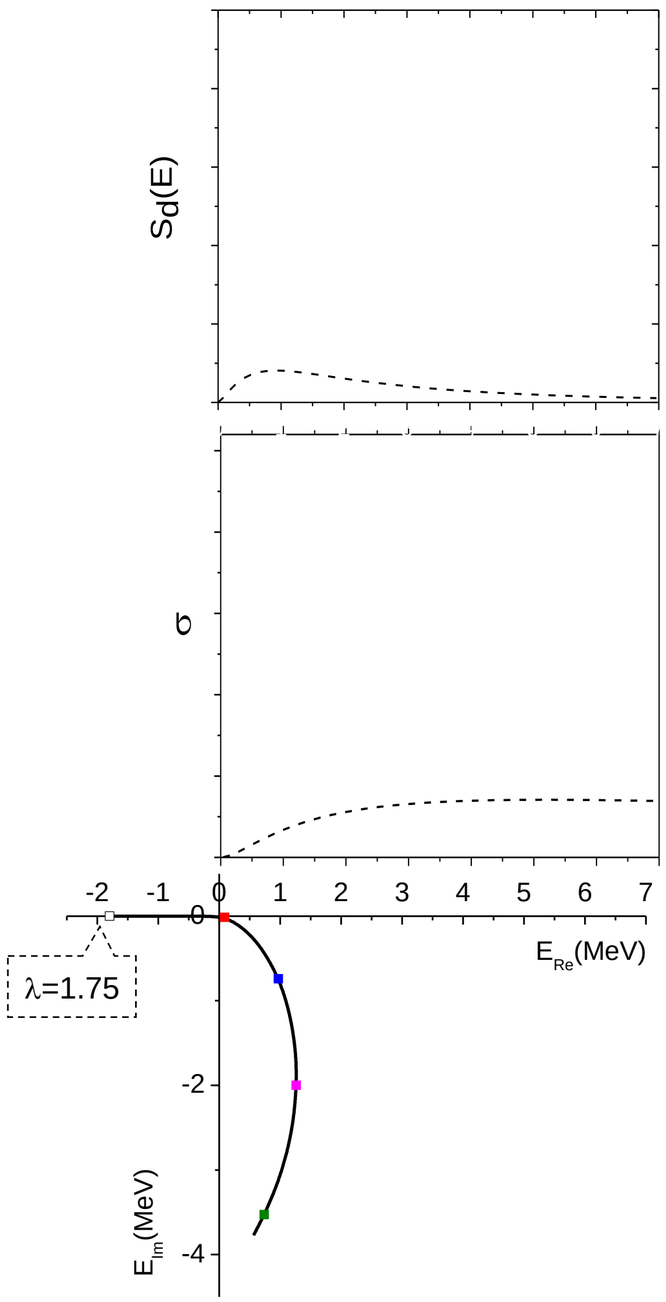
Local 2-body potential, describing n-$^3$H scattering in P-waves

$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$



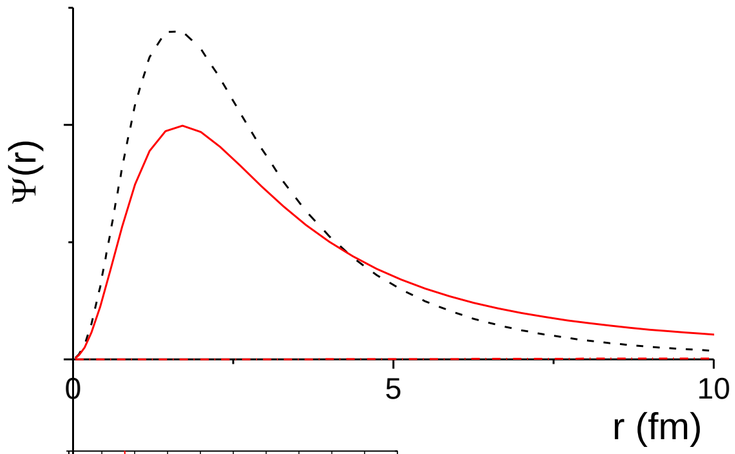



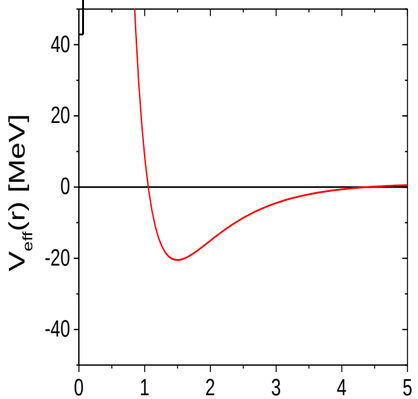
## Example

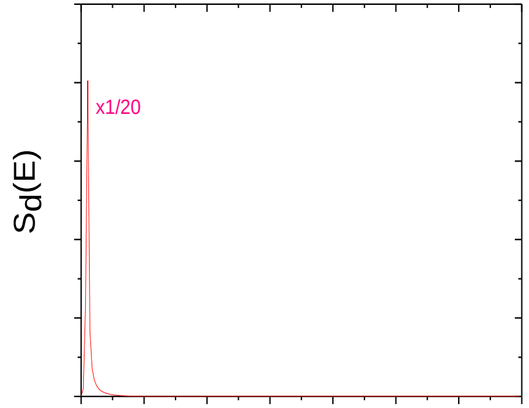
Local 2-body potential, describing n-$^3$H scattering in P-waves

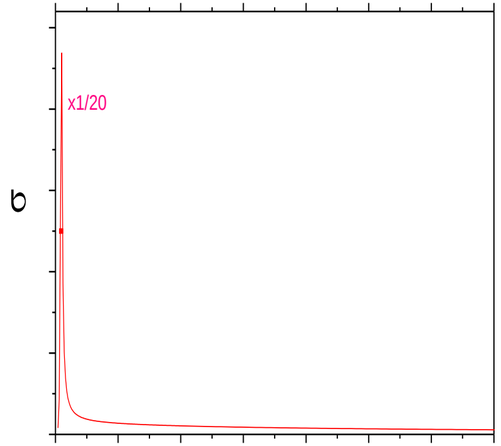
$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

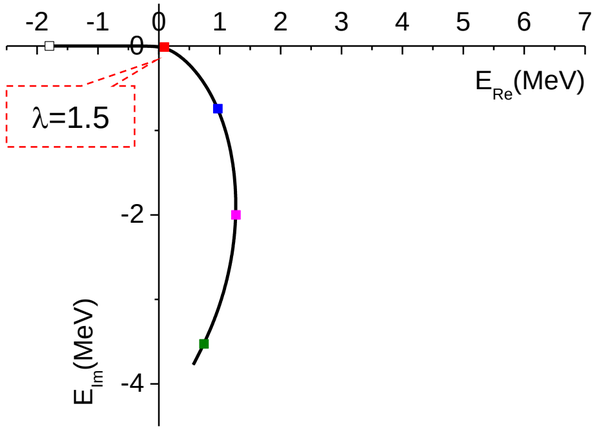
$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$



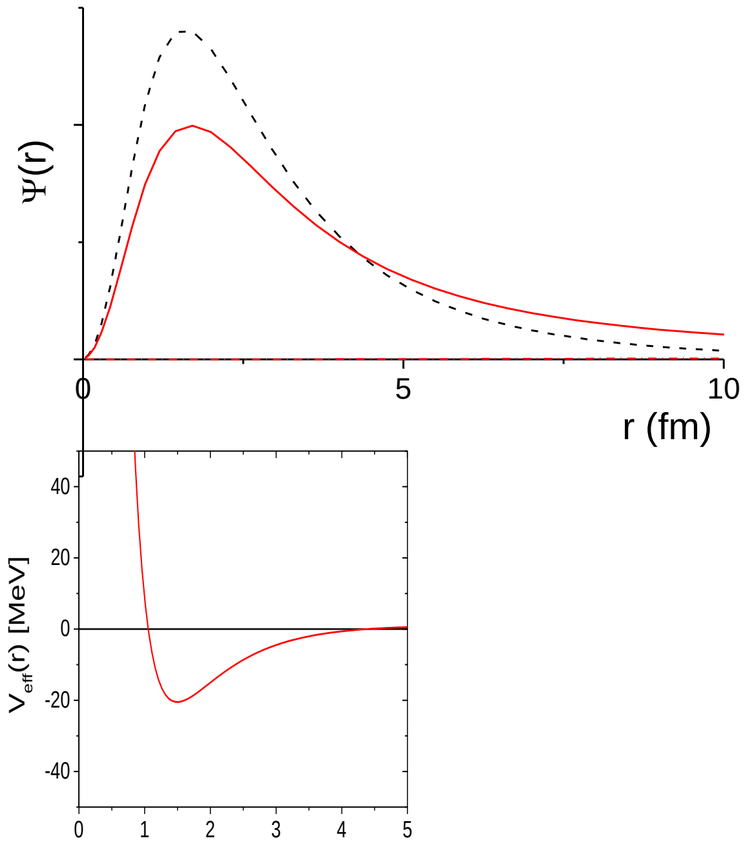



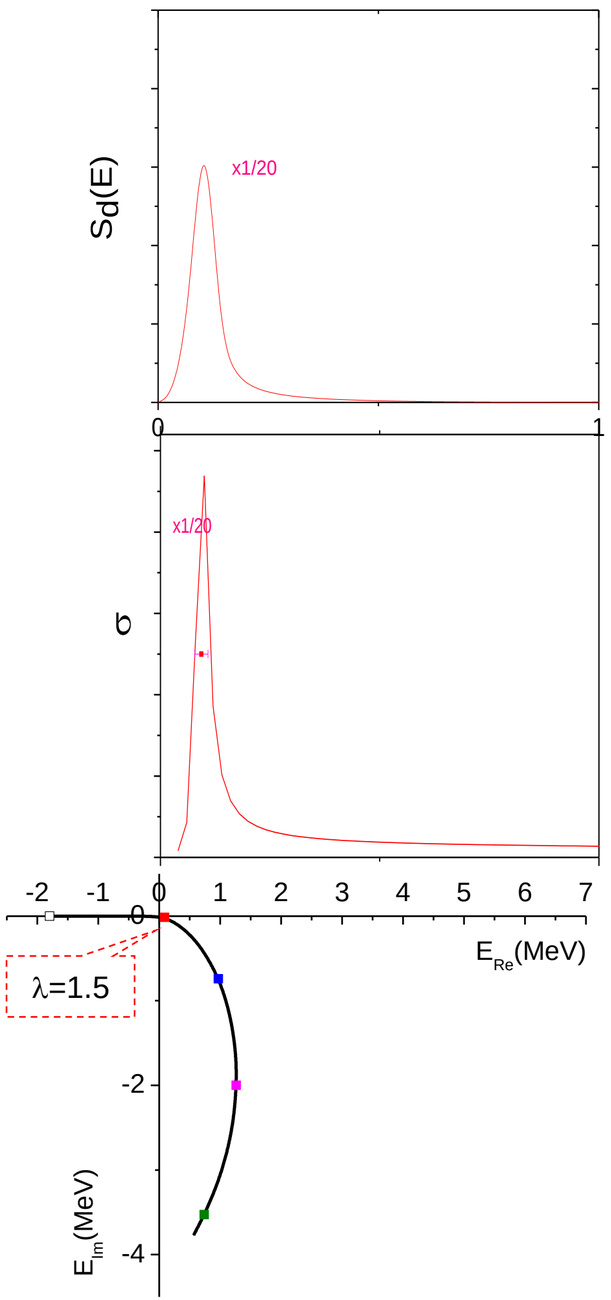
## Example

Local 2-body potential, describing n-$^3$H scattering in P-waves

$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$



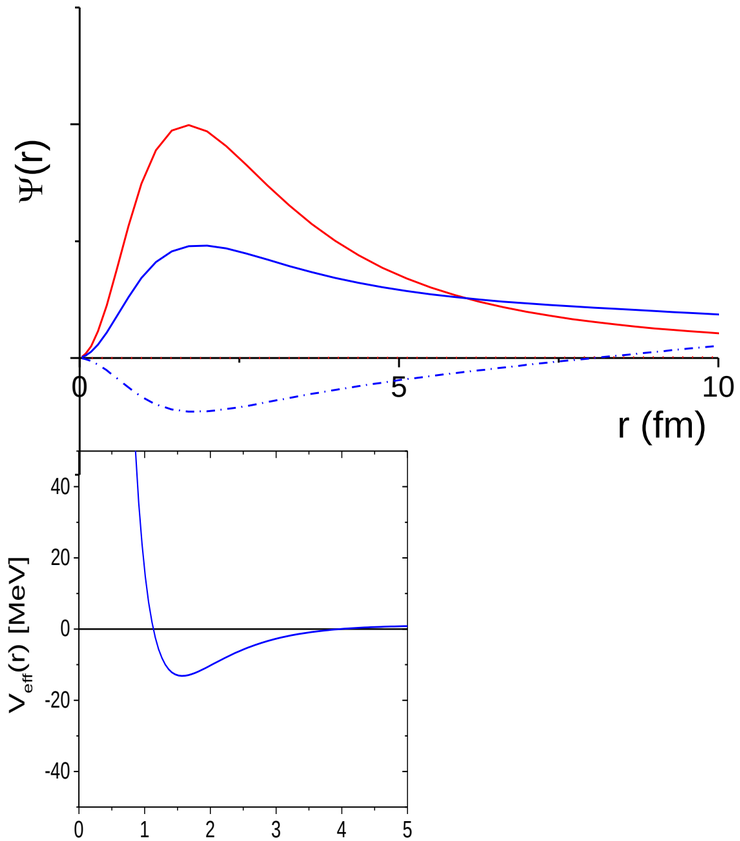



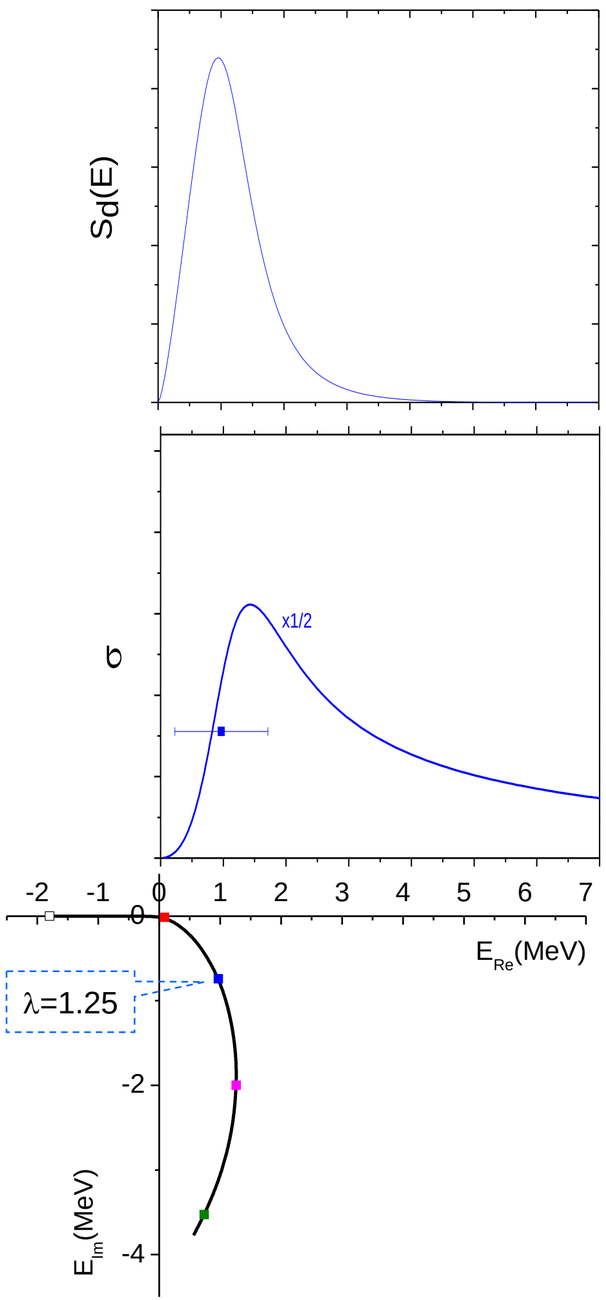
## Example

Local 2-body potential, describing n-$^3$H scattering in P-waves

$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$



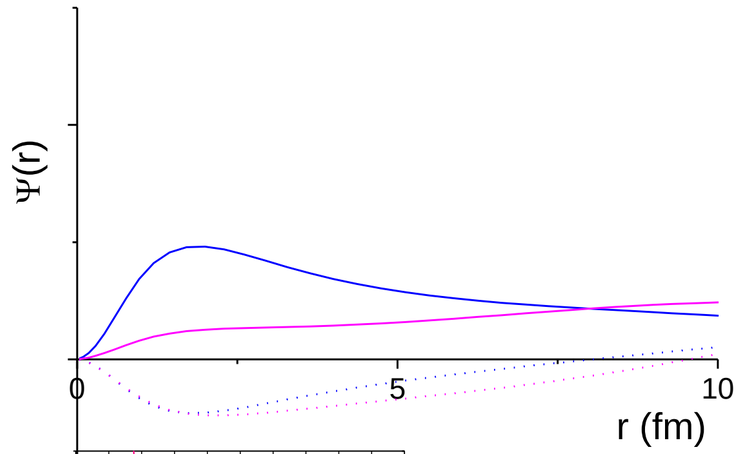



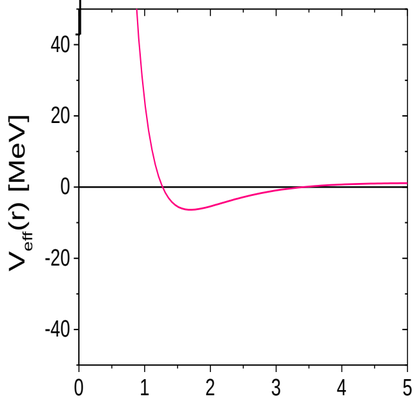
## Example

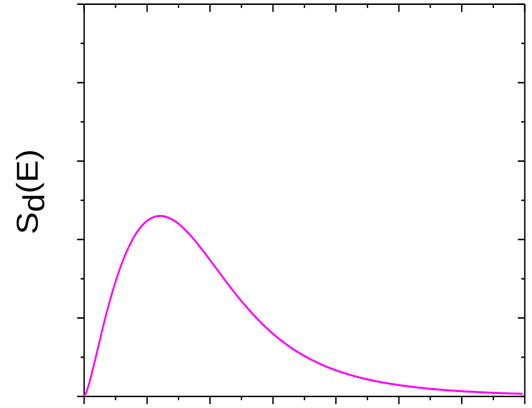
Local 2-body potential, describing n-$^3$H scattering in P-waves

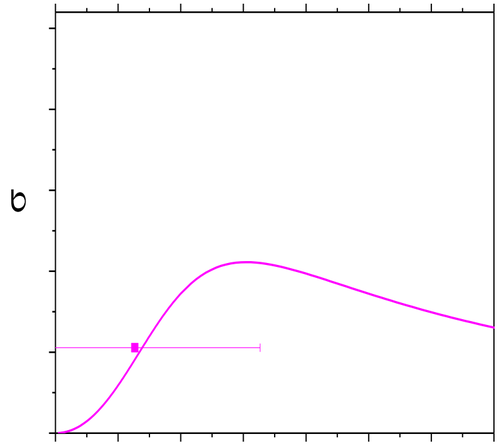
$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

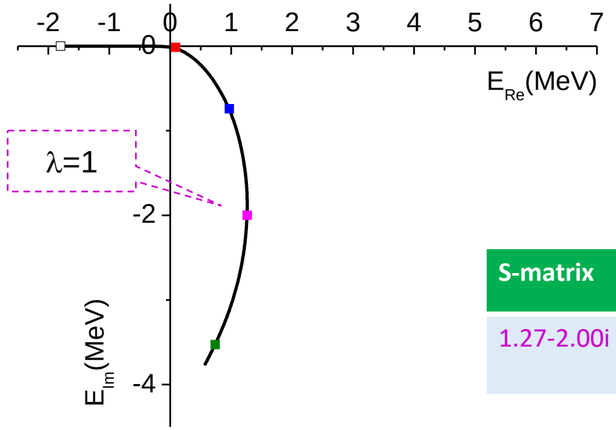
$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$



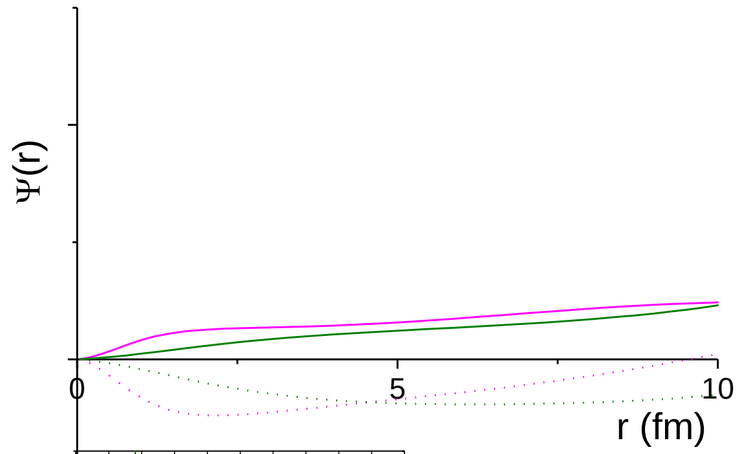

| S-matrix | R-matrix |
|---|---|
| 1.27-2.00i | 3.19-2.71i |

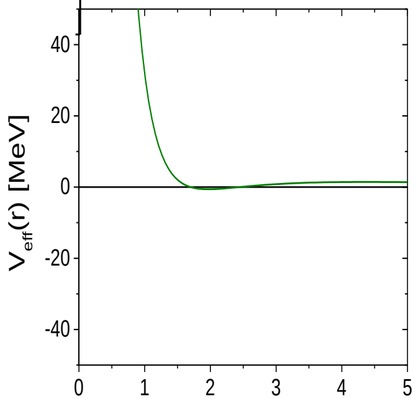
## Example

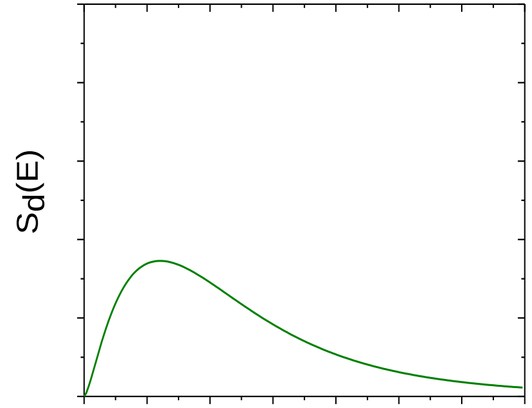
Local 2-body potential, describing n-$^3$H scattering in P-waves

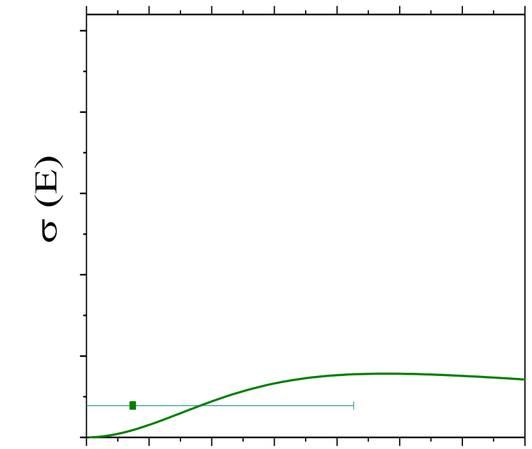
$$V_{l=1}(r) = \lambda\left[678.1\frac{e^{-2.55r}}{r} - 166.0\frac{e^{-0.68r}}{r}\right]$$

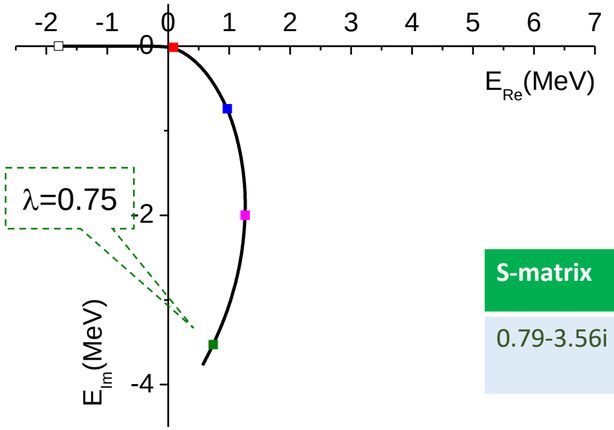
$$\frac{\hbar^2}{2\mu} = 27.647\ MeV.fm^2$$




| S-matrix | R-matrix |
|---|---|
| 0.79-3.56i | 5.27 -4.46i |

# 1. Boundary condition method

We consider radial Schrödinger equation (SE)

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\right]\psi_\ell(k,r)=k^2\psi_\ell(k,r); \qquad k^2=\frac{2\mu}{\hbar^2}E \qquad (1)$$

Likewise bound states resonances are regular solutions of the SE with purely outgoing wave function

$$\psi_\ell(k,r=0)=0; \quad \psi_\ell(k,r\to\infty)=C\hat{h}_\ell^+(\mathrm{kr}) \qquad (2)$$

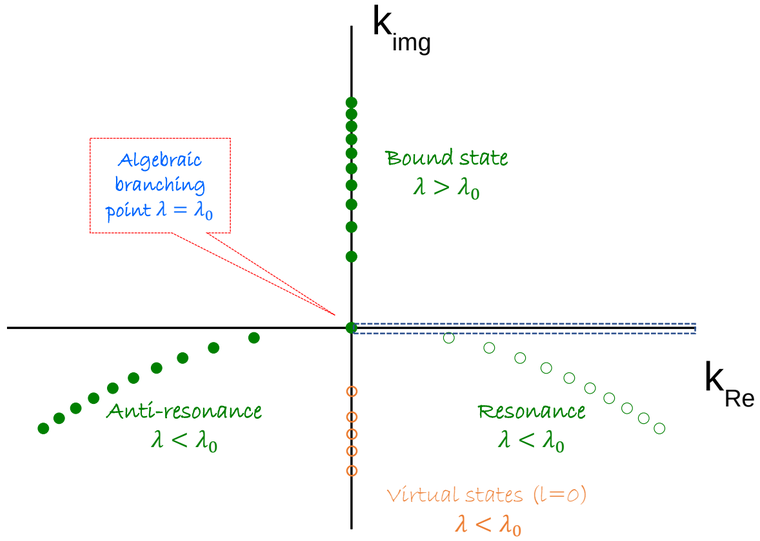
However since $Im(k_{res})<0$, the outgoing wave is exponentially divergent:

$$\hat{h}_\ell^+(k_{res}r)\sim e^{ik_{res}r}\sim e^{|Im(k_{res})|r} \qquad (3)$$

General solution of the radial Schrödinger behaves as:

$$\psi_\ell(k,r=0)=0; \quad \psi_\ell(k,r\to\infty)=f_\ell(k)\hat{h}_\ell^-(\mathrm{kr})-f_\ell(-k)\hat{h}_\ell^+(\mathrm{kr}) \qquad (4)$$

If one has a method of solving boundary condition problem, one may readily scan complex momentum plane for presence of zeroes of the Jost function $f_l(k)$:




## Method

- Solve (1) by imposing boundary condition (4) using your favorite method to solve differential equations: Numerov, Spline/collocation, calculable R-matrix, variational,… and calculate $S_\ell^{-1}(k)=f_\ell(k)/f_\ell(-k)$
- Repeating calculations for different k, scan the complex momentum plane searching the zeroes of the Jost function, or zeroes of $S_\ell^{-1}(k)=f_\ell(k)/f_\ell(-k)$. Use Newton or Lagrange interpolation methods to find iteratively $k_{res}$, where $S_\ell^{-1}(k_{res})=0$.

| λ | E (MeV) |
|---|---|
| 1.75 | -1.7914 |
| 1.50 | (9.33-1.50i)·10$^{-2}$ |
| 1.25 | 0.9712-0.7446i |
| 1.00 | 1.2666-2.0023i |
| 0.75 | 0.7868-3.5605i |

# 1. Boundary condition method

## Particular case, l=0

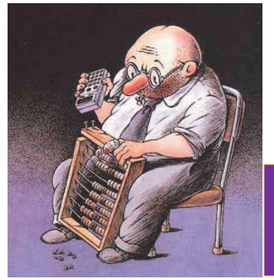
$\hat{h}_l^+(\mathrm{kr}) = e^{ikr}$

One looks for the solution: $\psi_0(k,r) = u(r)e^{ikr}$, by setting it to SE and factoring out $e^{ikr}$ one gets:

$$\left[-\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}V(r)\right]u(r) = k\left[2i\frac{d}{dr}\right]u(r)$$

$$u(0)=0; \quad \frac{d}{dr}u(r\to\infty)=0$$

| λ | 20 | 30 | 40 | E (MeV) |
|---|---|---|---|---|
| 1.75 | -1.8208 | -1.7929 | -1.7913 | -1.7914 |
| 1.50 | (5.73-2.82i) $\cdot 10^{-2}$ | (9.50-2.01i) $\cdot 10^{-2}$ | (9.62-1.51i) $\cdot 10^{-2}$ | (9.33-1.50i)$\cdot 10^{-2}$ |
| 1.25 | 0.92-0.71i | 1.02-0.71i | 1.12-0.82i | 0.9712-0.7446i |
| 1.00 | 1.55-1.91i | 1.16-1.78i | 0.96-1.57i | 1.2666-2.0023i |
| 0.75 | | | | 0.7868-3.5605i |



## Life hack for l>0

In order to estimate resonance position for l>0, one may cut centrifugal barrier term in SE beyond certain $r = R_{max}$
As for l=0, one may look for solution in a form $\psi_l(k,r) = u_l(r)e^{ikr}$ and solve:

$$\left[-\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}V(r) + Hn(R_{max}-r)\frac{2\mu}{\hbar^2}\frac{l(l+1)}{r^2}\right]u_l(r) = k_{res}\left[2i\frac{d}{dr}\right]u_l(r)$$

$$u_l(0)=0; \quad \frac{d}{dr}u(r>R_{max})=0$$

Numerical Heavside Delta function

# 1. Boundary condition method

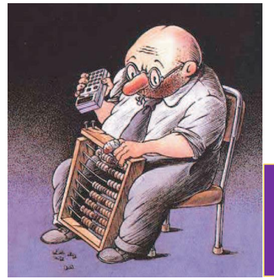
## Particular case, l=0

$\hat{h}_l^+(\mathrm{kr}) = e^{ikr}$

One looks for the solution: $\psi_0(k,r) = u(r)e^{ikr}$, by setting it to SE and factoring out $e^{ikr}$ one gets:

$$\left[-\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}V(r)\right]u(r) = k\left[2i\frac{d}{dr}\right]u(r)$$

$$u(0)=0; \quad \frac{d}{dr}u(r\to\infty)=0$$



## Life hack for l>0

In order to estimate resonance position for l>0, one may cut centrifugal barrier term in SE beyond certain $r = R_{max}$
As for l=0, one may look for solution in a form $\psi_l(k,r) = u_l(r)e^{ikr}$ and solve:

$$\left[-\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}V(r) + Hn(R_{max}-r)\frac{2\mu}{\hbar^2}\frac{l(l+1)}{r^2}\right]u_l(r) = k_{res}\left[2i\frac{d}{dr}\right]u_l(r)$$

$$u_l(0)=0; \quad \frac{d}{dr}u(r>R_{max})=0$$

Numerical Heavside Delta function

## Drawbacks

- Boundary condition problem
- Extension beyond 2-body case due to complexity of the boundary conditions, possibility of presence of more than one open channels, break up,…

- Stability of the resonances

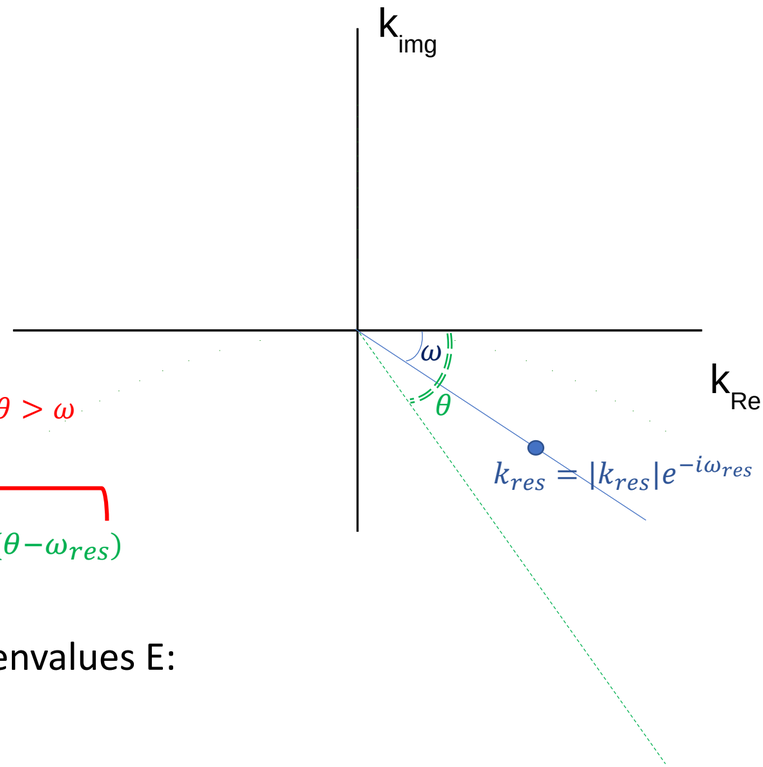
$$\hat{h}_\ell^\pm(kr) \sim e^{\pm ikr} \sim e^{ik_R r}e^{\pm k_{Im} r}$$

$$\frac{\hat{h}_\ell^+(\mathrm{kr})}{\hat{h}_\ell^-(\mathrm{kr})} \sim e^{2k_{Im} r}$$

$$\delta f_\ell(k) \sim \int \delta V(r) e^{2k_{Im} r} dr$$

As $k_{Im}$ increases resonance moves away from the real axis, its position becomes more and more affected by the details of the potential tail

# 2. Complex scaling (CS) method




## Idea

*Credits to R. Hartree et al., (1946)*
*J. Nuttal and H. L. Cohen, Phys. Rev. **188** (1969) 1542*

- If one scales $r \to\to re^{i\theta}$

$$\psi_\ell(r \to \infty) = C\hat{h}_\ell^+(k_{res}\mathrm{r}) \to\to C\hat{h}_\ell^+\left(k_{res}\mathrm{r}e^{i\theta}\right) \sim e^{ik_{res}re^{i\theta}} = e^{i|k_{res}|r\cdot\cos(\theta-\omega_{res})}\underbrace{e^{-|k_{res}|r\cdot\sin(\theta-\omega_{res})}}_{\text{Exp. bound if } \theta > \omega}$$

- Thus finding resonance positions is limited to solving CS Schrödinger's eq. for the complex eigenvalues E:

$$\frac{\hbar^2}{e^{2i\theta}2\mu}\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right]\tilde{\psi}_\ell(r) - V(re^{i\theta})\tilde{\psi}_\ell(r) = E\tilde{\psi}_\ell(r);$$

$$\tilde{\psi}_\ell(0) = 0; \qquad \tilde{\psi}_\ell(r \to \infty) = 0$$

## Method

- Expand c.s. outgoing wave in your favorite basis:

$$\tilde{\psi}_\ell(r) \approx \sum_{i=1}^{N} c_i \phi_i$$

Complex coefficients

square integrable basis functions

- Convert c.s. Schrödinger equation into algebraic eigenvalue problem

$$[\widetilde{H}_{j,i} - E\delta_{i,j}]c_i = 0$$

When calculating ME's of $[\widetilde{H}_{j,i}]$ one should multiply kinetic energy ME's of bound state problem by $\frac{1}{e^{2i\theta}}$ & recalculate potential energy ME's for $V(re^{i\theta})$

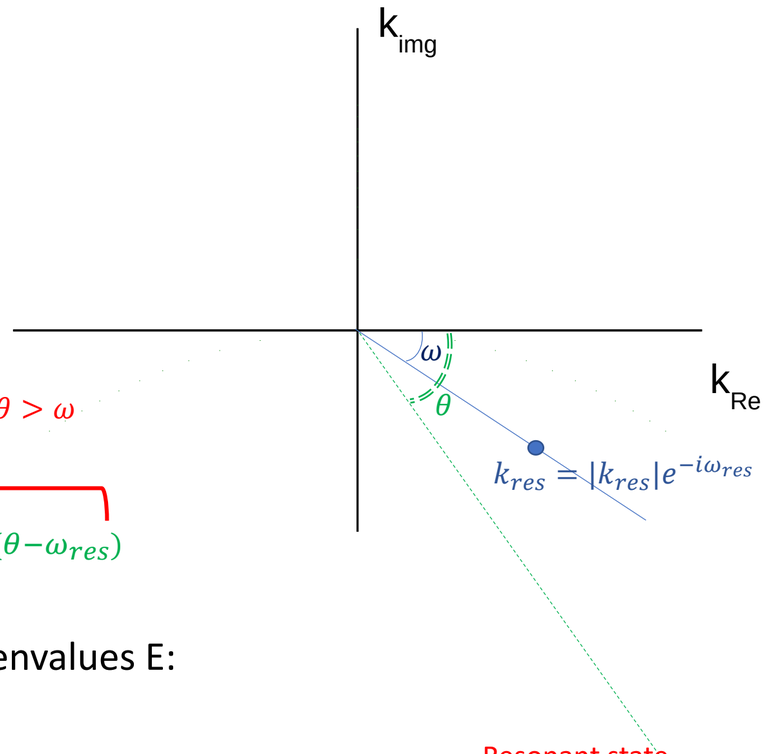
# 2. Complex scaling (CS) method

## Idea

*Credits to R. Hartree et al., (1946)*
*J. Nuttal and H. L. Cohen, Phys. Rev. **188** (1969) 1542*

- If one scales $r \to\to re^{i\theta}$

$$\psi_\ell(r \to \infty) = C\hat{h}_\ell^+(k_{res}\mathrm{r}) \to\to C\hat{h}_\ell^+\left(k_{res}\mathrm{r}e^{i\theta}\right) \sim e^{ik_{res}re^{i\theta}} = e^{i|k_{res}|r\cdot\cos(\theta-\omega_{res})}\underbrace{e^{-|k_{res}|r\cdot\sin(\theta-\omega_{res})}}_{\text{Exp. bound if } \theta > \omega}$$




- Thus finding resonance positions is limited to solving CS Schrödinger's eq. for the complex eigenvalues E:

$$\frac{\hbar^2}{e^{2i\theta}2\mu}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right]\tilde{\psi}_l(r) - V(re^{i\theta})\tilde{\psi}_l(r) = E\tilde{\psi}_l(r);$$

$$\tilde{\psi}_\ell(0) = 0; \qquad \tilde{\psi}_\ell(r \to \infty) = 0$$

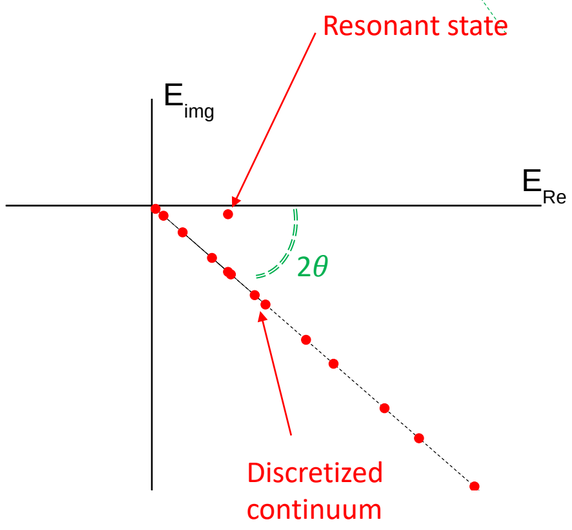
## Spectrum

- Resonant state is exposed if $\theta > \omega_{res}$

- Discretized continuum states are rotated by a angle $2\theta$ in energy plane, since:

$$\psi_{sc}(r \to \infty) \sim \sin(kr + \delta) \to\to \sin\left(\tilde{k}r + \delta\right)$$

thus k $\to\to$ $\tilde{k} = ke^{-i\theta}$; E $\to\to$ $Ee^{-i2\theta}$



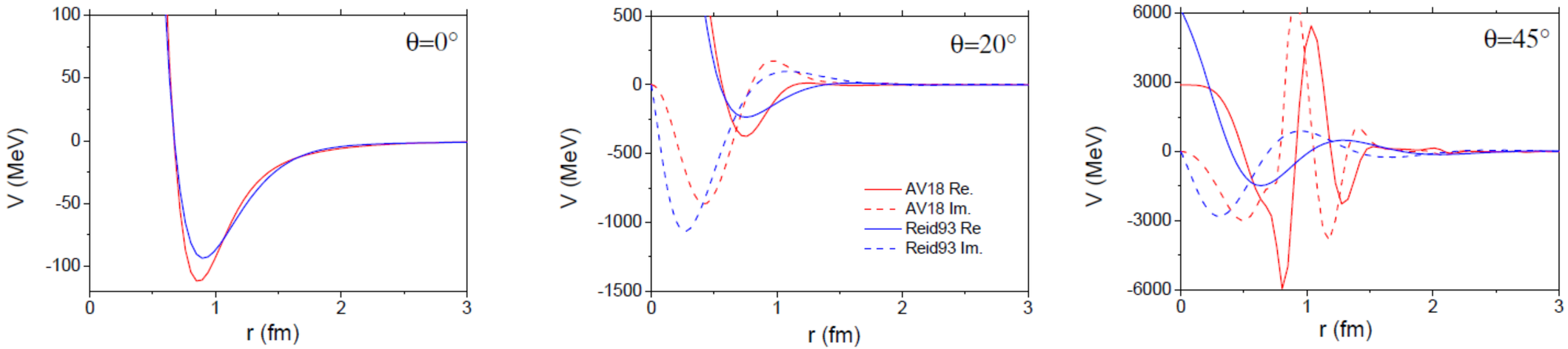
## Drawbacks

- Solution identical to bs problem, only in complex arithmetics

- One should work with CS potential $V(re^{i\theta})$ or CS basis functions
  - 😐 Potential and/or basis should be analytic
  - 👎 Transformed potential is usually more complicated than the original one (hardcore oscillations)
  - 👎 May imposes upper limit for the θ to be used, since

$$V(r) \sim \exp(-\mu r^n) \rightarrow\rightarrow \underbrace{\exp(-\mu r^n e^{in\theta})}_{\text{diverges for } \theta > \pi/(2n)}$$

Typical ill-behaved example χEFT potentials in nuclear physics, which involve high-momentum regulators $\exp(-cp^n)$ $with\ n = 6 - 8$




Usually basis functions are better behaved than pot. under CS, therefore one may use Cauchy theorem to calculate matrix elements:

$$V_{ij}^{\theta} = \int_0^\infty \phi_i(r) V(re^{i\theta}) \phi_j(r)\, \mathrm{dr} = e^{-i\theta} \int_0^\infty \phi_i(re^{-i\theta}) V(r) \phi_j(re^{-i\theta}) \mathrm{dr}$$

# 2. Complex scaling (CS) method

## Drawbacks

- Solution identical to bs problem, only in complex arithmetics

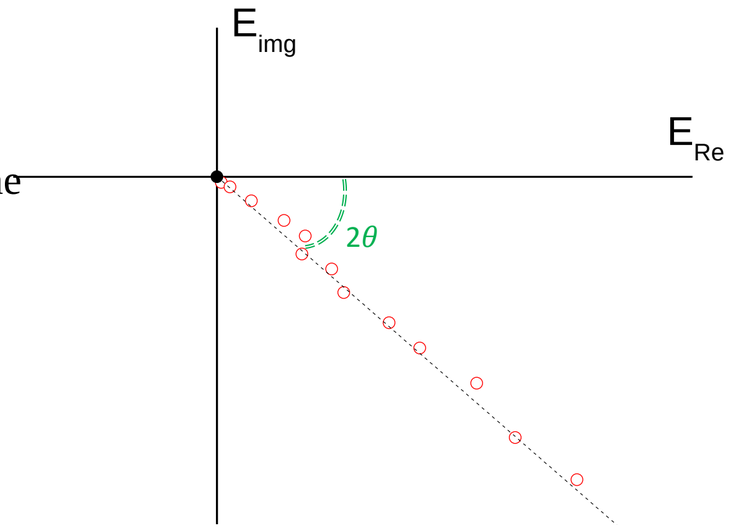
- One should work with CS potential $V(re^{i\theta})$ or CS basis functions
  - 😐 Potential and/or basis should be analytic
  - 👎 Transformed potential is usually more complicated than the original one (hardcore oscillations)
  - 👎 May imposes upper limit for the θ to be used, since

$$V(r) \sim \exp(-\mu r^n) \rightarrow\rightarrow \underbrace{\exp(-\mu r^n e^{in\theta})}_{\text{diverges for } \theta > \pi/(2n)}$$

- Eigenvalues distributed in a complicated way for A>2 systems

Extended ABC theorem: E. Balslev et al., Commun. Math. Phys., vol. **22**, (1971) 280

1) Discretized scattering states are spread around $2\theta$ tangent line in energy plane (due to the numerics $E_{sc} = |E_{sc}|e^{-i2\theta}$ relation is not perfectly preserved)



# 2. Complex scaling (CS) method

## Drawbacks

- Solution identical to bs problem, only in complex arithmetics

- One should work with CS potential $V(re^{i\theta})$ or CS basis functions
  - 😐 Potential and/or basis should be analytic
  - 👎 Transformed potential is usually more complicated than the original one (hardcore oscillations)
  - 👎 May imposes upper limit for the θ to be used, since

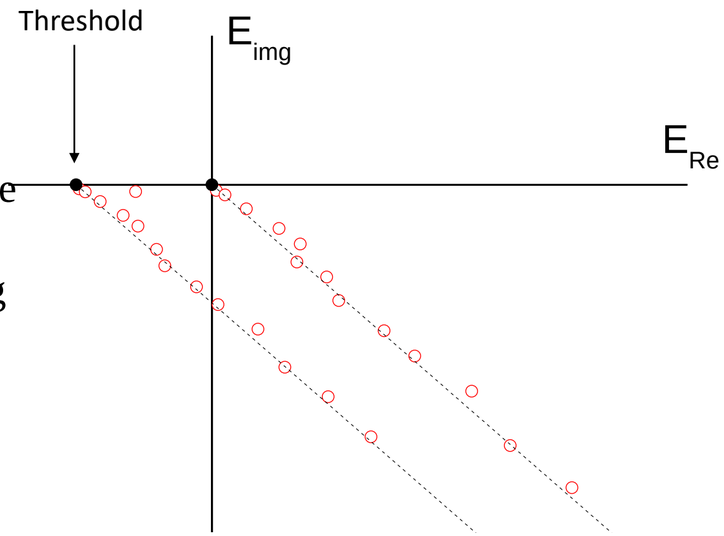
$$V(r) \sim \exp(-\mu r^n) \rightarrow\rightarrow \underbrace{\exp(-\mu r^n e^{in\theta})}_{\text{diverges for } \theta > \pi/(2n)}$$

- Eigenvalues distributed in a complicated way for A>2 systems

Extended ABC theorem: E. Balslev et al., Commun. Math. Phys., vol. 22, (1971) 280

1) Discretized scattering states are spread around $2\theta$ tangent line in energy plane (due to the numerics $E_{sc} = |E_{sc}|e^{-i2\theta}$ relation is not perfectly preserved)

2) Additional scattering eigenvalues are spread around $2\theta$ tangent lines emerging from each threshold, representing systems breaking into subsystems



## Drawbacks

- Solution identical to bs problem, only in complex arithmetics

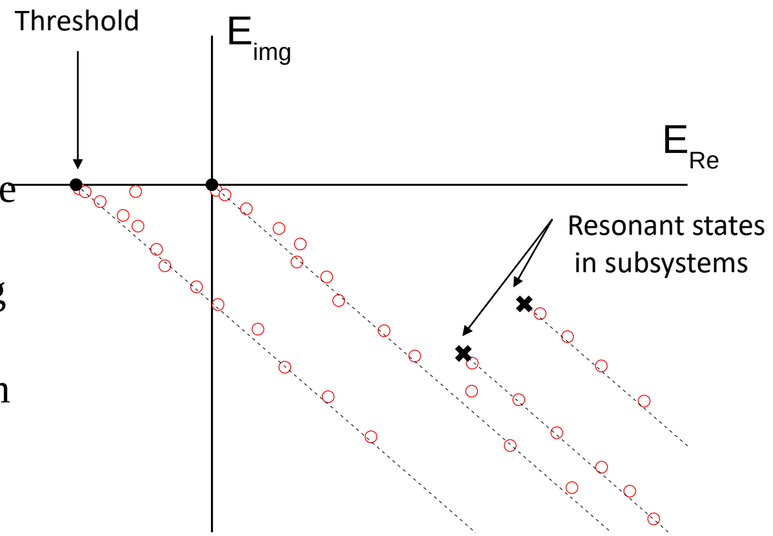
- One should work with CS potential $V(re^{i\theta})$ or CS basis functions
  - 😐 Potential and/or basis should be analytic
  - 👎 Transformed potential is usually more complicated than the original one (hardcore oscillations)
  - 👎 May imposes upper limit for the θ to be used, since

$$V(r) \sim \exp(-\mu r^n) \rightarrow\rightarrow \underbrace{\exp(-\mu r^n e^{in\theta})}_{\text{diverges for } \theta > \pi/(2n)}$$

- Eigenvalues distributed in a complicated way for A>2 systems

Extended ABC theorem: E. Balslev et al., Commun. Math. Phys., vol. 22, (1971) 280

1) Discretized scattering states are spread around $2\theta$ tangent line in energy plane (due to the numerics $E_{sc} = |E_{sc}|e^{-i2\theta}$ relation is not perfectly preserved)
2) Additional scattering eigenvalues are spread around $2\theta$ tangent lines emerging from each threshold, representing systems breaking into subsystems
3) Third kind of eigenvalues are scattered around $2\theta$ tangent lines emerging from each resonant state, present in subsystems, position

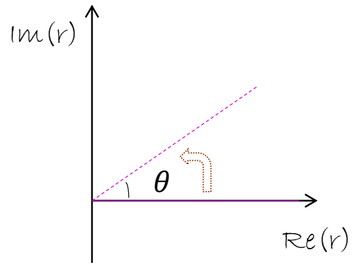



One usually performs a few independent calculations to clarify stability of the resonant eigenvalues: relative to $\theta$, basis size, basis parameters, etc..

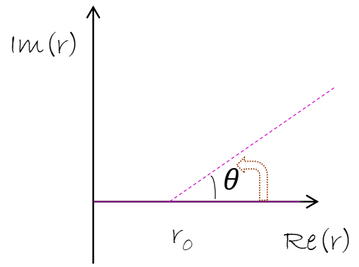
# 2. Exterior complex scaling (ECS) methods

B. Simon, Phys. Lett. A 71 (1979) 211

## Idea



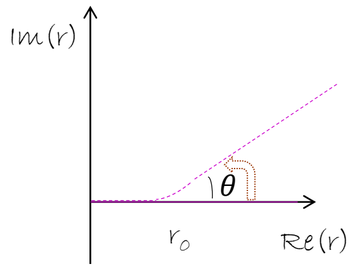

- The original CS is based on the transformation $r \to\to re^{i\theta}$
  For short range potential it may lead however to complications, when evaluating CS potential energy matrix elements




- To avoid former complications one may introduce exterior complex scaling (ECS) procedure, which avoids transforming sharp short range potential parts (within $r_0$ radius)

$$\begin{cases} r \to\to r & r < r_0 \\ r \to\to r_0 + (r - r_0)e^{i\theta} & r > r_0 \end{cases}$$

This may lead however to numerical instabilities due to non-analyticity (sharpness) of the transformation




- One may make a compromise by making the transformation smoother (SECS)

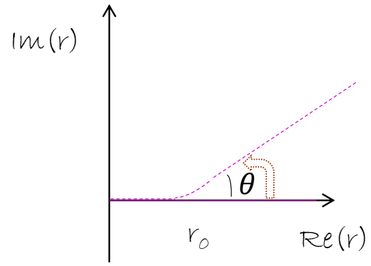
$$r \to\to R(r)$$
$$\begin{cases} r \to\to r & r \ll r_0 \\ r \to\to r_0 + (r - r_0)e^{i\theta} & r \gg r_0 \end{cases}$$

$$\hat{H}^{secs}(r) = -\frac{\hbar^2}{2\mu[R'(r)]^2}\left[\frac{d^2}{dr^2} - \frac{R''(r)}{R(r)}\frac{d}{dr}\right] + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{[R(r)]^2} + V(R(r))$$

For a Square integ. basis:

$$V_{ij}^{secs} = \int_0^\infty \phi_i(r)V(R(r))\phi_j(r)R'(r)\,\mathrm{d}r; \quad H_{ij}^{secs} = \frac{\hbar^2}{2\mu}\int_0^\infty \phi'_i(r)\frac{1}{R'(r)}\phi'_j(r)\,\mathrm{d}r$$

# 2. Exterior complex scaling (ECS) methods




- SECS works perfectly for 2-body case. One may use transformations with large $\theta$ angles and thus determine positions of broad resonances.

## Drawbacks

- Formally SECS or ECS is difficult (even impossible) to generalize beyond A=2 case.

## Life-hack:

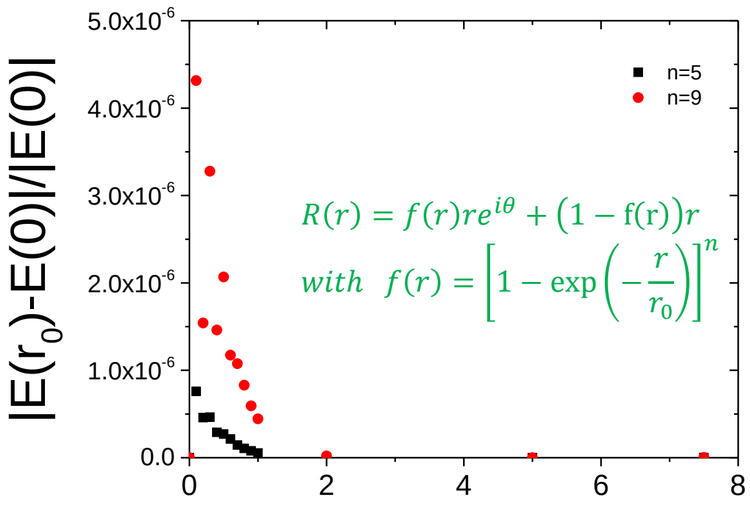
"Dirty" exterior complex scaling method:

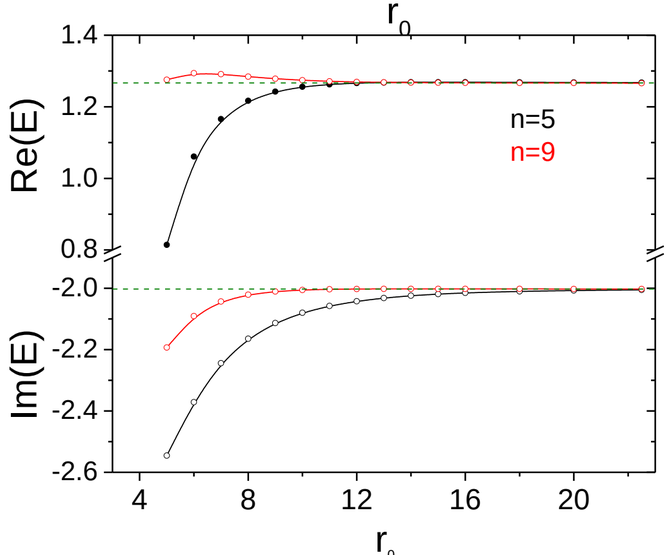
- Use sufficiently sharp external scaling
- Employ proper kinetic energy transformation
- Avoid transformation of the potential energy

$$\hat{H}^{secs}(r) = -\frac{\hbar^2}{2\mu[R'(r)]^2}\left[\frac{d^2}{dr^2} - \frac{R''(r)}{R(r)}\frac{d}{dr}\right] + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{[R(r)]^2} + V(R(r))$$

$$\approx -\frac{\hbar^2}{2\mu[R'(r)]^2}\left[\frac{d^2}{dr^2} - \frac{R''(r)}{R(r)}\frac{d}{dr}\right] + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{[R(r)]^2} + V(r)$$

Calculation of $\lambda = 1$ resonance with $\theta$=40°




Application beyond A=2: R. Lazauskas et al., Phys.Lett. B 791 (2019) 335-341

## 2. Extrapolation of effective range formulae (ERF)

- Let consider ERF for scattering by short range potential

$$k^{2\ell+1}cot\delta_\ell(k) = -\frac{1}{a_\ell}+\frac{1}{2}r_\ell \mathrm{k}^2 + \mathrm{c}_\ell^{(4)}\mathrm{k}^4 + \cdots$$

or

$$ik^{2\ell+1}\frac{S_l(k)+1}{S_l(k)-1} = -\frac{1}{a_\ell}+\frac{1}{2}r_\ell \mathrm{k}^2 + \mathrm{c}_\ell^{(4)}\mathrm{k}^4 + \cdots \quad (1)$$

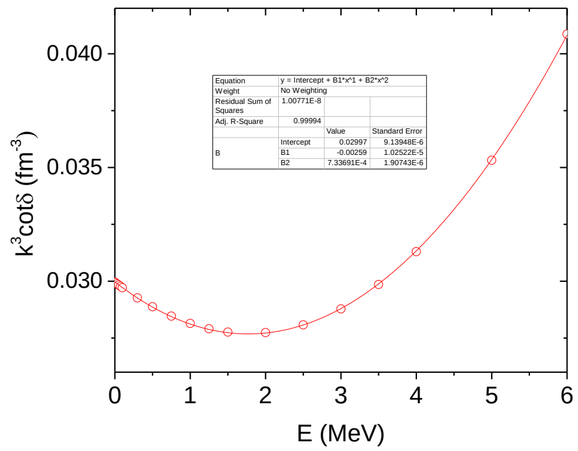
At pole $S_l(\mathrm{k_{res}})$ is singular, and thus:

$$ik_{res}^{2\ell+1} = -\frac{1}{a_\ell}+\frac{1}{2}r_\ell \mathrm{k}_{res}^2 + \mathrm{ck}_{\mathrm{res}}^4 + \cdots \quad (2)$$

**Method**

Thus we have a ready method to find resonance positions:

1. Calculate scattering observables $k^{2\ell+1}cot\delta_\ell(k)$ at several energies
2. Determine ERF parameters: $a_\ell, r_\ell, \mathrm{c}_\ell^{(2i+2)}$ with i=1,2,.. by fitting eq.(1)
3. Solve eq.(2) to determine $k_{res}$ and thus $E_{res} = \frac{\hbar^2}{2\mu}k_{res}^2$




| i | $E_{res}$ (MeV) |
|---|---|
| 0 | 1.4864-1.9780i |
| 1 | 1.2612-1.9981i |
| 2 | 1.2661-2.0031i |
| | 1.2666-2.0023i |

- Former expression does not allow to take into account other singularieties (cuts), in this case one may employ Padé parametrization

$$k^{2\ell+1}cot\delta_\ell(k) = \frac{P_N(k^2)}{Q_M(k^2)}$$

$P_N(x)$ *and* $Q_M(x)$ are polynomials of *x* of order *N* and *M* respectively

At pole, one should have: $ik_{res}^{2\ell+1} = -\frac{1}{a_\ell}\frac{P_N(k_{res}^2)}{Q_M(k_{res}^2)}$, or

$$ik_{res}^{2\ell+1}Q_M(k_{res}^2) = P_N(k_{res}^2)$$

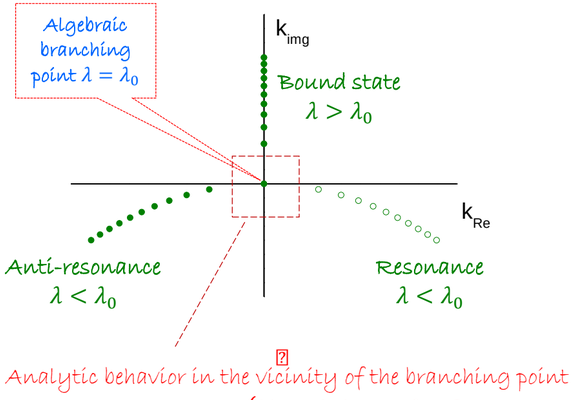
## 2. Extrapolation of effective range formulae (ERF)

### Drawbacks

- Requires to solve scattering problem : complicated for A>2 case
- When extrapolating - one should take into account presence of the thresholds due to multiple open channels & its effect on scattering amplitude (discontinuities in its energy derivative)

# 2. Analytic continuation in coupling constant (ACCC)

- For a resonant states evolving from a bound state, one may artificialy bind a state and try to use bound state energy input to evaluate position of a resonant state
- However, when moving from bound state region (axis!) to continuum it **is crucial** to account for **proper analytical behavior**: branching points & non-trivial pole trajectory




Analytic behavior in the vicinity of the branching point

$$k(\lambda \to \lambda_0) \sim \begin{cases} i(\ \ -\lambda_0), & \ell = 0 \\ i\sqrt{\lambda - \lambda_0}, & \ell > 0 \end{cases}$$

## Method

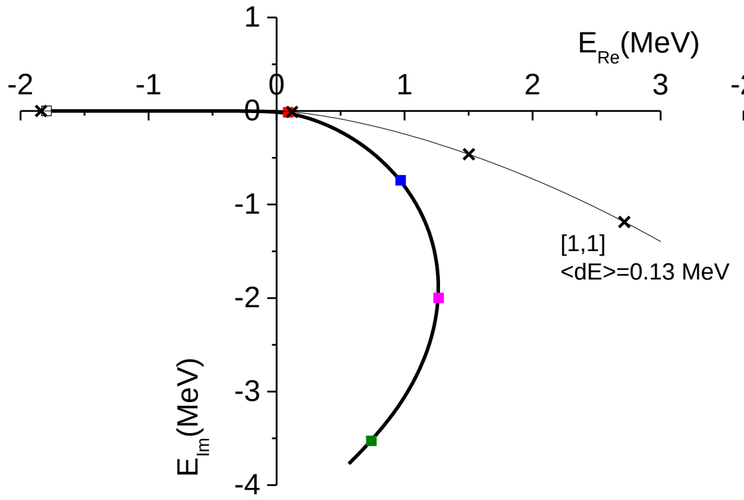
**ACCC** (V.I. Kukulin et al., « Theory of resonances », Kluwer AP 1989)

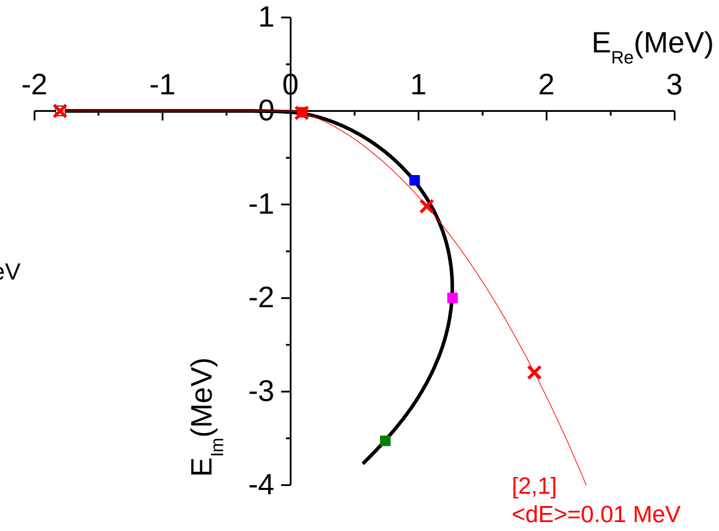
- Add artificial binding potential to Hamiltonian $\lambda V(r)$
- Calculate several binding energies of the system $E_i(\lambda_i)$
- Determine accurately $\lambda_0$ such that $E(\lambda_0) = 0$ !
- Extrapolate $E_{res} = E(\lambda = 0)$ using $E_i(\lambda_i)$ and $\lambda_0$ values, knowing that close to branching point:

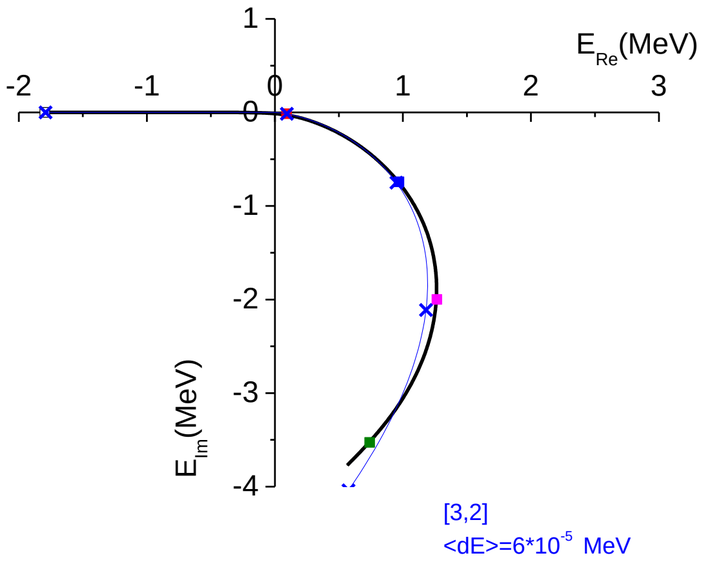
$$\mathrm{k_{res}} = \frac{2\mu}{\hbar^2}\sqrt{E_{res}(\lambda \to \lambda_0)} \sim \begin{cases} i(\lambda - \lambda_0), & \ell = 0 \\ i\sqrt{\lambda - \lambda_0}, & \ell > 0 \end{cases}$$

- Since pole trajectory is tricky – use Padé parametrization: $k_i = i\frac{P_N(v_i)}{Q_M(v_i)}$; $\quad v_i = \begin{cases} i(\lambda_i - \lambda_0), & \ell = 0 \\ i\sqrt{\lambda_i - \lambda_0}, & \ell > 0 \end{cases}$

# 2. Analytic continuation in coupling constant (ACCC)

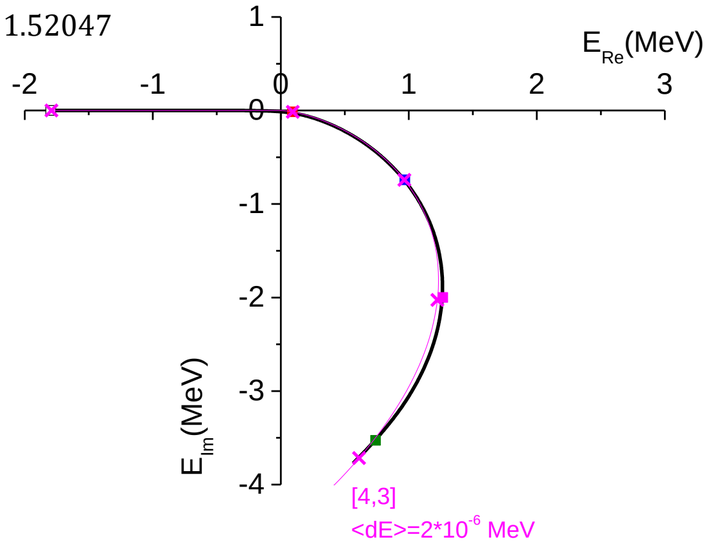


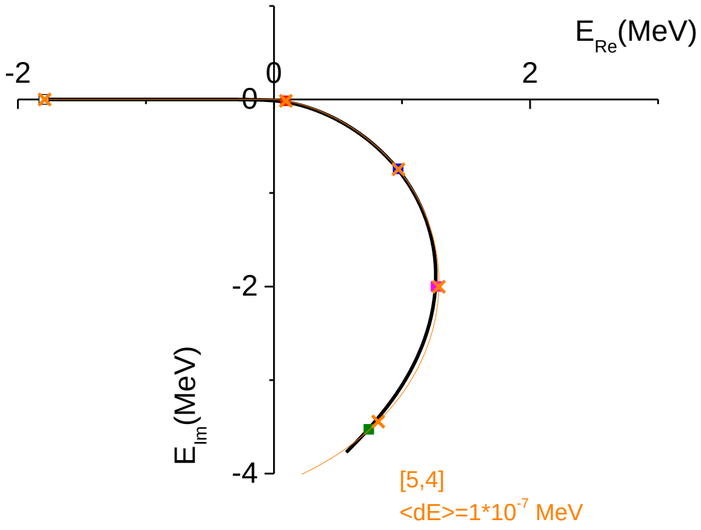



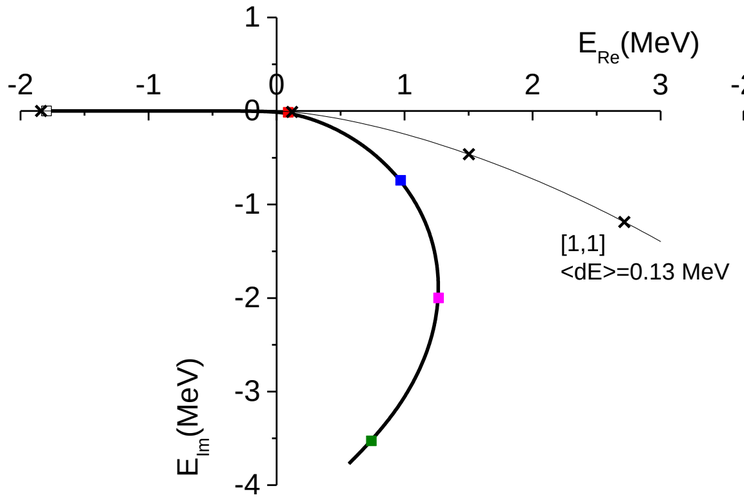

Binding energy input:
(0,20) MeV/19 points

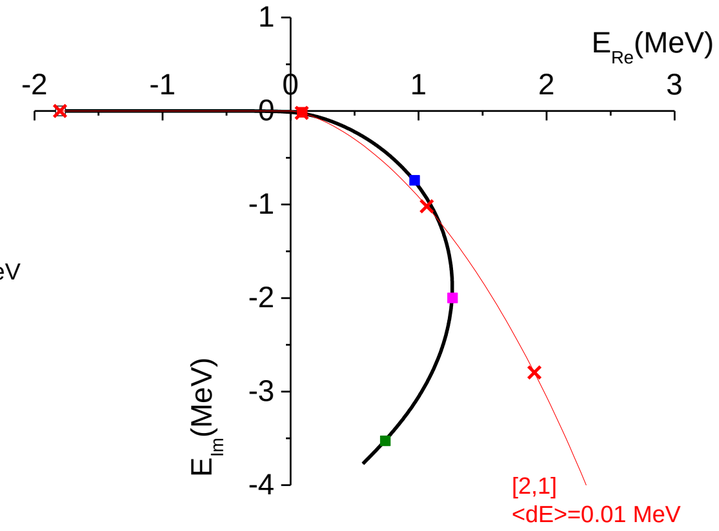
$\lambda_0 = 1.52047$

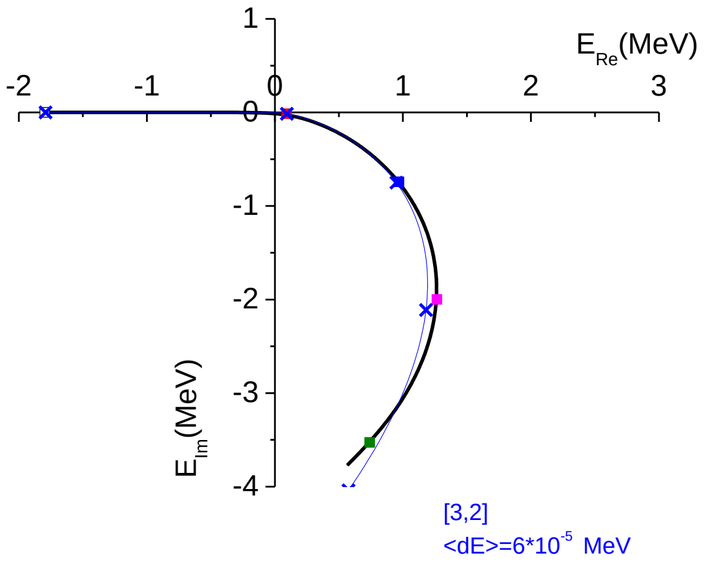


# 2. Analytic continuation in coupling constant (ACCC)

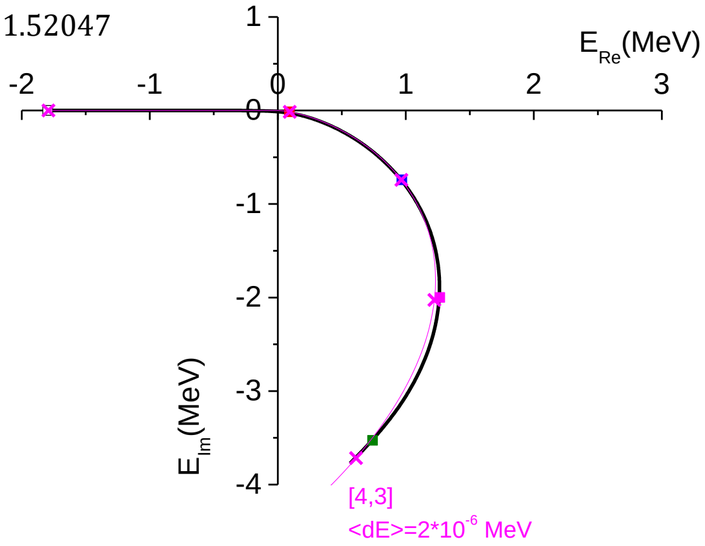


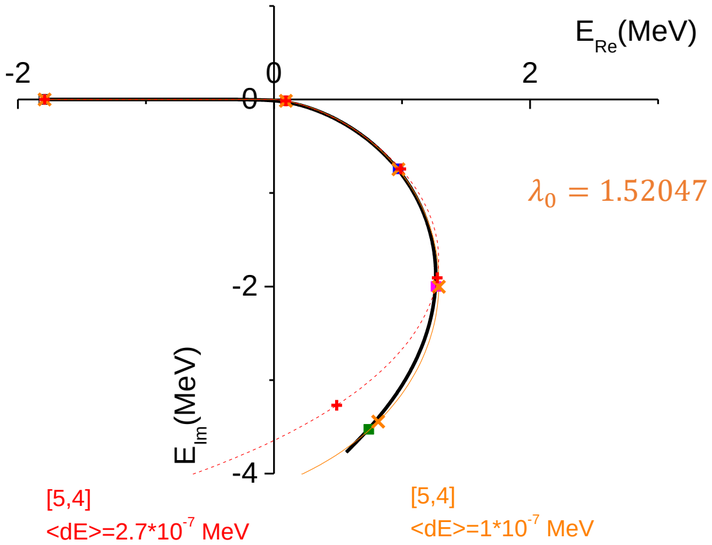




Binding energy input:
(0,20) MeV/19 points

$\lambda_0 = 1.52047$



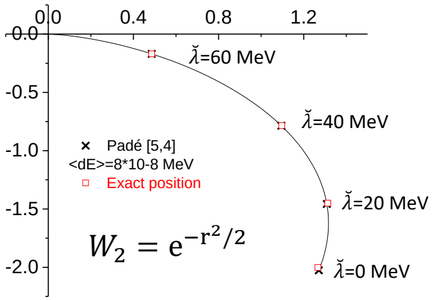

$\lambda_0 = 1.52047 \rightarrow 1.521$

# 2. Analytic continuation in coupling constant (ACCC)

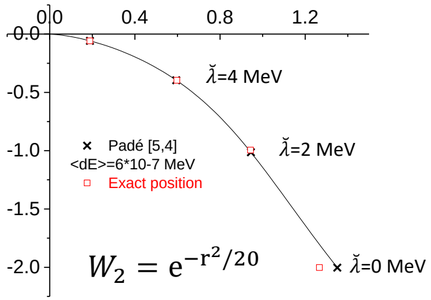
## Drawbacks

- Requires very accurate input of bound state energies $E(\lambda)$ & accurate determination of critical(unitarity) point $\lambda_0$.

- For A>2 systems one can not scale $V_{ij}$ to avoid emergence of new thresholds! Introduce auxiliary A-body interaction $\check{\lambda}\, W_A$, which does not act in subsytems.




Binding energy input:
(0,20) MeV/20 points




Binding energy input:
(0,30) MeV/19 points

- Dangers of $W_A$ not matching region of original interaction & producing spurious bound states.

- Presence of multiple thresholds in subsystems = multiple branching point singularities = necessity of more than one $\lambda_0$ (not resolved)

# 3. Stabilization method

## Idea

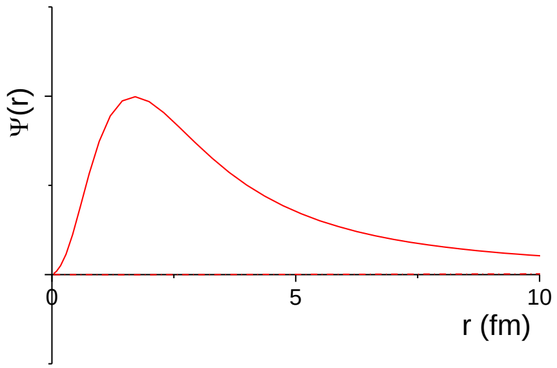
- If there exist a narrow resonance then for the energies in the $\mathrm{Re}(E_{res}) - \Gamma$, $\mathrm{Re}(E_{res}) + \Gamma$ range, the wave function will be localized inside, and will have a small amplitude outside the attractive well. Therefore square integrable basis might well approximate internal part of this wave function and allows to evaluate $\mathrm{Re}(E_{res})$.

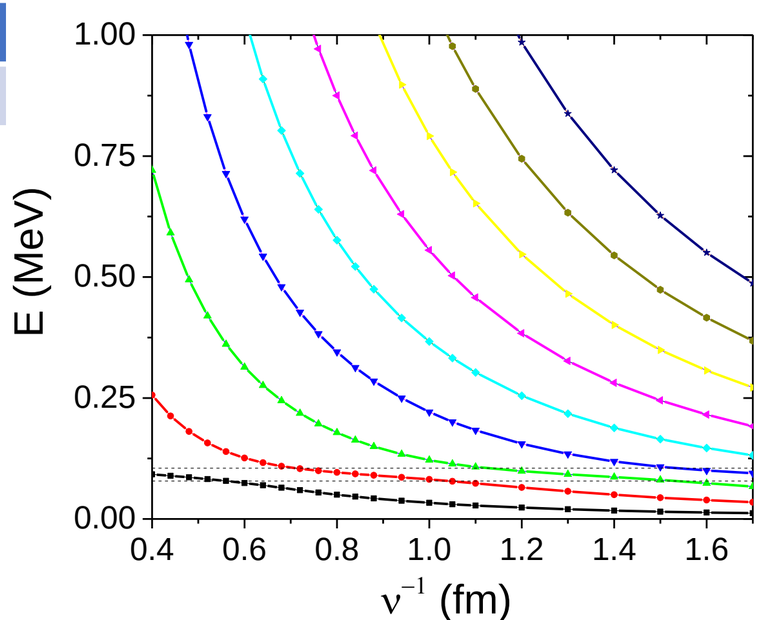
- Moreover eigenvalues representing scattering states will vary rapidly by changing basis parameters, whereas there should be a 'more stable' eigenvalue representing resonance in the interval $\mathrm{Re}(E_{res}) - \Gamma$, $\mathrm{Re}(E_{res}) + \Gamma$

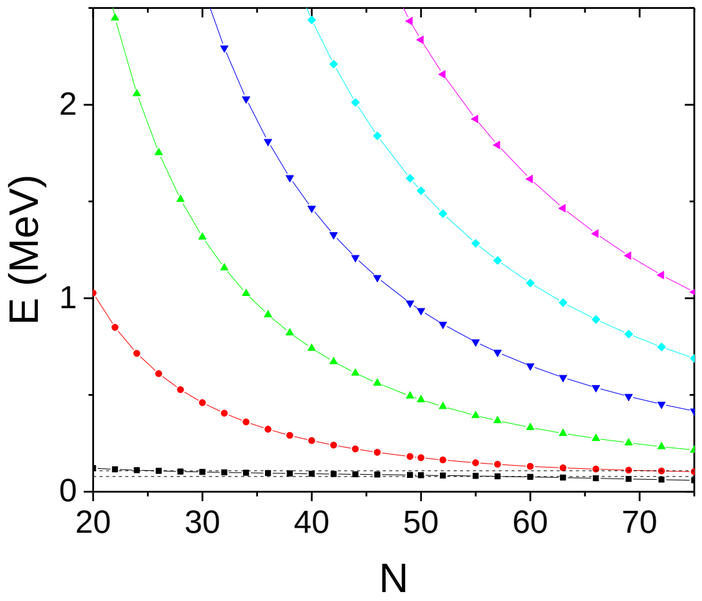
- Resonance position might be evaluated by analyzing bound state calculations & looking for stability of the eigenvalue relative to: basis size, non-linear parameters of the basis, …

- Finally, there should be only one eigenvalue representing resonant state. Consequently, in the vicinity of a resonant state phenomenon of 'avoided level crossing' takes place

| $\lambda$ | $E_{res}$ (MeV) |
|---|---|
| 1.50 | $(9.33-1.50i)\cdot 10^{-2}$ |




# 3. Stabilization method

## Idea

- If there exist a narrow resonance then for the energies in the $\mathrm{Re}(E_{res}) - \Gamma$, $\mathrm{Re}(E_{res}) + \Gamma$ range, the wave function will be localized inside, and will have a small amplitude outside the attractive well. Therefore square integrable basis might well approximate internal part of this wave function and allows to evaluate $\mathrm{Re}(E_{res})$.

- Moreover eigenvalues representing scattering states will vary rapidly by changing basis parameters, whereas there should be a 'more stable' eigenvalue representing resonance in the interval $\mathrm{Re}(E_{res}) - \Gamma$, $\mathrm{Re}(E_{res}) + \Gamma$

- Resonance position might be evaluated by analyzing bound state calculations & looking for stability of the eigenvalue relative to: basis size, non-linear parameters of the basis, …

- Finally, there should be only one eigenvalue representing resonant state. Consequently, in the vicinity of a resonant state phenomenon of 'avoided level crossing' takes place

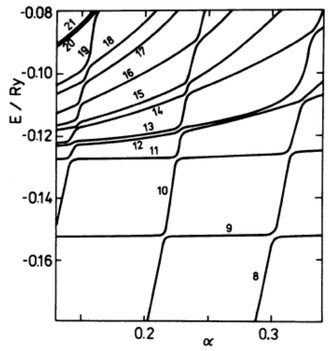
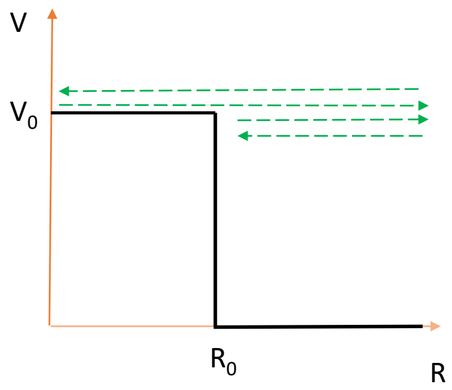



Fig. 7.2
The dependence of the energy eigenvalues on the variational parameter α. The stable behaviour of several eigenvalues is clearly seen.

## Drawbacks

- Only qualitative results
- Works only for very narrow resonances

# Repulsive potential and resonant behavior




Resonance positions may be found by solving:

$$\boxed{tanh\left[i\sqrt{k^2-\frac{m}{\hbar^2}V_0\mathrm{R}_0}\right]+k=0}$$

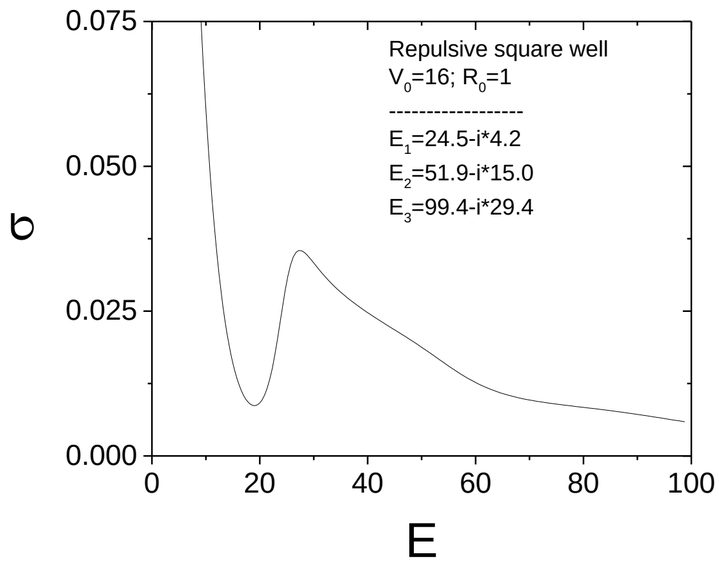
$$k=\sqrt{\frac{m}{\hbar^2}E}$$

Interference of the waves reflected from the origin and edge of the square

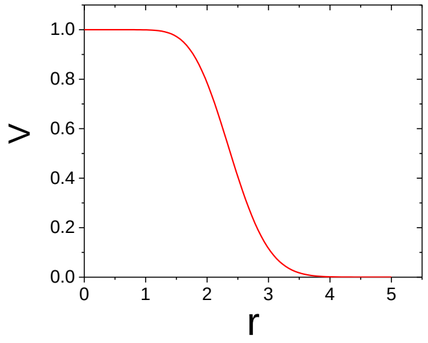
$$\sqrt{k^2-\frac{m}{\hbar^2}V_0\mathrm{R}_0}\approx n\pi$$

S. Flügge, "Practical Quantum Mechanics", Springer 1999 p.218

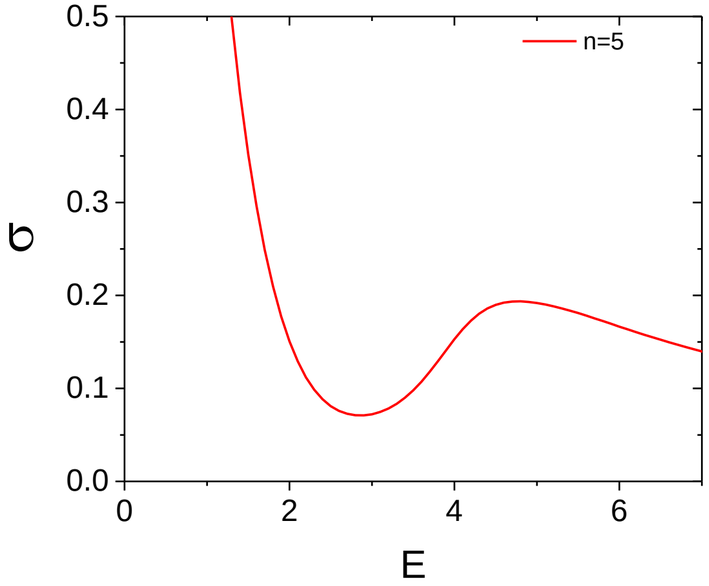
$$\frac{\hbar^2}{m}=1$$




$$V_n(r)=2.8\sum_{i=0}^{n}\frac{r^{2i}}{i!}\exp(-r^2)$$




| $n_{max}$ | $E_{res}$ |
|---|---|
| 0 | - |
| 1 | 3.41-3.84i |
| 2 | 3.94-2.71i |
| 3 | 4.02-1.99i |
| 4 | 3.98-1.52i |
| 5 | 3.90-1.21i |



# Biblical canons

1) V.I. Kukulin, V.M. Krasnopolsky., « Theory of resonances », Kluwer AP 1989

2) N. Moiseyev. "Quantum theory of resonances: calculating energies, widths and cross-sections by complex scaling." *Physics reports* **302.5-6** (1998): 212-293.