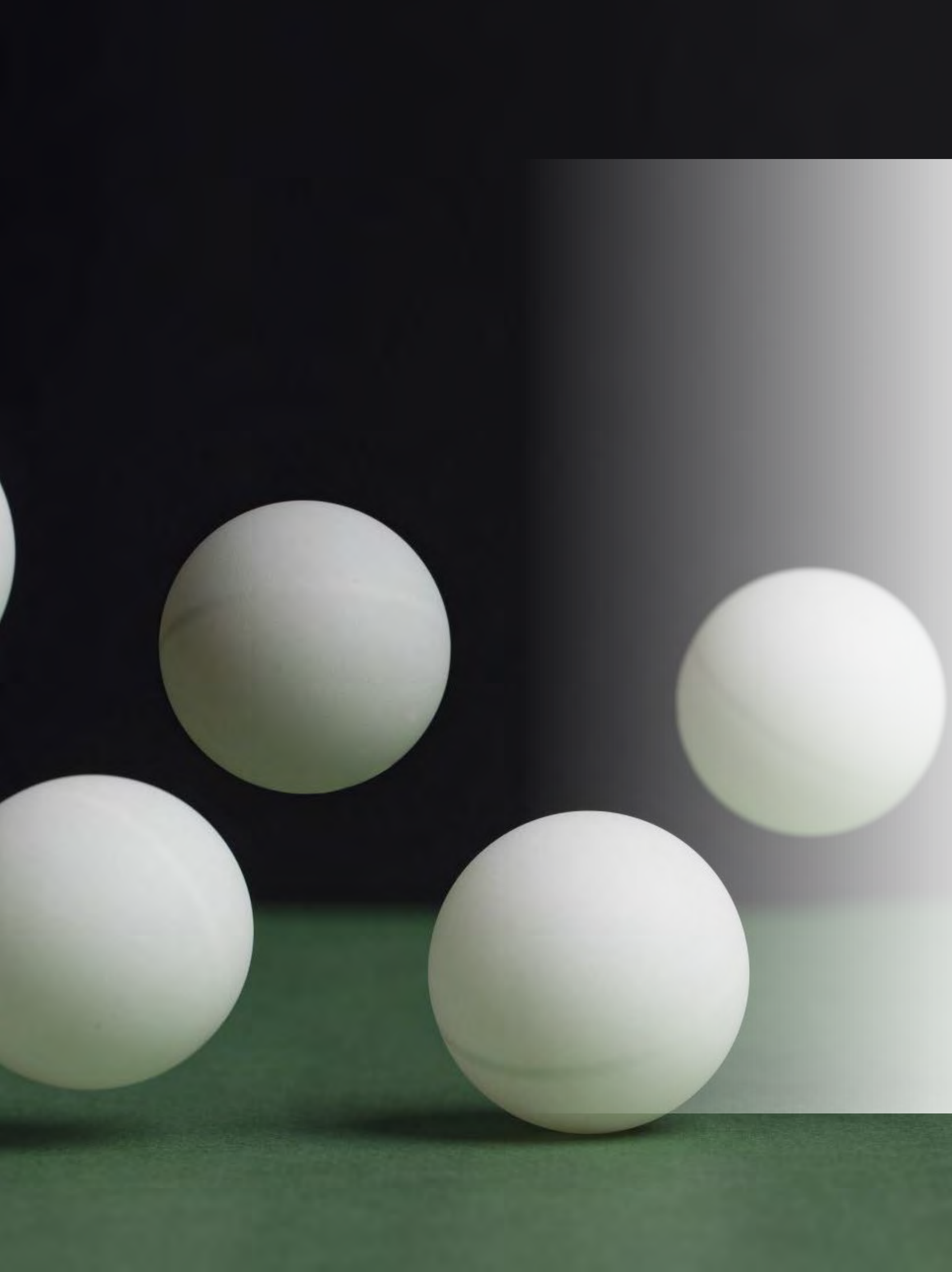

Hadron Resonances

from Experiment and Lattice QCD

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Lecture III of
“Lectures on scattering resonances”

September 6-10, 2021

Organized by:
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- Johannes Kirscher (Univ. Manchester)

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Literature & Resources

- Part on quantum mechanical scattering: Some pictures & formulas taken from
  - Helmut Haberzettl, "Quantum Mechanics with Introduction to Quantum Field Theory", Lecture Notes, to be published; indicated as [HZ] (helmut@gwu.edu)
- Example codes in Mathematica, partially coming from lectures at GW on computational physics:
  - Dropbox https://www.dropbox.com/sh/7h9bxxcvu124z2x/AAAiL2S5ISj8yVGYGiKejFxSa?dl=0
- Several slides borrowed from Maxim Mai [MM] and Deborah Rönchen [DR]
- References are hyperlinks; usually, only reviews with didactic components are cited (this is a lecture, not a review)
- What this lecture is
  - highlight of interesting aspects of arguable relevance with some useful links
  - ….and what it isn't (systematic & self-contained)
  - But, still, with some explicit derivations and in-depth examples & connections

Your Input

- Transmutation mechanisms between different types of states: bound, resonance, virtual, scattering. ✓
  - QM scattering in spherical well & animation; how to solve the Lippman-Schwinger equation; application to light scalar resonances
- How can one decide that a rather strong and narrow bump in a perturbation induced cross section (by photon or electron) is a resonance of the target ?
  - This entails two aspects: Significance of resonance signals and what else could cause bumps? ✓
  - Statistical aspects of resonances, Complex branch points,✓ Triangle singularities $\emptyset$,...
- EFT with resonance fields [Habashi et al., 2012.14995] [Habashi et al., 2007.07360] ($\emptyset$)
- The future of experimental programs.(Discussion)
- Physical interpretation whenever it is possible. (✓ but not focused on models)
- I am specially interested in few-body resonant states with more than two constituents (✓)

Content

1. Scattering basics:
   1. Scattering theory basics & application to spherical well
   2. Resonances as poles: Analytic continuation (2-body), crossing, causality
   3. Mathematica animation & example code (bound state vs. resonances)
   4. Extension to light scalar resonance
2. Phenomenology of resonances:
   1. Spectrum of excited baryons from experiment
   2. Statistical aspects: Model selection and other techniques
3. Few-body resonances and their decays from lattice QCD
   1. Three-body unitarity
   2. Analytic continuation for three-body amplitudes
   3. The finite-volume problem and recent progress

Interesting light hadrons

$\Delta(1232)3/2^-$
First excited baryon discovered
Standard Breit-Wigner (BW) resonance [Crede]

$\pi_1(1600)$
Isovector exotic (COMPASS/ GlueX,…) [Meyer]

$f_0(500)$ "σ"
Debated whether resonance or not, intricate connection to chiral dynamics; non-BW [Pelaez]

$N(1440)1/2^+$, "Roper"
Enigmatic; absent in many Lattice QCD and quark model calculations; non-BW [Burkert]

$\Lambda(1405)$
Two pole structure complicated production [Mai]

$f_0(980)$
Resonance close to threshold: molecule? Flatté-like, non-BW [Baru]

$N(1535)1/2^-$, $N(1650)1/2^-$
Nearby, overlapping resonances with same quantum numbers

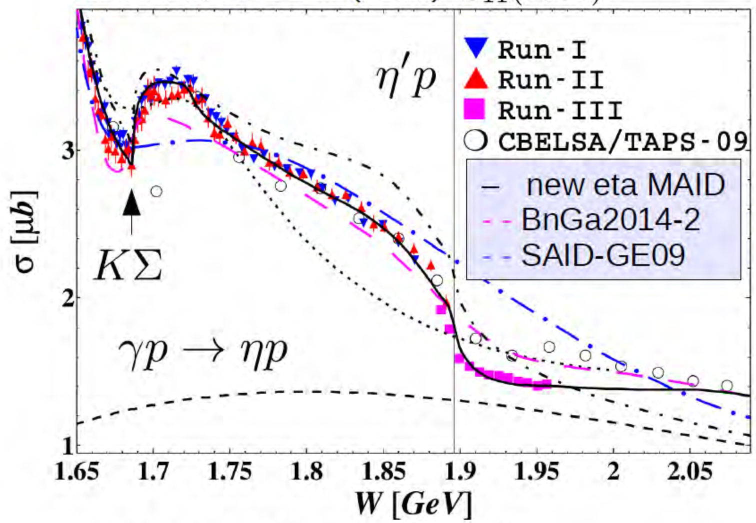
$N(1900)3/2^+$
Recently discovered in large experimental baryon searches for "missing resonance"

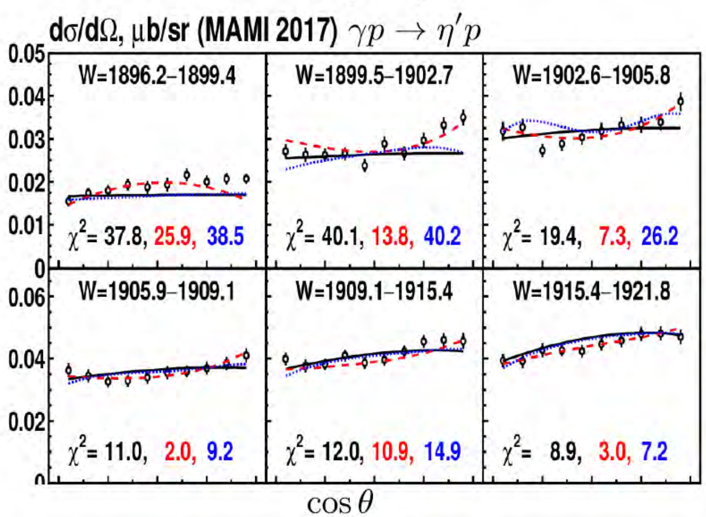
$a_1(1260)$
Clean production; three-body dynamics

$X(3875)$, $P_c^+(4\ldots)$
C, B resonances (no time)

Resonances or not?

A2 MAMI, PRL 118 (2017) $S_{11}(1895)$





BnGa
PLB785 (2018):

No narrow resonance

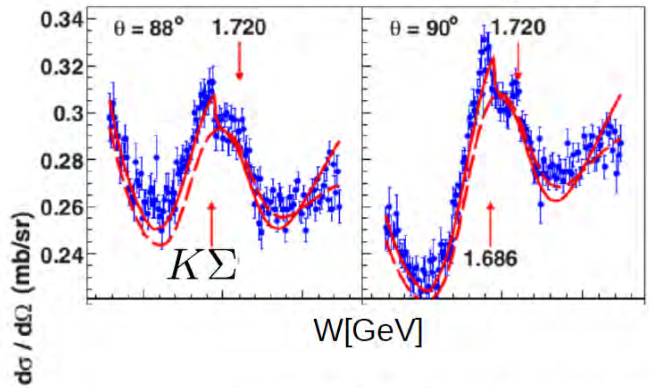
3/2⁻ narrow Resonance

5/2⁻ narrow Resonance

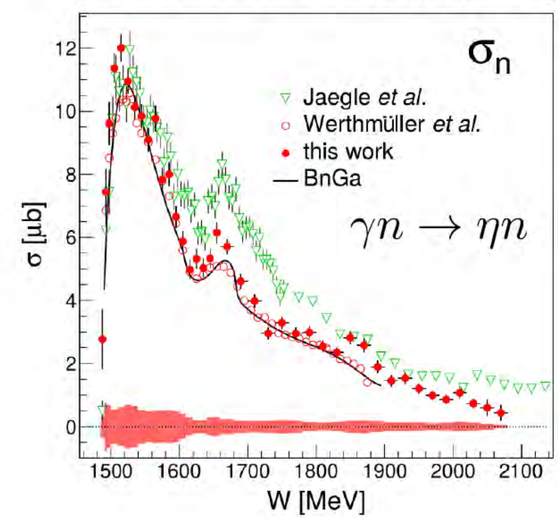
Data: A2.Mami PRL 118 (2017)

$\pi N \to \pi N$

EPECUR/SAID PRC 93 (2016)



[CBELSA/TAPS EPJA 53 (2017)]



1.1. QM Scattering: Baseline

- Radiation condition: $\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) \xrightarrow{r\to\infty} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{e^{ikr}}{r} f(\theta)$
- Scattering amplitude & partial-wave (PW) expansion:

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, t_\ell\, P_\ell(\xi) \;, \quad t_\ell = \frac{1}{k\cot\delta_\ell - ik}$$

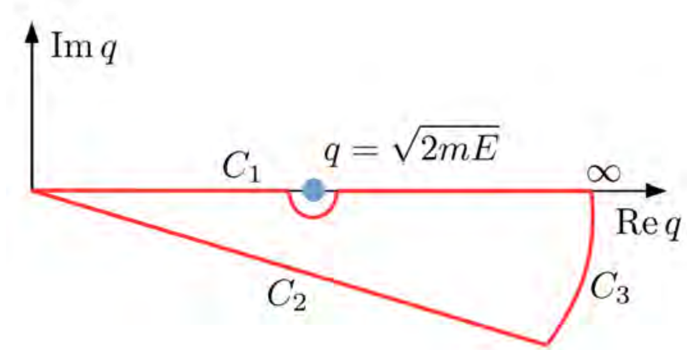
ı Legendre polynomials P_ℓ and $\xi = \cos\theta$.

- Lippmann-Schwinger equation (LSE)

$$T(\boldsymbol{p}',\boldsymbol{p}) = V(\boldsymbol{p}',\boldsymbol{p}) + \int \mathrm{d}^3 q\, V(\boldsymbol{p}',\boldsymbol{q}) \frac{1}{E - \frac{q^2}{2m} + i\epsilon} T(\boldsymbol{q},\boldsymbol{p})$$

- PW-projected LSE

$$T_\ell(p',p) = V_\ell(p',p) + \int_0^\infty \mathrm{d}q\, q^2 \frac{V_\ell(p',q)}{E - \frac{q^2}{2m} + i\epsilon} T_\ell(q,p)$$



- Solve, e.g., by contour deformation

How to solve the LSE

- Example code in Mathematica: Implementation of Haftl-Tabakin scheme [Haftl]

$$T_\ell(p',p) = V_\ell(p',p) + \int_0^\infty \mathrm{d}q\, q^2 \frac{V_\ell(p',q)}{E - \frac{q^2}{2m} + i\epsilon} T_\ell(q,p)$$

- Gauss integration $\int f(x)dx \approx \sum_{i=1}^{n} f(x_i)w_i$ for n off-shell momenta and one on-shell momentum $n+1$

$$\bar{V} = \begin{pmatrix} V_{11} & \cdots & V_{1n} & V_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ V_{n1} & \cdots & V_{nn} & \vdots \\ V_{n+1,1} & \cdots & \cdots & V_{n+1,n+1} \end{pmatrix} \qquad \bar{G} = \begin{pmatrix} \frac{q_1^2 w_1}{z-E_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \frac{q_n^2 w_n}{z-E_n} & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$\bar{T} = \begin{pmatrix} T_{11} & \cdots & T_{1n} & T_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ T_{n1} & \cdots & T_{nn} & \vdots \\ T_{n+1,1} & \cdots & \cdots & T_{n+1,n+1} \end{pmatrix}$$

[DR]

How to solve the LSE (2)

$$T_\ell(p',p) = \boxed{V_\ell(p',p) + \int_0^\infty \mathrm{d}q\, q^2 \, \frac{V_\ell(p',q)}{E - \frac{q^2}{2m} + i\epsilon} \, T_\ell(q,p)}$$

Discretize the integral: $\int dq\, q^2\, V(p',q)\, G(q,E)\, T(q,p) \quad \rightarrow \quad \bar{V}\bar{G}\bar{T}$

Gauss integration $\int f(x)dx \approx \sum_{i=1}^{n} f(x_i) w_i$ for n off-shell momenta and one on-shell momentum $n+1$

$$\bar{V}\bar{G}\bar{T} = \begin{pmatrix} \sum_{i=1}^{n} V_{1i} \frac{q_i^2 w_i}{z-E_i} T_{i1} & \cdots & \sum_{i=1}^{n} V_{1i} \frac{q_i^2 w_i}{z-E_i} T_{in} & \sum_{i=1}^{n} V_{1i} \frac{q_i^2 w_i}{z-E_i} T_{i,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ \sum_{i=1}^{n} V_{ni} \frac{q_i^2 w_i}{z-E_i} T_{i1} & \cdots & \sum_{i=1}^{n} V_{ni} \frac{q_i^2 w_i}{z-E_i} T_{in} & \vdots \\ \sum_{i=1}^{n} V_{n+1,i} \frac{q_i^2 w_i}{z-E_i} T_{i1} & \cdots & \cdots & \sum_{i=1}^{n} V_{n+1,i} \frac{q_i^2 w_i}{z-E_i} T_{i,n+1} \end{pmatrix}$$

[DR]

How to solve the LSE (3)

- On-shell → on-shell for physical amplitude

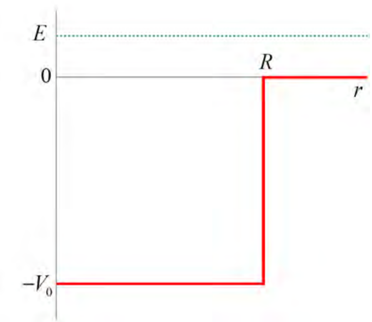
$$
\begin{aligned}
T_{ik} &= V_{ik} + \sum_{j=1}^{n} V_{ij} \frac{q_j^2 w_j}{z - E_j} T_{jk} && \text{off-shell} \to \text{off-shell} \\
T_{n+1,k} &= V_{n+1,k} + \sum_{j=1}^{n} V_{n+1,j} \frac{q_j^2 w_j}{z - E_j} T_{jk} && \text{off-shell} \to \text{on-shell} \\
T_{i,n+1} &= V_{i,n+1} + \sum_{j=1}^{n} V_{ij} \frac{q_j^2 w_j}{z - E_j} T_{j,n+1} && \text{on-shell} \to \text{off-shell} \\
\boxed{T_{n+1,n+1}} &= V_{n+1,n+1} + \sum_{j=1}^{n} V_{n+1,j} \frac{q_j^2 w_j}{z - E_j} T_{j,n+1} && \text{on-shell} \to \text{on-shell}
\end{aligned}
$$

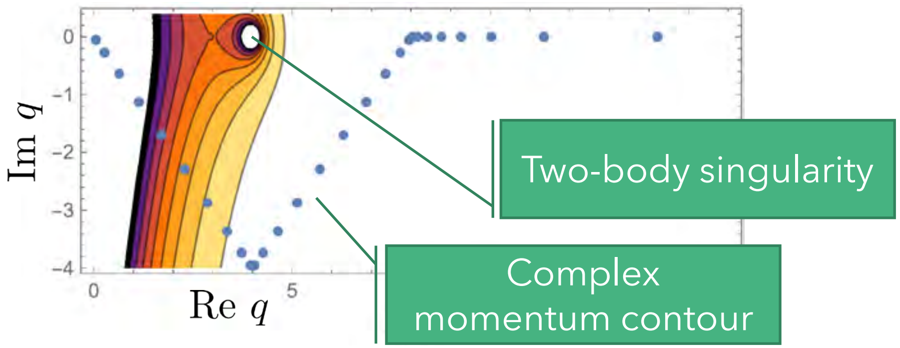
- We can now invert the matrix:

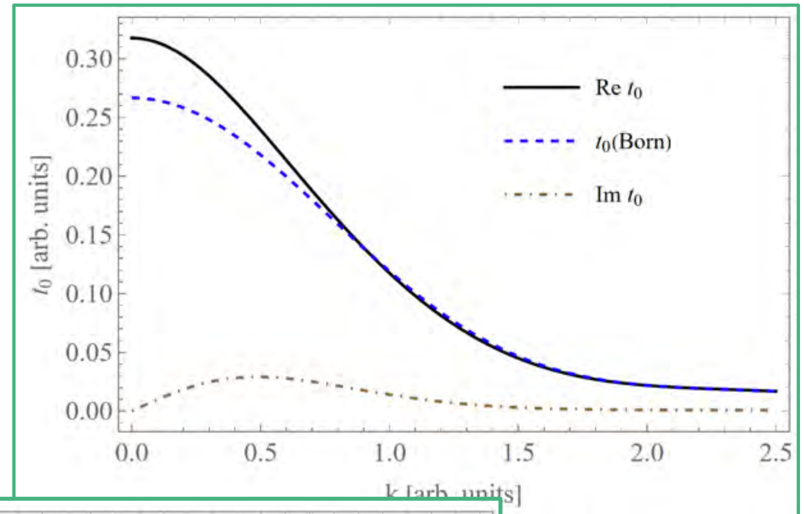
$$\bar{\mathrm{T}} = (\mathbb{1} - \bar{\mathrm{V}}\bar{\mathrm{G}})^{-1}\bar{\mathrm{V}}$$

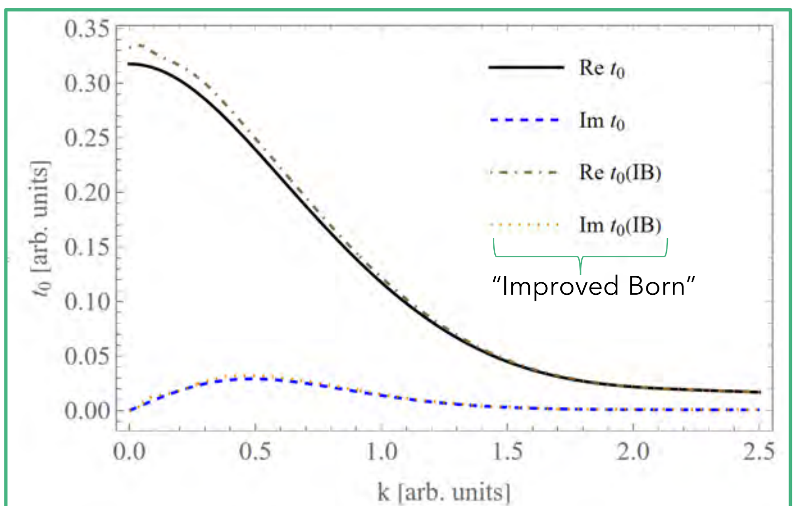
CompPhys-project

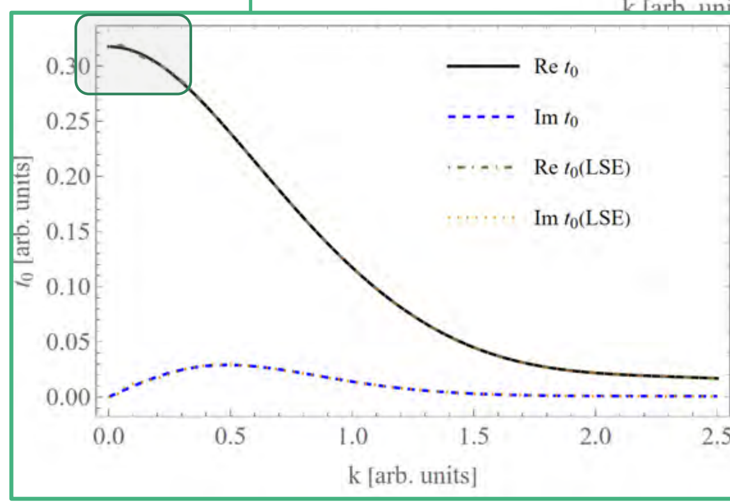
- Spherical well with LSE compared to analytic solution
- "LSE_for_Spherical_Well.nb"





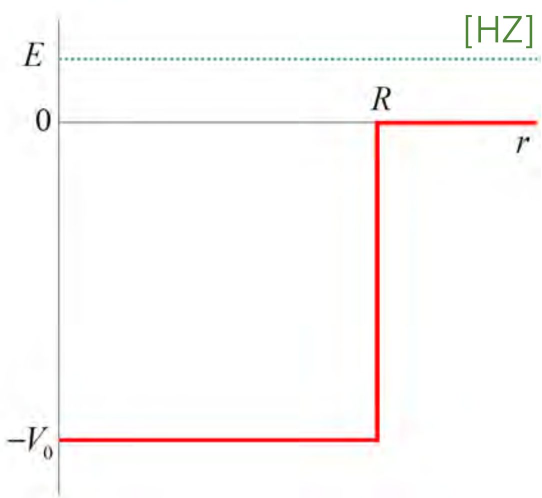






Spherical well

Potential in radial coordinates:



$$V(r) = \begin{cases} -V_0 & \text{for} \quad r < R \, , \\ 0 & \text{for} \quad r > R \, , \end{cases}$$

Scattering phase shifts (E>0)

Matching

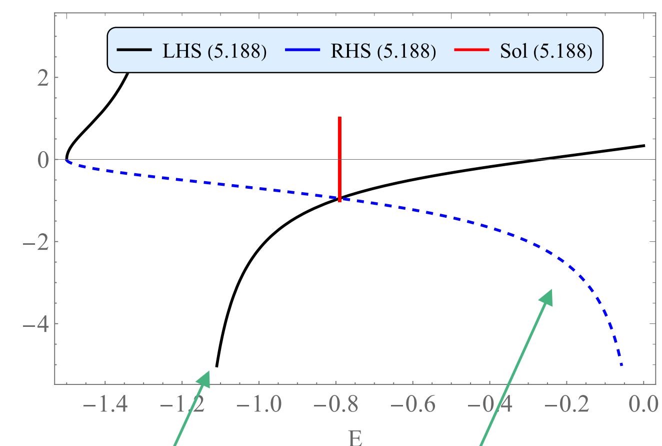
Sph. Bessel-f.

$$\tan \delta_\ell = \frac{k\, j_\ell'(kR) - \gamma_\ell\, j_\ell(kR)}{k\, n_\ell'(kR) - \gamma_\ell\, n_\ell(kR)}$$

Wave number

v. Neumann-f.

Bound state energies (E<0) [note: a=R]



$$v = a^2 v_0 = \frac{2m}{\hbar^2} V_0 a^2$$

$$\lambda = a^2 \kappa^2 = -\frac{2ma^2}{\hbar^2} E > 0$$

$$\tan \sqrt{v - \lambda} = -\sqrt{\frac{v - \lambda}{\lambda}}$$

Breit-Wigner Resonances

- Small energies $kR \ll 1$

$$\tan\delta_\ell \approx \underbrace{\frac{\ell - \gamma_\ell R}{\ell + 1 + \gamma_\ell R}}_{=0} \frac{(kR)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}$$

- Expansion around the pole:

$$\tan\delta_\ell \approx \frac{1}{(E - E_{\mathrm{R}})g'_\ell(E_{\mathrm{R}})} \frac{(kR)^{2\ell+1}}{[(2\ell-1)!!]^2}$$

- Or:

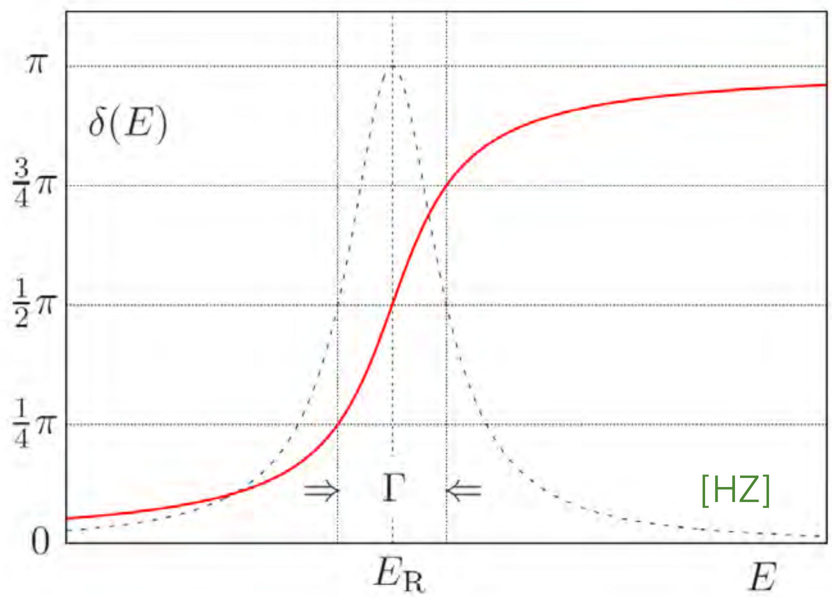
$$\tan\delta_\ell \approx -\frac{\Gamma_\ell}{2(E - E_{\mathrm{R}})} \quad \text{where} \quad \Gamma_\ell = -\frac{2(kR)^{2\ell+1}}{g'_\ell(E_{\mathrm{R}})[(2\ell-1)!!]^2}$$

- For t-matrix and cross section:

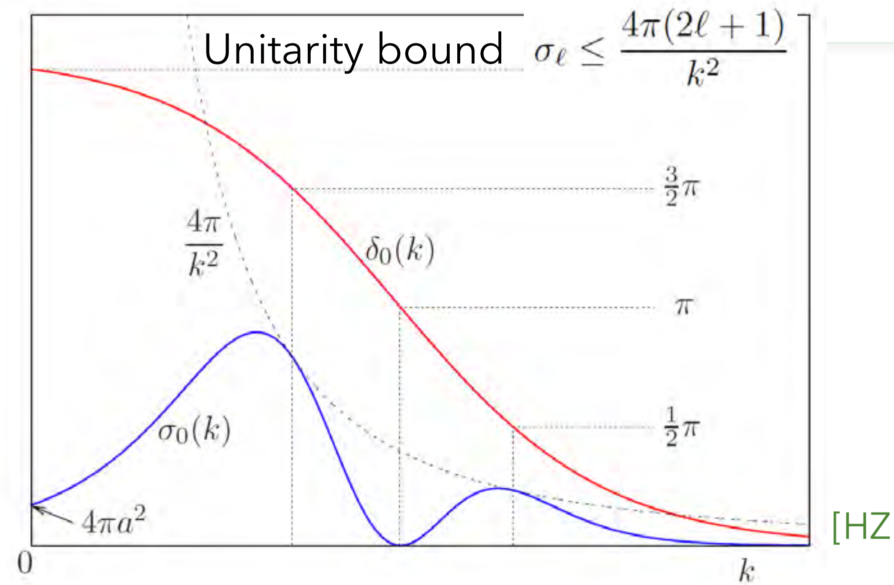
$$\boxed{t_\ell \approx \frac{1}{k} \frac{\frac{\Gamma_\ell}{2}}{E_{\mathrm{R}} - E - i\frac{\Gamma_\ell}{2}}} \qquad \sigma \approx \frac{4\pi}{k^2}(2\ell_0 + 1) \frac{\frac{\Gamma_{\ell_0}^2}{4}}{(E - E_{\mathrm{R}})^2 + \frac{\Gamma_{\ell_0}^2}{4}}$$

[HZ]

BW resonances and Ramsauer-Townsend



$$\tan \delta_\ell \approx -\frac{\Gamma_\ell}{2(E - E_\mathrm{R})}$$



S-wave, low energy:

$$f(\theta) \xrightarrow{k \to 0} t_0 P_0 = t_0 = \frac{\sin \delta_0}{k} \mathrm{e}^{i\delta_0} \approx -a\, \mathrm{e}^{i\delta_0}$$

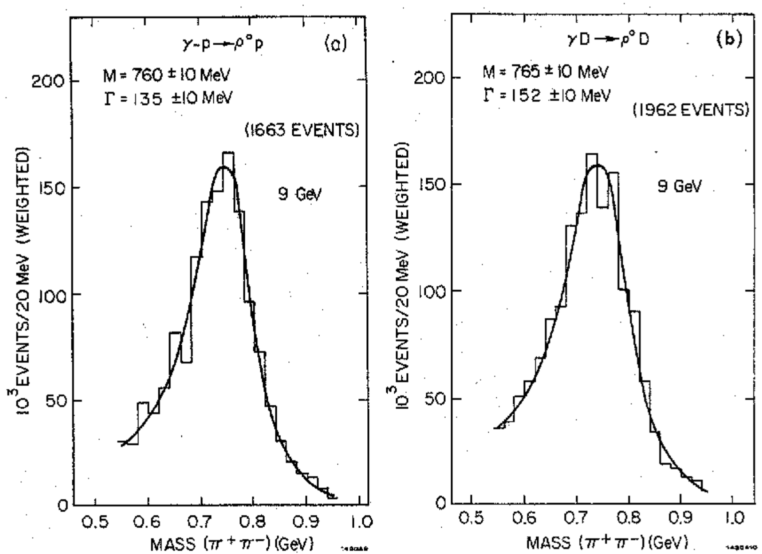
$$\sigma_0(k) = \frac{4\pi}{k^2} \sin^2 \delta_0(k)$$

Typical Breit-Wigner resonances

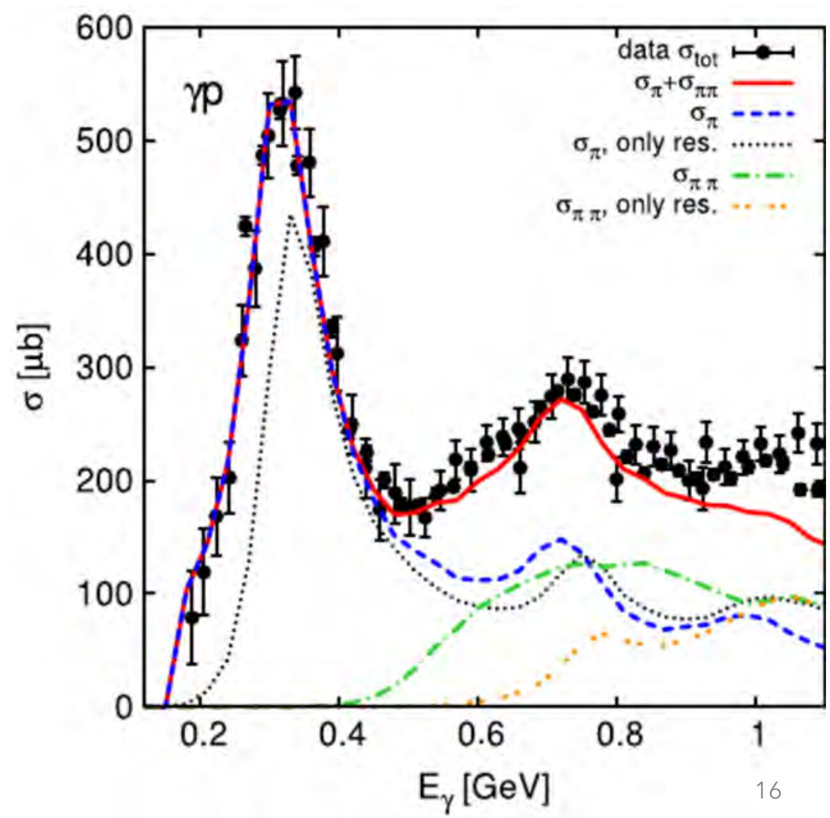
ρ-meson photoproduction

$$\left.\begin{array}{l} M = 760 \pm 10 \text{ MeV} \\ \Gamma = 135 \pm 10 \text{ MeV} \end{array}\right\} \quad \text{Hydrogen, 9 BeV.}$$

$$\left.\begin{array}{l} M = 765 \pm 10 \\ \Gamma = 152 \pm 10 \end{array}\right\} \quad \text{Deuterium, 9 BeV.}$$



Δ-baryon photoproduction



Deficiencies of Breit-Wigner

- Breit-Wigner resonances are an idealized case
  - No background (see Laurent expansion previous slide)
  - Reaction dependent: Shape changes in different channels
  - No energy-energy dependent width in simplest BW form. Width MUST be energy dependent even for S-wave (unitarity)
  - Adding Breit-Wigner resonances violates unitarity
  - Close-by threshold have an influence (Generalization: Flatté)

$$A_i \sim \frac{M_R\sqrt{\Gamma_0\Gamma_i}}{M_R^2 - E^2 - iM_R(\Gamma_1+\Gamma_2)}, \quad i=1,2 \qquad 1:\ \pi\eta,\ 2:\ \bar{K}K$$

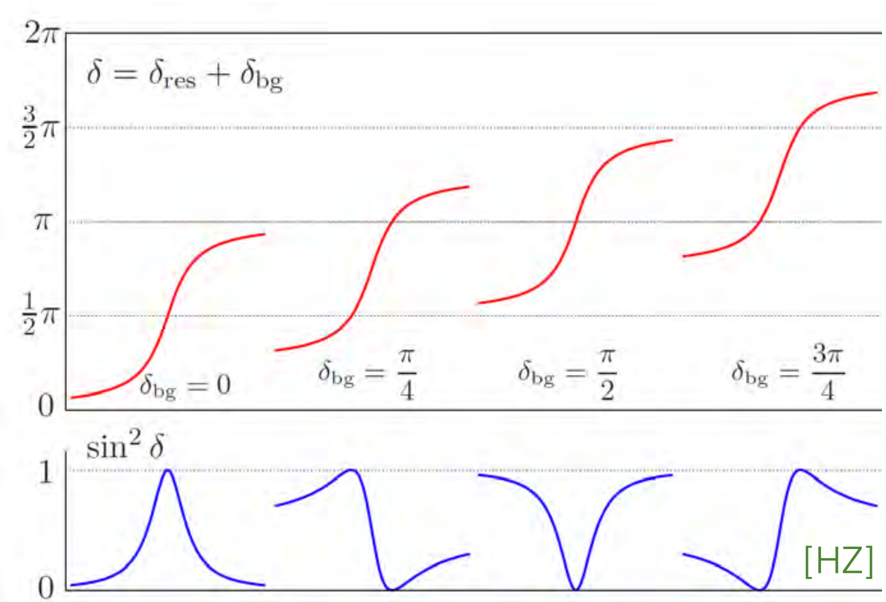
$$\Gamma_1 = g_1 k_1, \quad k_1 = \frac{1}{2E}\sqrt{[E^2-(m_\eta+m_\pi)^2][E^2-(m_\eta-m_\pi)^2]} \qquad \Gamma_2 = g_2 k_2 \quad k_2 = \sqrt{\frac{E^2}{4} - m_K^2}$$

[Lesniak]

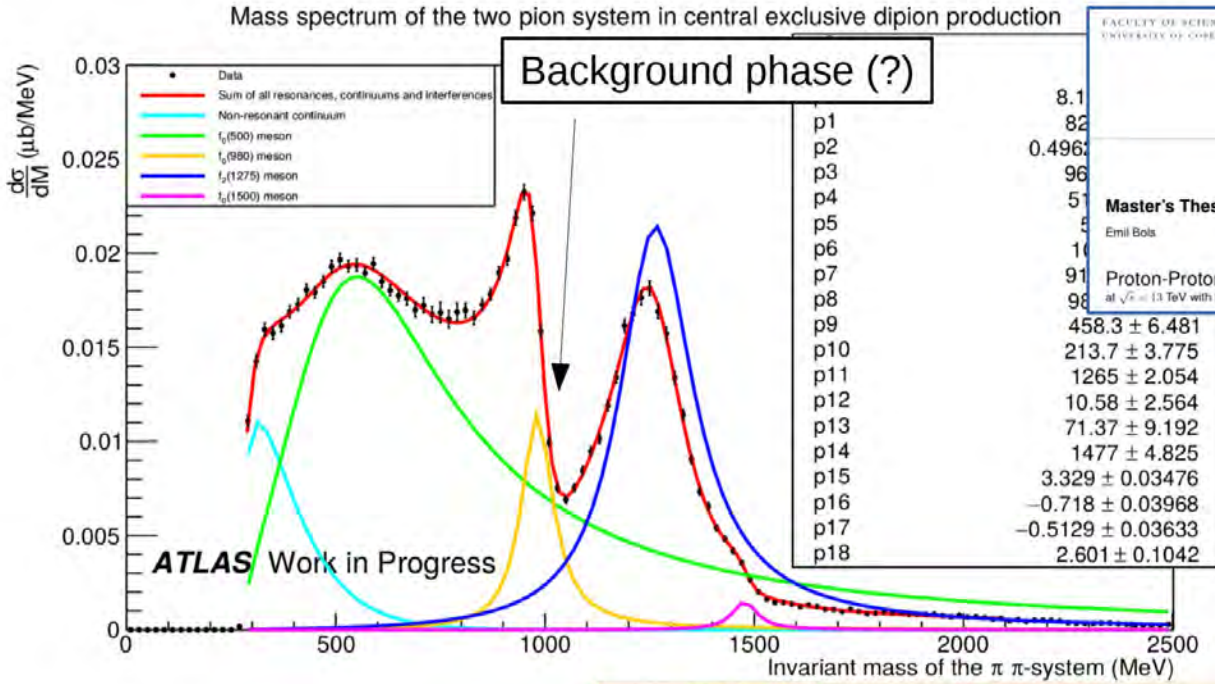
  - Coupled-channel environment respected
  - Unitarity respected (as long as no other background is added ;)
  - Example of "analytic continuation": k_2 is complex below $\bar{K}K$ threshold!

Background

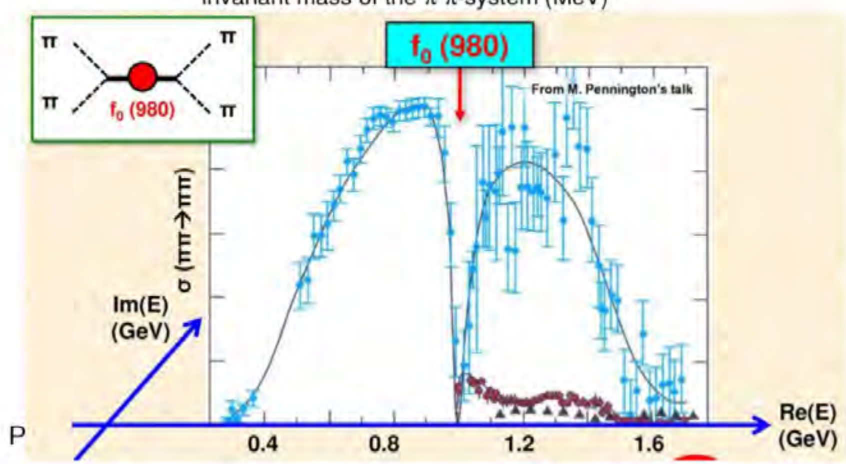
- Refers to non-resonant contributions to scattering amplitude (= physical effects), not experimental background
  - Sometimes resonances and background are added at the level of cross sections, but, of course, they add at level of amplitudes (interference)
- Resonances are by no means bumps in cross sections:



More complicated cases



Opposite pictures

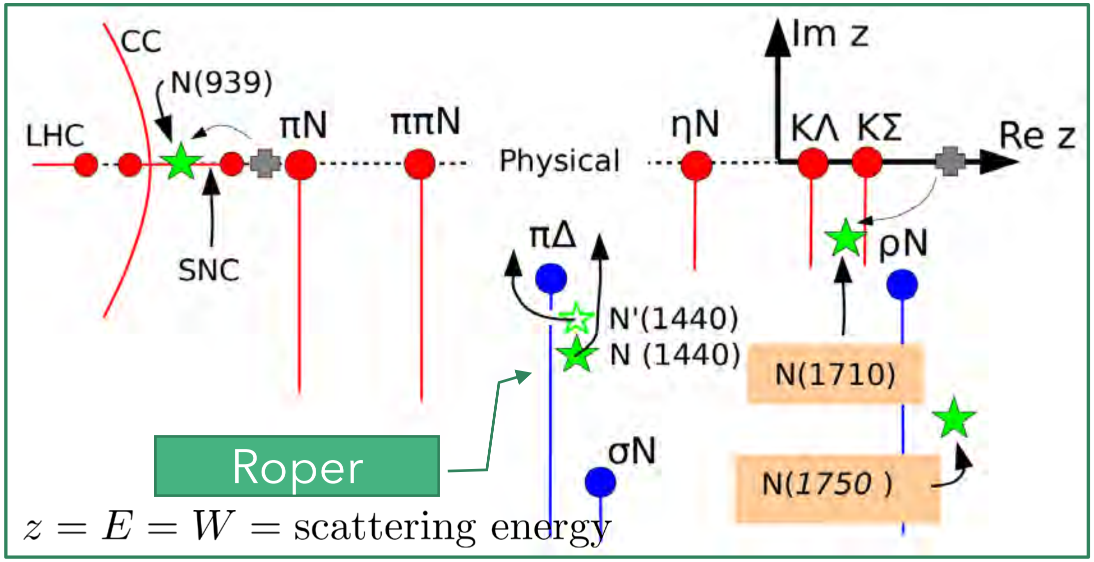


Source:
H. Kamano

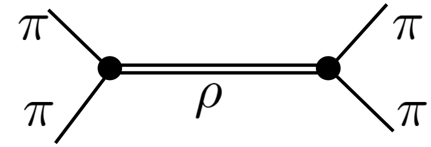
1.2 Hadronic resonances as poles

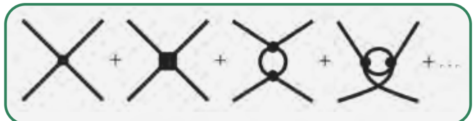
- Defining resonances as poles in amplitudes at complex energies resolves all mentioned problems
  - Real part of pole position ⟺ Mass
  - 2x Imaginary part of pole position ⟺ Width
  - Pole residue ⟺ Branching ratio into different channels because amplitudes factorize at poles

  - Next goal: What is this?
    - Red: Real thresholds
    - Blue: sub-channel thres.
    - Why is Roper double?
    - What happens below threshold?



Analytic continuation (2B)



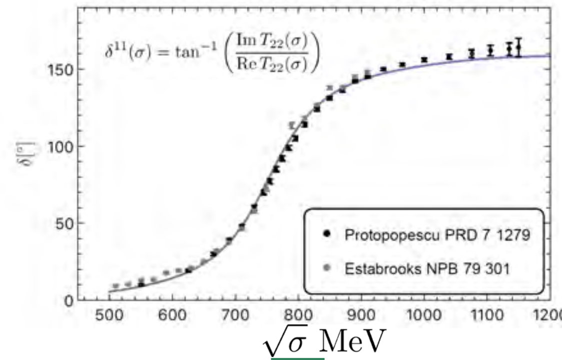


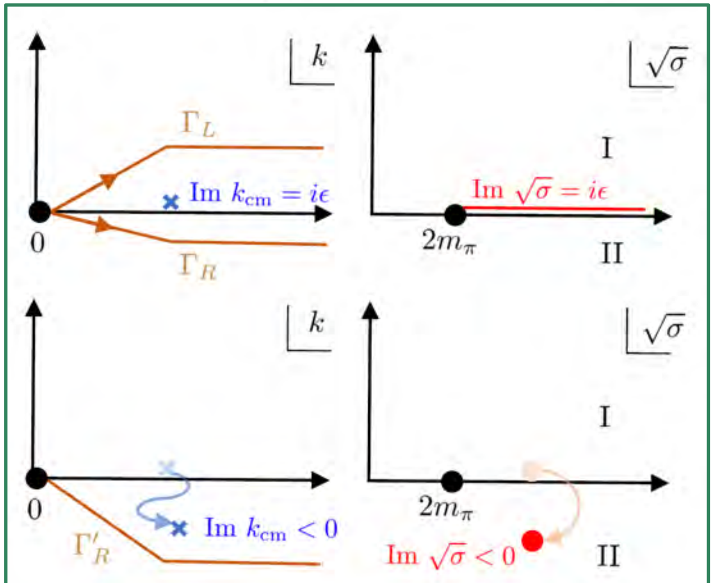
Inverse Amplitude Method [Dobado]

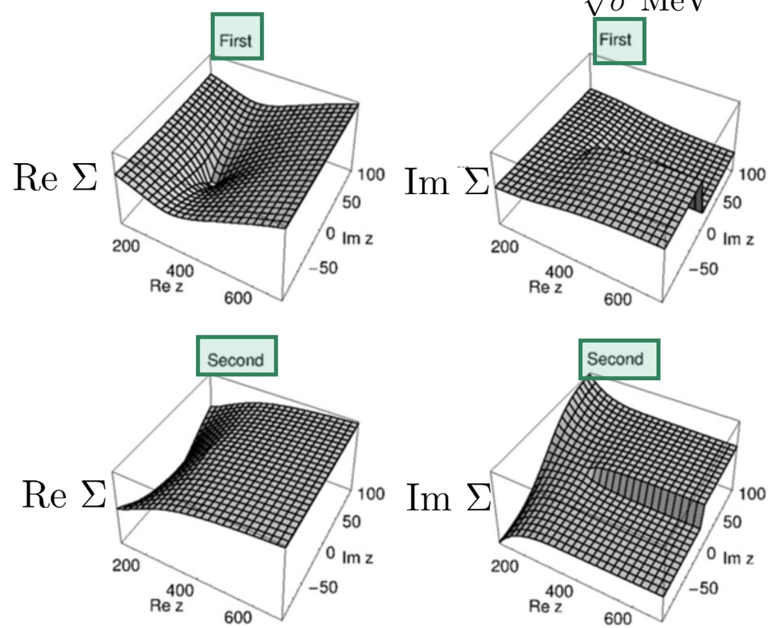
$$T_{22}(\sigma) = \tilde{v}(k_{\rm cm})\tau(\sigma)\tilde{v}^*(k_{\rm cm}),\ \ k_{\rm cm} = \sqrt{\frac{\sigma}{4} - m_\pi^2}\,,$$

$$\tau^{-1}(\sigma) = \boxed{K^{-1}} - \Sigma\,,$$

$$\Sigma = \int_0^\infty \frac{{\rm d}k\, k^2}{(2\pi)^3}\frac{1}{2E_k}\frac{\sigma^2}{\sigma'^2}\frac{\tilde{v}(k)^*\tilde{v}(k)}{\boxed{\sigma - 4E_k^2 + i\epsilon}} \qquad E_k = \sqrt{k^2+m_\pi^2}$$



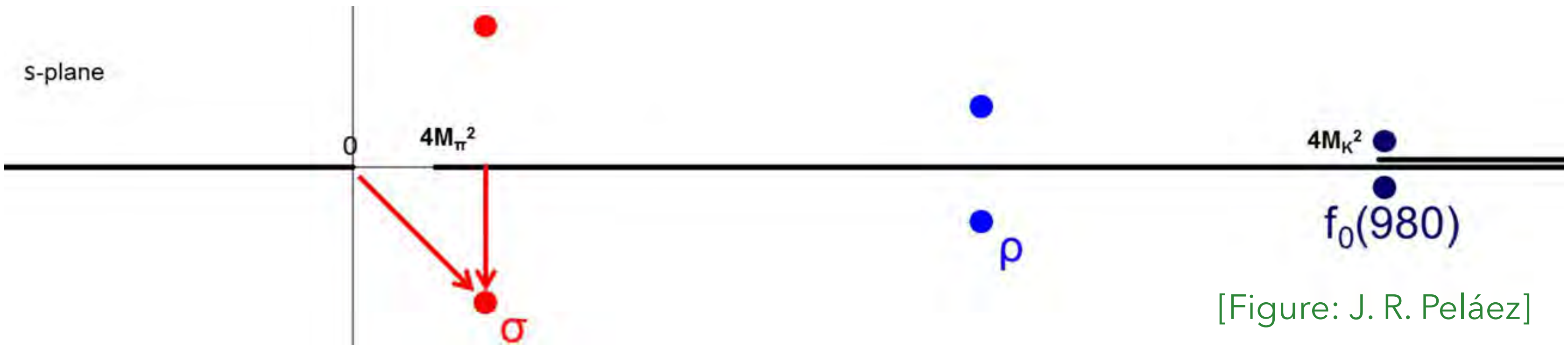




"Adiabatic" contour deformation
(remember for 3-body case later!)

Right-hand and left-hand cuts

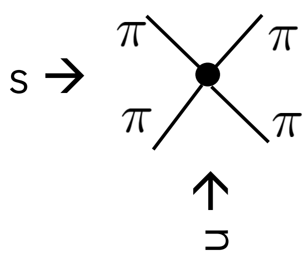
- Pole positions of wide resonances might be distorted if "left-hand cut" is not taken properly into account (and: analyticity in *s*, not $\sqrt{s}$)



[Figure: J. R. Peláez]

- Build in crossing symmetry manifestly through Roy-(like equations)

[Peláez]

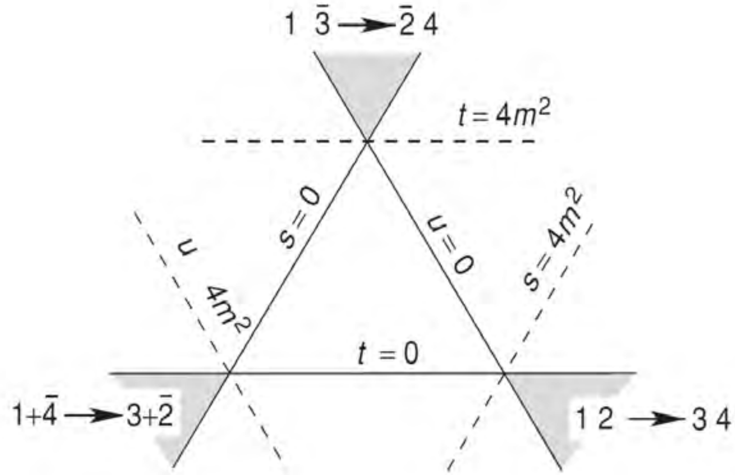


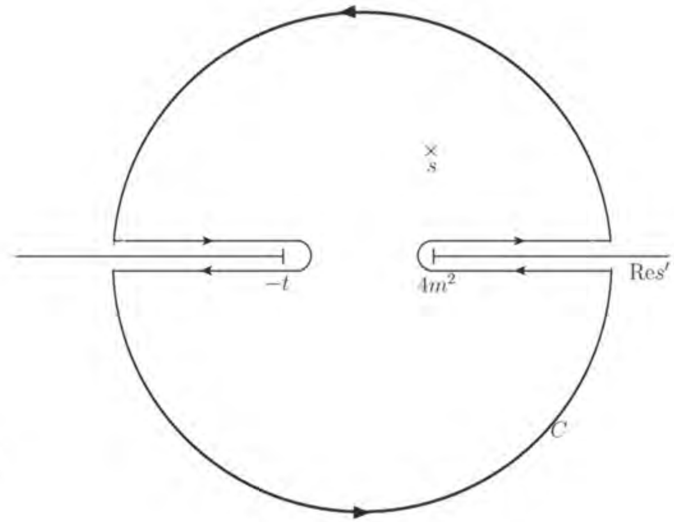
Advantage: $\pi\pi$ scattering in u-channel is still $\pi\pi$

πN: [Hoferichter]

Roy(-like) equations

[Figure & formulas: J. R. Peláez] [Gribov]





Unphysical region

$$T(s,t,u) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}T(s',t,u')}{s'-s} + \frac{1}{\pi}\int_{-\infty}^{-t} ds' \frac{\mathrm{Im}T(s',t,u')}{s'-s}$$

Subtraction

$$T^{(I)}(s,t) = T^{(I)}(0,t) + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left[\frac{\mathrm{Im}\, T^{(I)}(s',t)}{s'(s'-s)} - \frac{\mathrm{Im}\, T^{(I)}(u',t)}{u'(u'-s)}\right]$$

Crossing relations:

$$T^{(I)}(u',t,s') = \sum_{I'} C_{su}^{II'} T^{(I')}(s',t,u'), \quad T^{(I)}(0,t) = \sum_{I''} C_{st}^{II''} T^{(I'')}(t,0)$$

$s \leftrightarrow u$ crossing

Only physical Region!

Partial-wave expansion

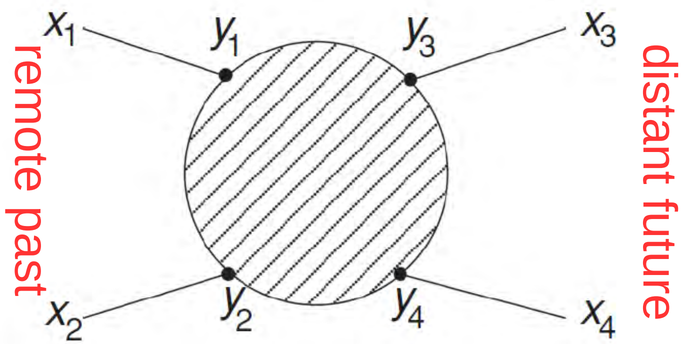
Roy-Eq.: $$t_\ell^{(I)}(s) = \overline{ST}_\ell^I(s) + \sum_{I'=0}^{2}\sum_{\ell'=0}^{\ell_{max}} \int_{4M_\pi^2}^{s_{max}} ds' \overline{K}_{\ell\ell'}^{II'}(s,s')\mathrm{Im}\, t_{\ell'}^{I'}(s') + \overline{DT}_\ell^I(s)$$

Coupled partial waves

Causality and analyticity (1)

- 4-point Green function $A(x_1, x_2; x_3, x_4)$

$$A(x_1, x_2; x_3, x_4) =$$



$$= \int f(y_1, y_2, y_3, y_4) \left\{ \prod_{i=1}^{4} D(y_i - x_i)\, d^4 y_i \right\}$$

- *D(y-x)* : free particle propagation

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp_0}{2\pi i} \frac{\exp\{-ip^\mu (y-x)_\mu\}}{m^2 - p^2 - i\epsilon}$$

[Gribov]

Causality and analyticity (2)

- As $y_0 > x_0$, pole at $p_0 = \sqrt{m^2 + \mathbf{p}^2}$

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\exp\{-ip^\mu(y-x)_\mu\}}{2p_0}$$
$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \psi_\mathbf{p}(y) \cdot \psi^*_\mathbf{p}(x), \qquad y_0 > x_0$$

- while for final state $x_{03} > y_{03},\ x_{04} > y_{04}$

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \psi_\mathbf{p}(x) \cdot \psi^*_\mathbf{p}(y), \qquad x_0 > y_0$$

- Truncates amplitude *f* gets multiplied by product of wave functions.

[Gribov]

Causality and analyticity (3): Amplitude in momentum space

- Fourier transform of f :

$$\mathcal{M}(p_i) = \int f(y_1, y_2, y_3, y_4)\, \mathrm{e}^{-i(p_1 y_1 + p_2 y_2) + i(p_3 y_3 + p_4 y_4)} \prod d^4 y_i$$

- Make it simple:
  - Forward scattering $p_1 \approx p_3,\, p_2 \approx p_4$
  - Solve some integrals → only dependence on relative positions, here chosen: $y_{13} = y_1 - y_3$

$$\mathcal{M} \Longrightarrow (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \int \mathrm{e}^{ip_1(y_3 - y_1)} f(y_{13}; p_2) d^4 y_{13}$$

- Forward scattering → Only dependence on one variable

[Gribov]

Causality and analyticity (4):

- The amplitude is proportional to the absorption of a particle in y_1 and creation in y_3 (and reversely for anti-particle):

$$f(y_3, y_1) \propto \langle T\, \psi(y_3)\bar{\psi}(y_1)\rangle \qquad \Delta y^\mu = y_3^\mu - y_1^\mu$$

$$\equiv \vartheta(\Delta y_0)\cdot \psi(y_3)\bar{\psi}(y_1) \;\pm\; \vartheta(-\Delta y_0)\cdot \bar{\psi}(y_1)\psi(y_3)$$

$$= \vartheta(\Delta y_0)\left[\psi(y_3)\bar{\psi}(y_1) \mp \bar{\psi}(y_1)\psi(y_3)\right] \pm \bar{\psi}(y_1)\psi(y_3)$$

(compare to the time evolution operator U in QM which is a time-ordered product; the S-matrix is actually a time-evolution operator)

- Consider now a space-like interval $(\Delta y)^2 < 0$
- The operators $\psi(y_3)\bar{\psi}(y_1)$ have to commute; otherwise, a person at y_3 could tell what was measured at y_1 $\rightarrow$ Causality!
- Then: $f(y_3, y_1) \propto \vartheta(\Delta y_0)\vartheta((\Delta y)^2)\cdot f_1 \pm \bar{\psi}(y_1)\psi(y_3)$
- Insert one in the last term:

$$\langle 0|\,\bar{\psi}(y_1)\psi(y_3)\,|0\rangle = \sum_n \langle 0|\,\bar{\psi}(y_1)|n\rangle \cdot \langle n|\psi(y_3)|0\rangle = \sum_n |C_n|^2\, \mathrm{e}^{-iP_n(y_1-y_3)}$$

[Gribov]

Causality and analyticity (5)

- We still have to integrate over *y* to get *M* (see previous slides):

$$\sum_n |C_n^2| \int d^4 y_{31}\ \mathrm{e}^{ip_1 y_{31}} \cdot \mathrm{e}^{iP_n y_{31}} \propto \delta(p_{0,1} + P_{0,n}) \ = \ 0$$

- This has to be zero because all incoming, outgoing, intermediate particles have positive energy, e.g., $P_{0,n} > 0$
- Finally, as $p_1 y \equiv E_1 t - \mathbf{p}_1 \cdot \mathbf{y} = E_1 \cdot (t - v_1 z)$ Projection of **y** in $\mathbf{p}_1$ direction

$$\mathcal{M}(E_1) = \int d^4 y\, f_1(y) \cdot \vartheta(y_0)\vartheta(y_\mu^2)\, \mathrm{e}^{ip_1 y} = \int d^3\mathbf{y} \int_{\sqrt{\mathbf{y}^2}}^{\infty} dt\, \mathrm{e}^{iE_1(t - v_1 z)} f_1(y)$$

- Make use of all delta-functions →

$$t > 0, \quad t > \sqrt{z^2 + \mathbf{y}_\perp^2} \geq |z| > |v_1 z| \Longrightarrow (t - v_1 z) > 0$$

- If $\mathrm{Im}\, E_1 > 0$ and *f* increases less than expon., *M* converges in the upper half plane.

Causality and analyticity (6)

- Implies the so-called *polynomial boundary* for *M(s)*

$$|\mathcal{M}(s)| < |s|^N$$

- Absolut converging integral → Integration and differentiation can be interchanged.
- Cauchy relations:

$$u = u(x,y),\ v = v(x,y),\ z = x + iy \to \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- hold in the upper half plane with

$$u = \mathrm{Re}\,\mathcal{M}, v = \mathrm{Im}\,\mathcal{M},\ z = E_1$$

- Cauchy relations fulfilled ↔ function analytic.

Conclusion: Causality ensures that there are no resonance poles in the upper energy half-plane (1st Riemann sheet)

1.2 from resonances to bound state

A *computational physics* exercise:

In this exercise you will learn about analytic properties of the scattering amplitude. First, have a look at this video – you will produce something similar. The exercise serves to get intuition about scattering/bound state problems and the underlying analytic structure in terms of singularities that manifest themselves as resonances and bound states – and how one transforms into the other as the potential depth changes. For simplicity, you may set $\hbar = m = 1$ in the entire problem. This is also done in the video. Note: here we look at the S-wave only.
Our example is the spherical square well. We want to make an animation that shows the partial-wave amplitude $t_0(k)$ as a function of $k \in \mathbb{C}$. Treating the problem in the complex k-plane is slightly simpler because there is only one Riemann sheet while the complex $E = \hbar^2 k^2/(2m)$-plane has two Riemann sheets.

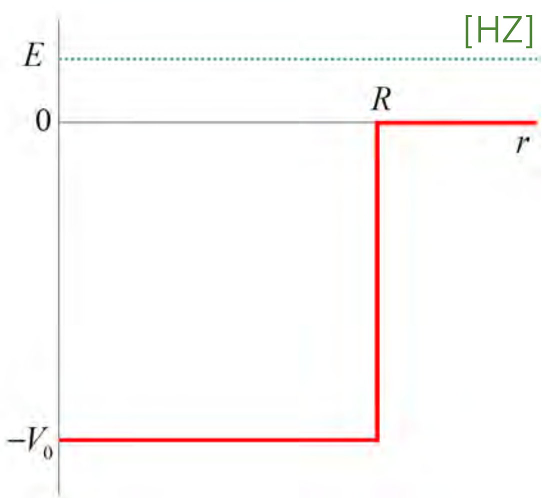
1. *Bound state problem:* From topic 5, solve the bound-state problem numerically for a well that allows for at least one S-wave bound state. Check the bound state condition to make sure the state exists. Make a plot in which you show the RHS and LHS of Eq. (5.188) for illustration.

2. *The power of analyticity:* Bound state energies are pole positions of t_0 on the positive imaginary k-axis. For the same well as before, search numerically for poles and confirm that their positions (or, position if you have a well with only one bound state) coincide with the bound state energies determined in 1.

3. *Pole trajectories*: Trace the pole movements ("trajectories") in the complex k-plane by plotting $\log|t_0|(k)$ for different $0 < V_0 < V_{\max}$ (make an animation). The logarithm only serves to make poles more visible in the contour plot. This would look like in the video, but you do not have to look for poles for every value of V_0 which is quite cumbersome and takes a lot of time. However, do the animation like in that video, i.e, complex plane to the left and phase shift to the right, to see what effects poles have on the phase shift. Choose the maximal depth of the well, $V_{\max}$, such that there you have at least two bound states.

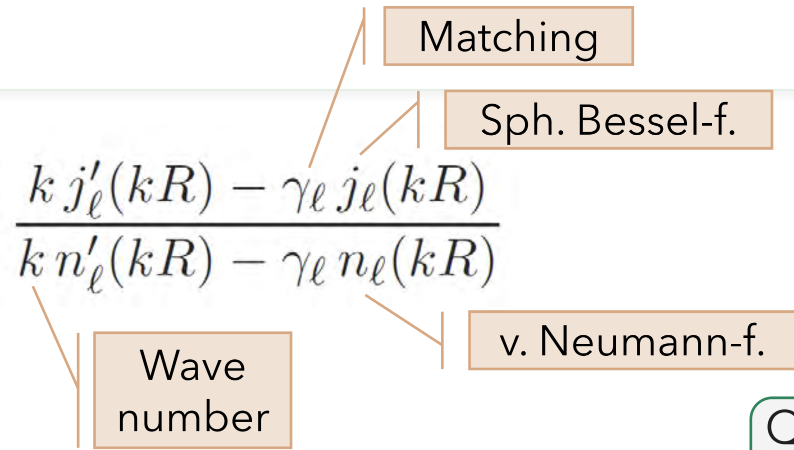
Reminder:

Potential in radial coordinates:



$$V(r) = \begin{cases} -V_0 & \text{for} \quad r < R\,, \\ 0 & \text{for} \quad r > R\,, \end{cases}$$

Scattering phase shifts (E>0)

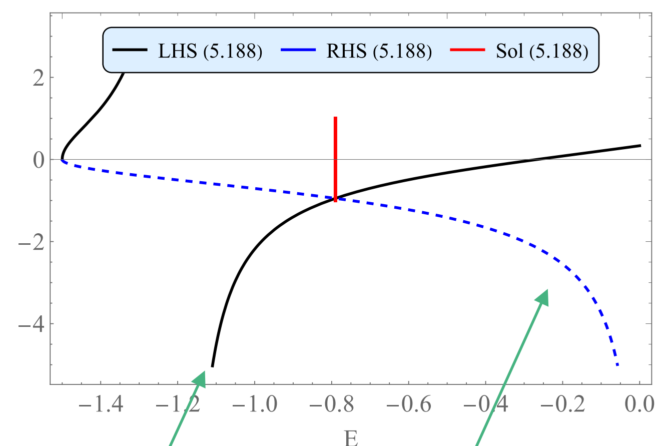


$$\tan\delta_\ell = \frac{k\,j_\ell'(kR) - \gamma_\ell\, j_\ell(kR)}{k\,n_\ell'(kR) - \gamma_\ell\, n_\ell(kR)}$$

QMII

QMI

Bound state energies (E<0) [note: a=R]



$$v = a^2 v_0 = \frac{2m}{\hbar^2} V_0 a^2$$

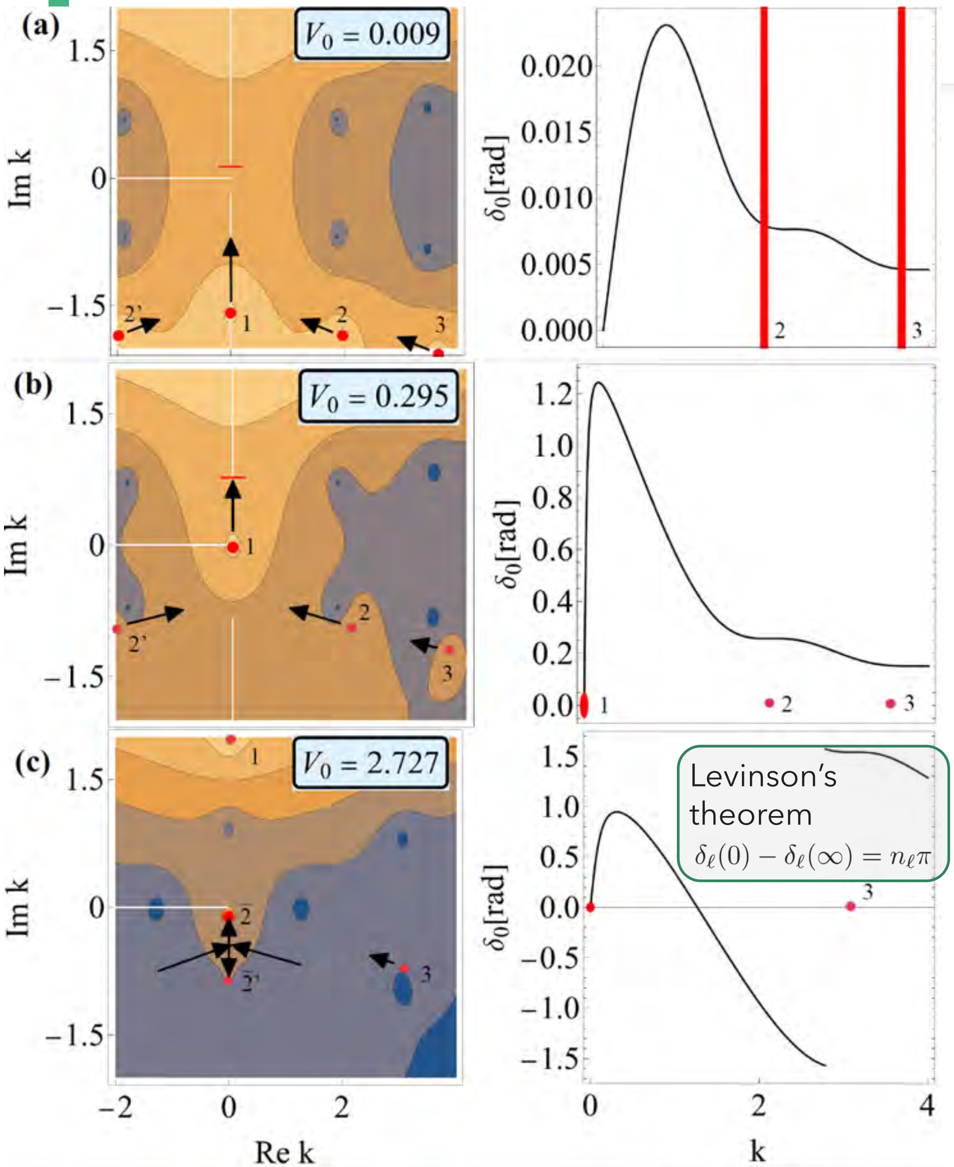
$$\lambda = a^2\kappa^2 = -\frac{2ma^2}{\hbar^2} E > 0$$

$$\tan\sqrt{v-\lambda} = -\sqrt{\frac{v-\lambda}{\lambda}}$$

1.3. Resonances and bound states

$$E = \frac{(\hbar k)^2}{2m}$$

1.3. From resonances to bound states



Left column: S-wave T-matrix, $|t_0|$, in the complex-momentum plane (arb. units). **Right column:** phase shift.

(a) For a shallow potential, there is no bound state, but only virtual state 1 and resonances 2 and 3.

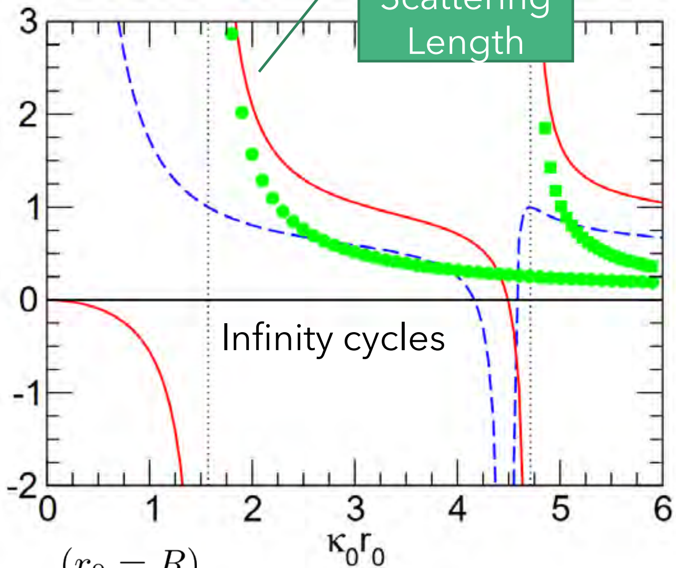
In **(b)**, infinite scattering length is reached which motivates a discussion of universality. [Braaten]

In **(c)**, pole 1 became a deeply bound state. Pole 2 and its mirror pole 2' have met on the imaginary k-axis and then separated again as virtual states $\bar{2}$ and $\bar{2}'$, with $\bar{2}$ on its way to become a bound state and $\bar{2}'$ a deeper-bound virtual state. Such intriguing S-wave pole trajectories have only been discovered ten years ago.

[Hanhart et al., 0801.2871]

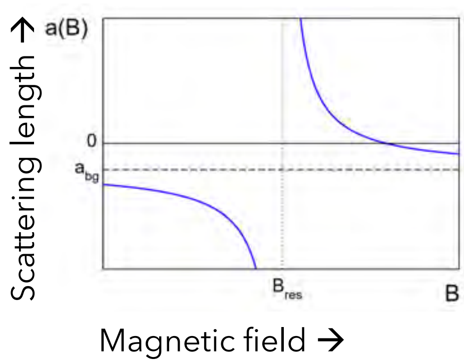
Feshbach resonances [Braaten]

Scattering Length

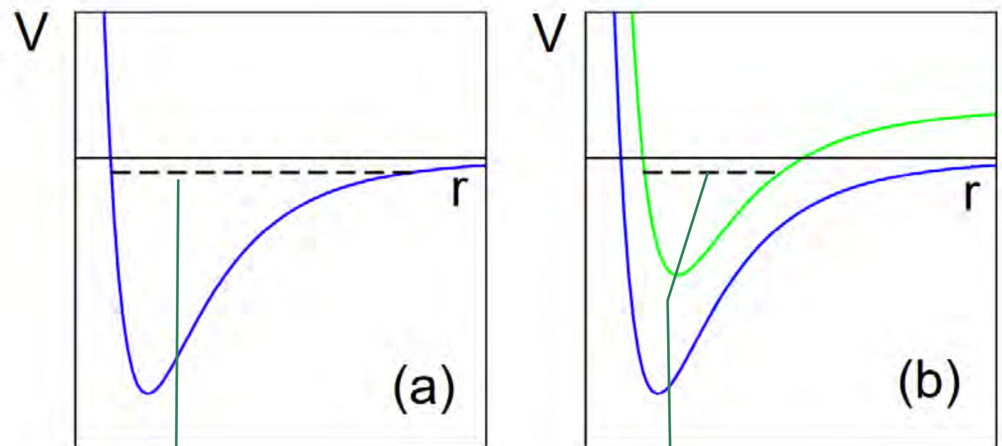


$(r_0 = R)$

$\kappa_0 = (mV_0/\hbar^2)^{1/2}$



Mechanisms to generate large scattering length



Shape resonance

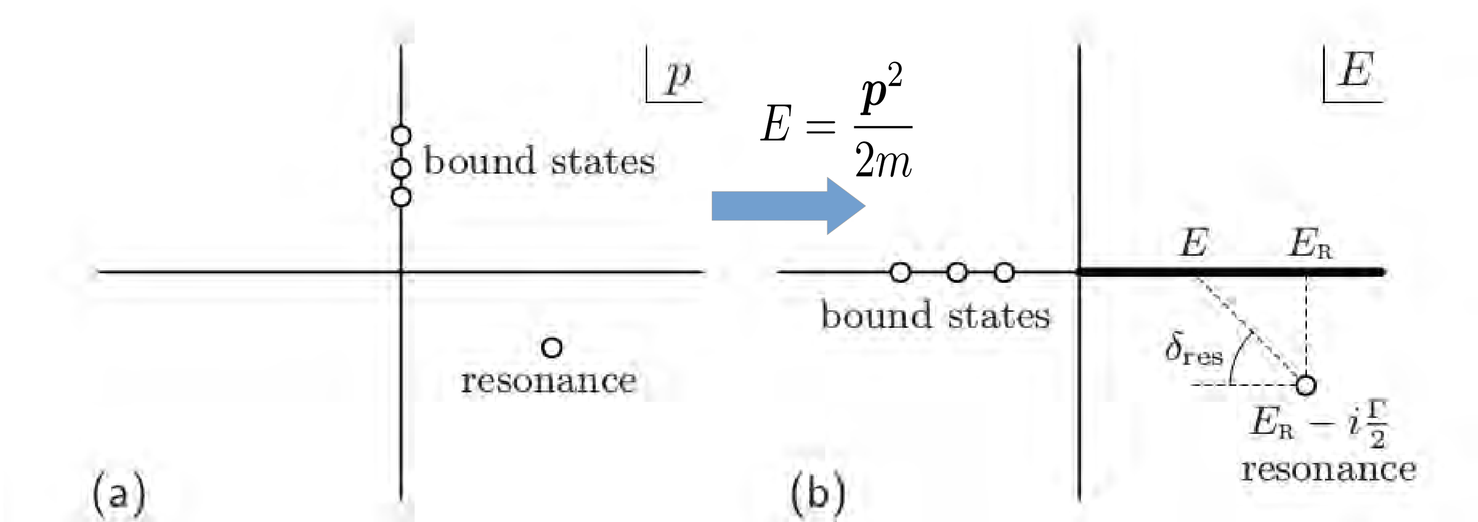
Feshbach resonance: bound state in a weakly-coupled, closed channel

Momentum vs. energy plane

(Preparation for next slide)

One Riemann sheet

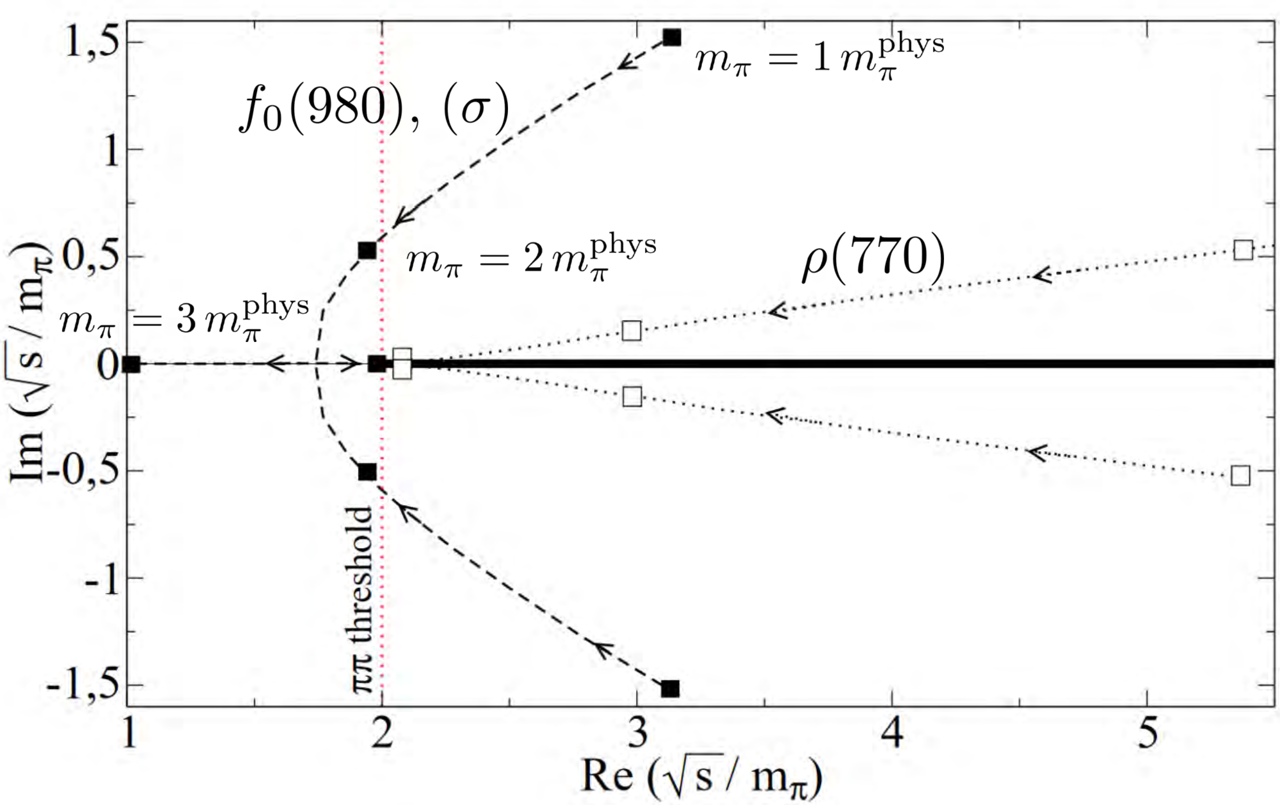
Two Riemann sheets



[HZ]

1.4 Chiral trajectories of light mesons

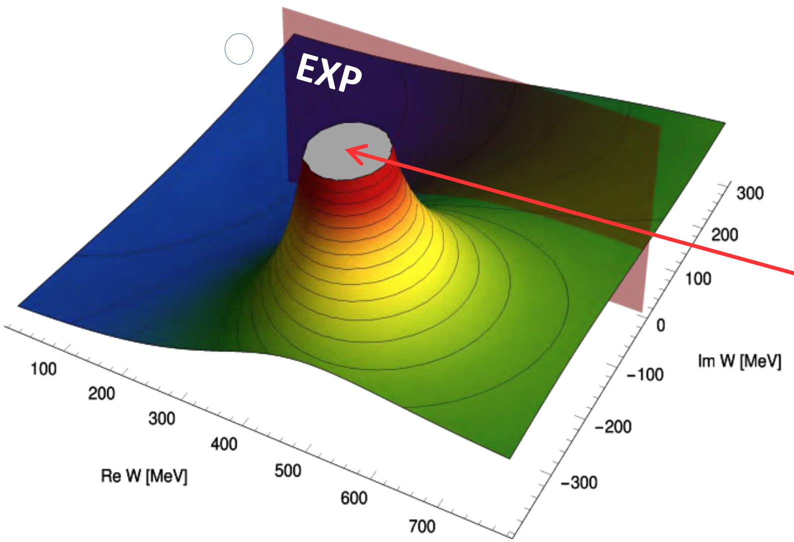
- Quark-mass dependence as predicted from “Inverse amplitude method” with one-loop ChPT [Hanhart et al., 0801.2871]

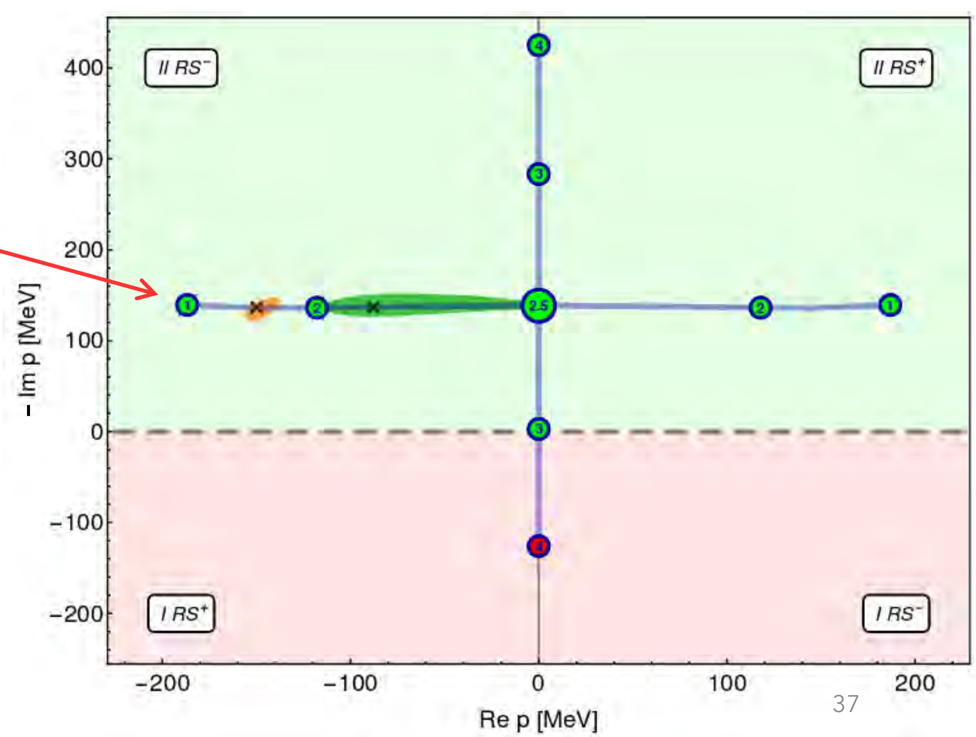


- Axes: $\sqrt{s}$ vs. k
- Resonances → Virtual state → bound state
- But rho-resonance: rather featureless conversion to bound state
- Wide scalar mesons are not at all conventional Breit-Wigner resonances
- Prominent molecular component [Morgan/Pennington] [Baru] [Guo]

Chiral trajectories in lattice QCD

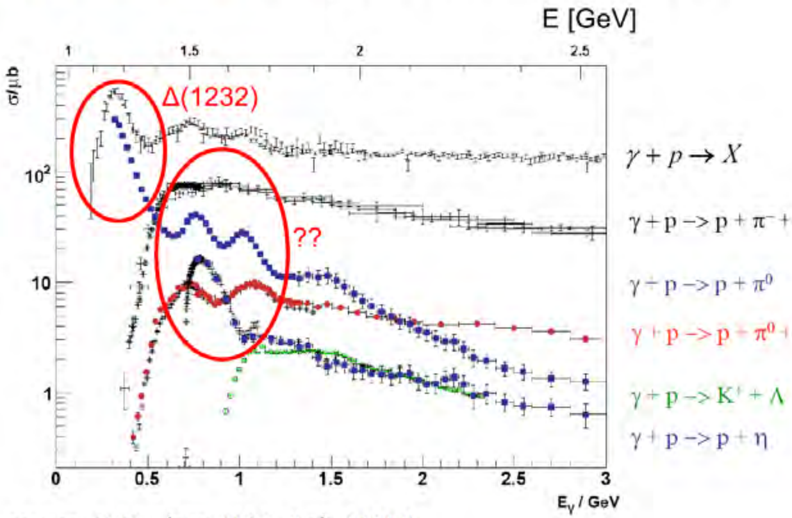
- A lattice calculation at M_π=227 MeV and 315 MeV [GWQCD, 1803.02897]
- σ becomes a (virtual) bound state @ *M_π = (345) 415 MeV*





2. Phenomenology of resonances
2.1 Spectrum of excited baryons

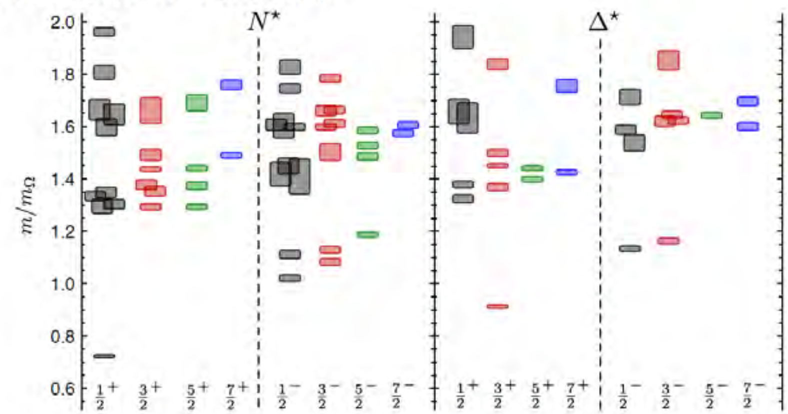
Experimental study of hadronic reactions



source: ELSA; data: ELSA, JLab, MAMI

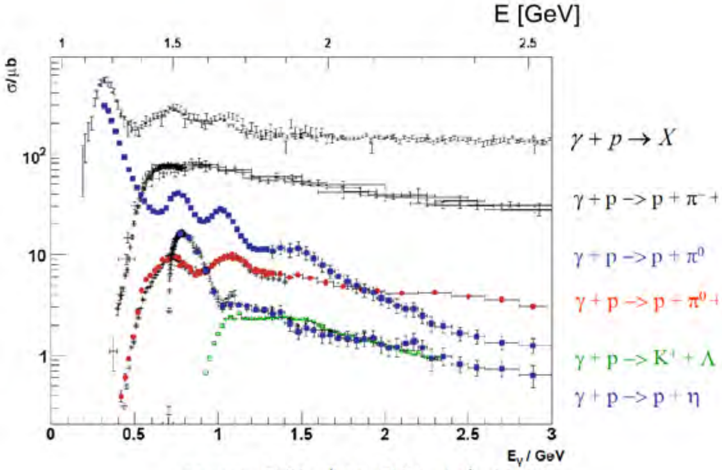
$\Longleftrightarrow$

Theoretical predictions of excited hadrons
e.g. from lattice calculations:
(with some limitations)



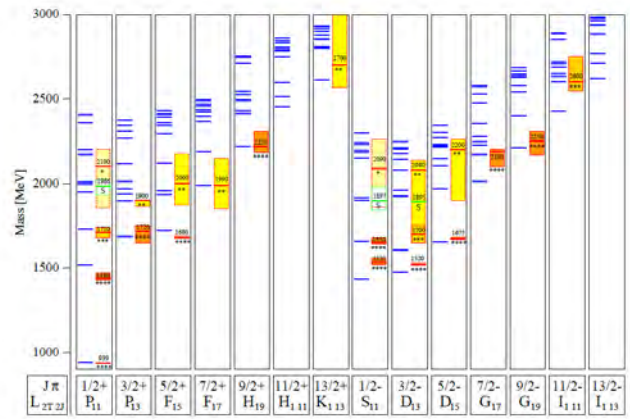
$m_\pi = 396$ MeV [Edwards *et al.*, Phys.Rev. D84 (2011)]

From experimental data to the resonance spectrum



source: ELSA; data: ELSA, JLab, MAMI

$\Longleftrightarrow$



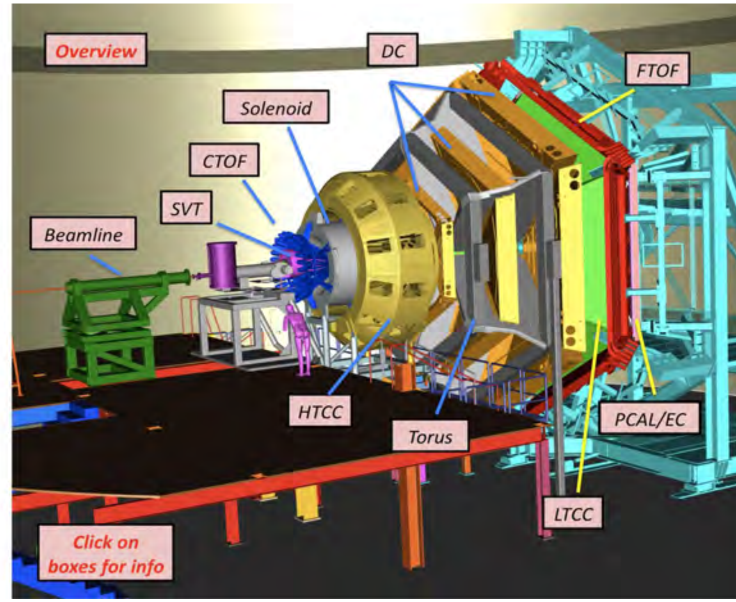
Löring *et al.* EPJ A 10, 395 (2001), experimental spectrum: PDG 2000

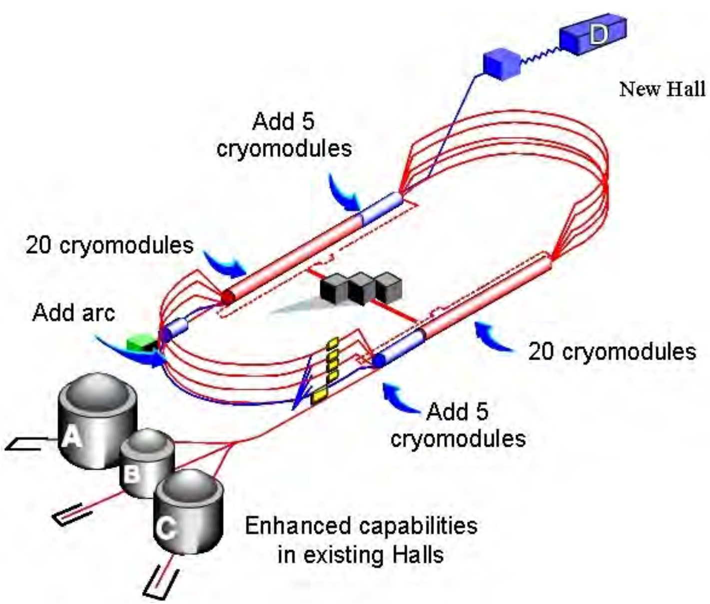
Different modern analyses frameworks:

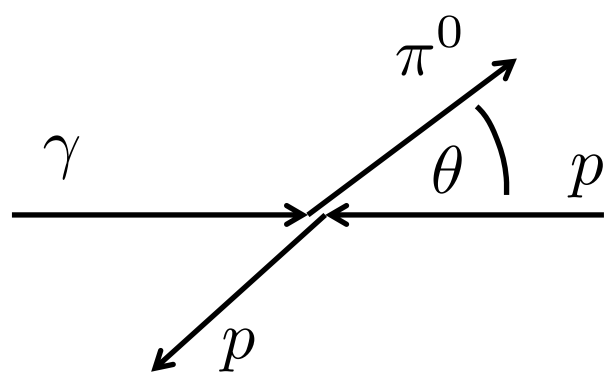
- **unitary isobar models:** unitary amplitudes + Breit-Wigner resonances
  MAID, Yerevan/JLab, KSU
- **(multi-channel) K-matrix:** GWU/SAID, BnGa (phenomenological),
  Gießen (microscopic Bgd)
- **dynamical coupled-channel (DCC):** 3d scattering eq., off-shell intermediate states
  ANL-Osaka (EBAC), Dubna-Mainz-Taipeh, Jülich-Bonn
- **other groups:** JPAC (high energies), Mainz-Tuzla-Zagreb PWA (MAID + fixed-t
  dispersion relations, L+P), Gent, truncated PWA

Photoproduction experiments

(Jlab, Mami, Elsa, GRAAL,…)







Degrees of freedom
- Energy
- Scattering angle
- Polarizations

Typical data situation

Reaction type

Observable:
Differential cross section
$\frac{d\sigma}{d\Omega}\,[\mu b/sr]$

Data from <u>different</u> experiments

Models from different analysis groups (GW-based and others)

Excluded regions

large variations
~ 2 orders of magnitude

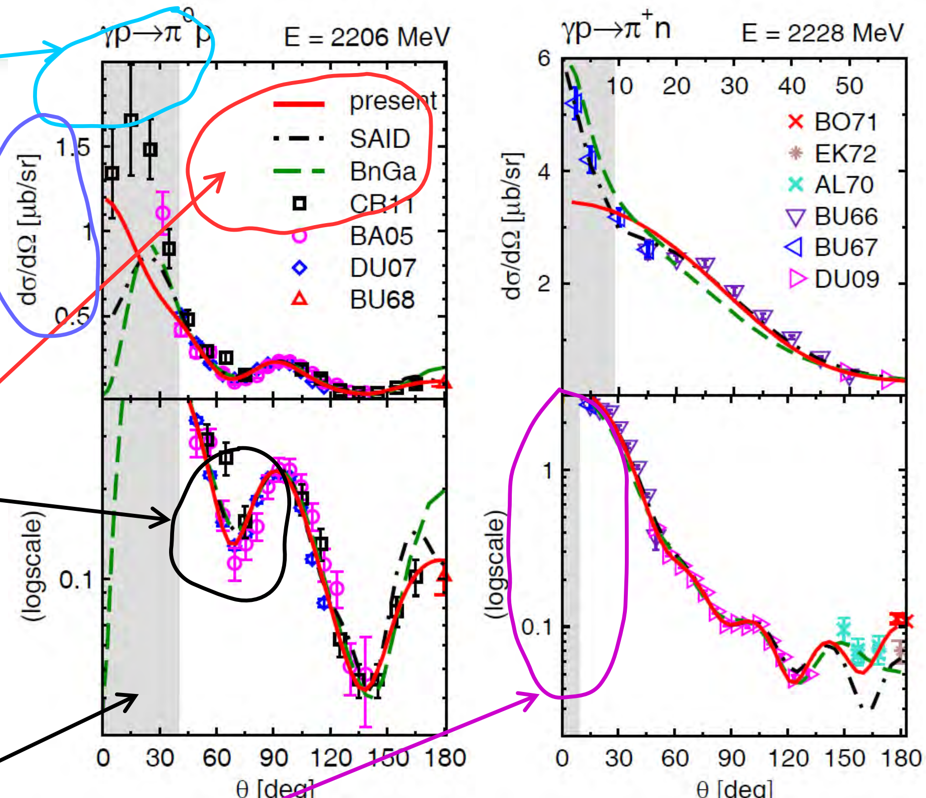


Fig. 2. High-energy behavior in the reaction $\gamma p \to \pi^0 p$ (left) and $\gamma p \to \pi^+ n$ (right). Solid (red) line: fit 2; dash-dotted (black) line: GWU/SAID CM12 [3]; dashed (green) line: Bonn-Gatchina [119]. Data $\pi^0 p$: CR11 [120], BA05 [126], DU07 [127], BU68 [128]. Data $\pi^+ n$: BO71 [129], EK72 [130], AL70 [131], BU66 [132], BU67 [133], DU09 [134]. The regions excluded in our fit are shown as shaded areas.

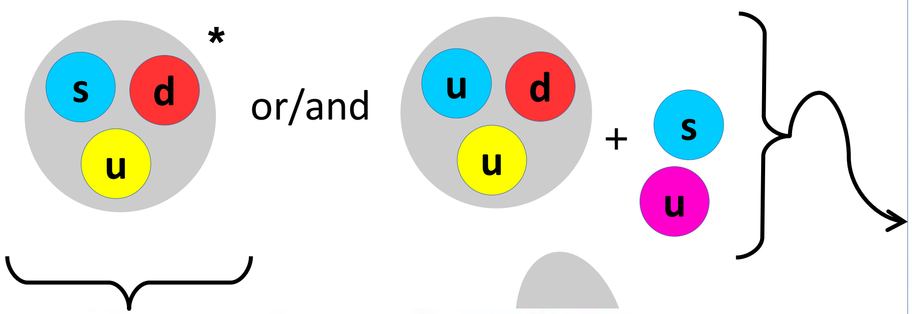
QCD at low energies → *mass generation & confinement*

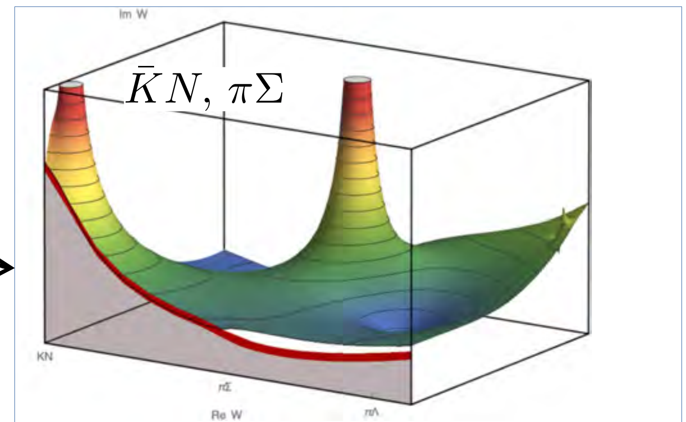
Non-perturbative dynamics → rich spectrum of excited states

How many are there? → missing resonance problem)

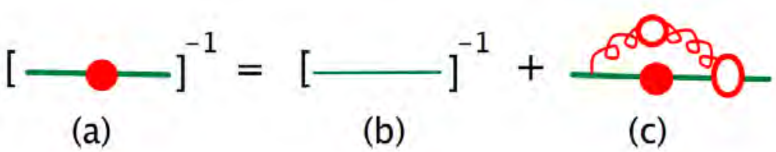
What are they? → 2-quark/3-quark, hadron molecules, …

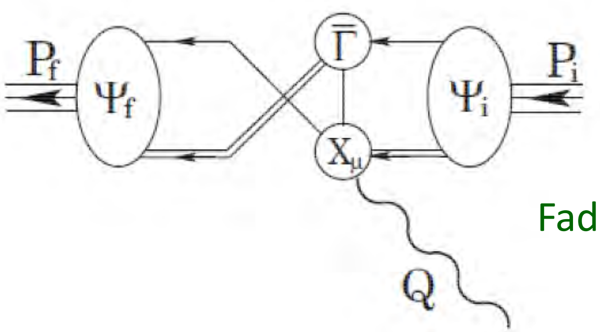
$\Lambda(1405)$





Hadronic molecule [Mai]





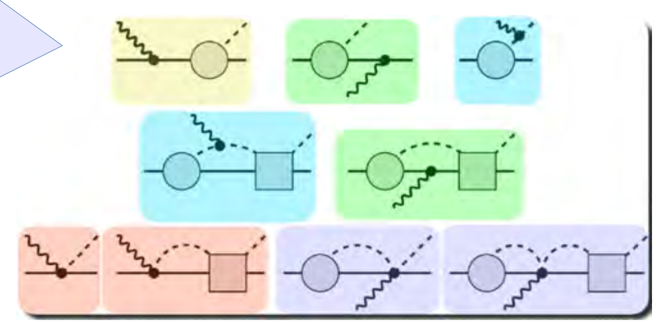
Faddeev Eq. / DSE (Binosi, Cloet, Chang, Roberts)

Using ONLY meson-baryon degrees of freedom (no explicit quark dynamics):

Manifestly gauge invariant approach based on full BSE solution

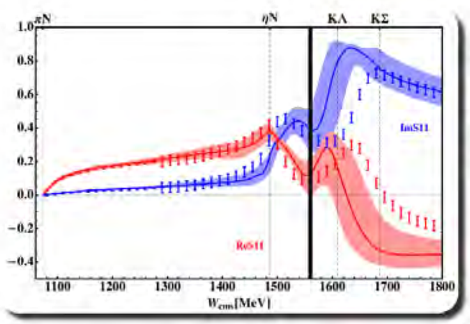
[Ruic, M. Mai, U.-G. Meissner PLB 704 (2011)]

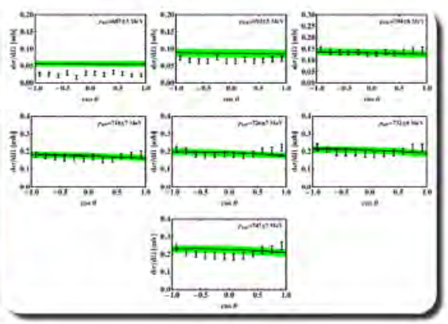
Gauge invariance

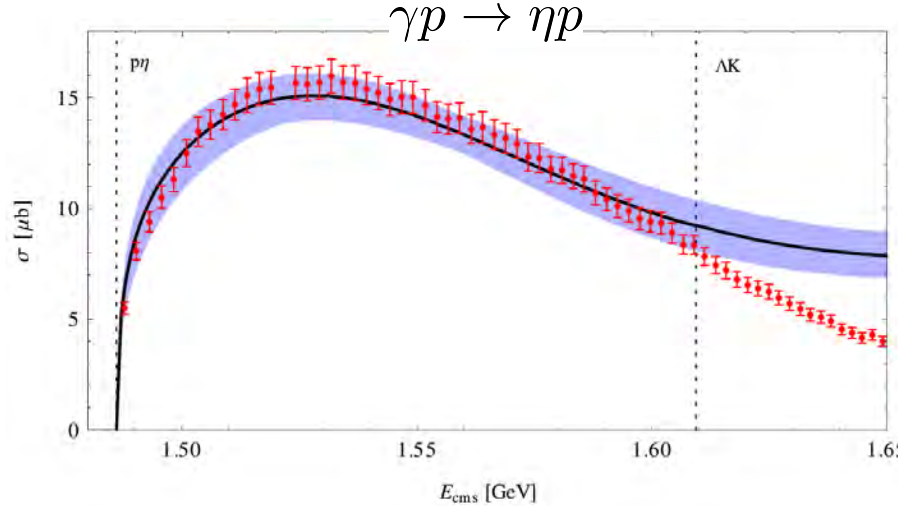


Fit

- Exact unitary meson-baryon scattering amplitude T with parameters, fixed to reproduce:
  - πN-partial wave S_{11} and S_{31} for $\sqrt{s} < 1560$ MeV
    Arndt et al. (2012)
  - $\pi^- p \to \eta n$ differential cross sections
    Prakhov et al. (2005)





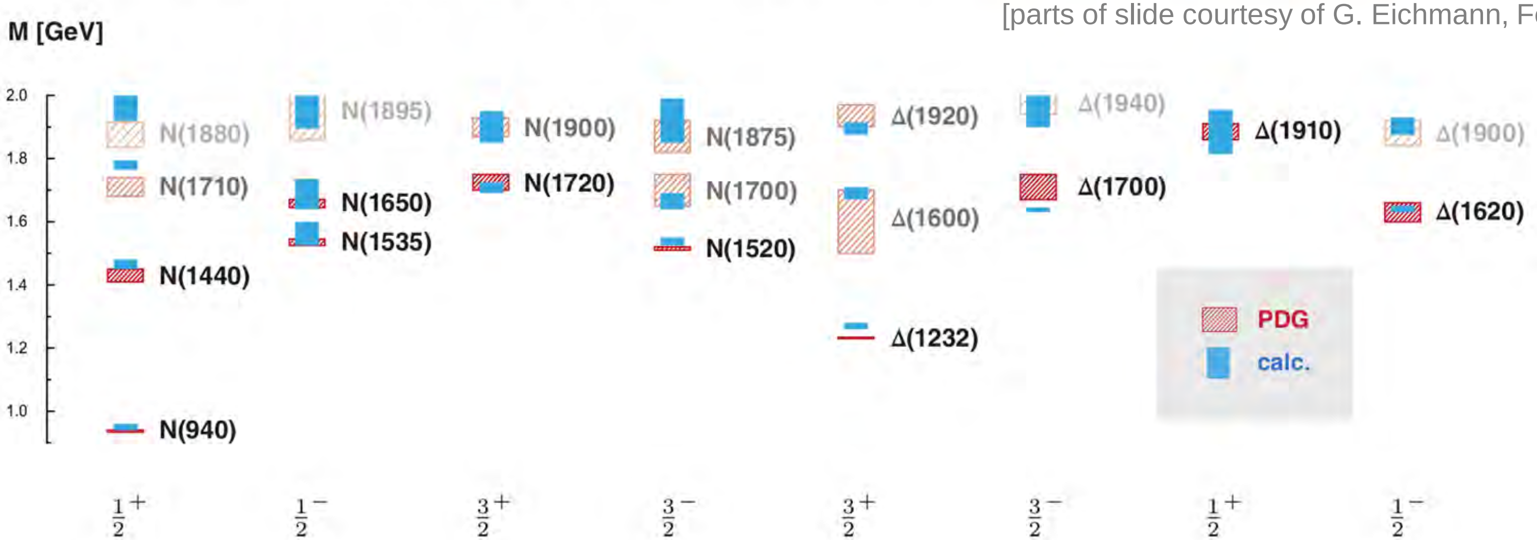


→ Making the “Missing resonance problem” worse ?!

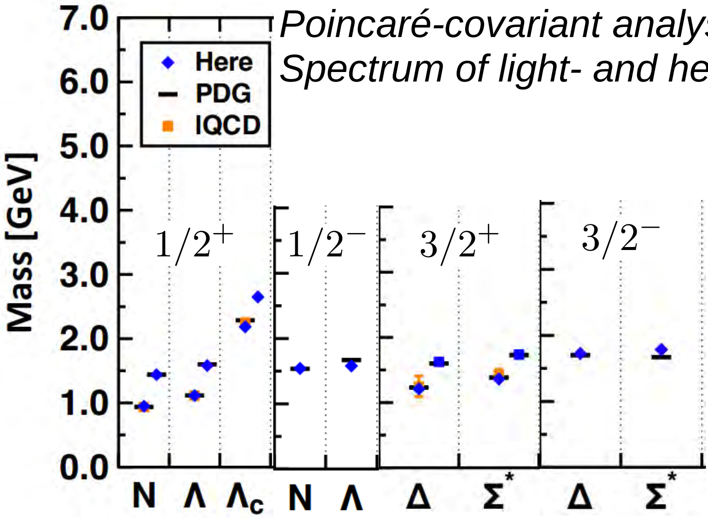
Results in dynamical quark picture

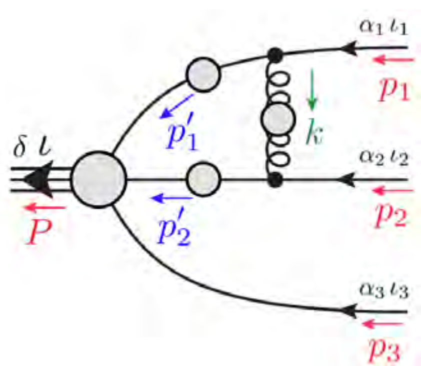
Quark-diquark with reduced pseudoscalar + vector diquarks: GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)

[parts of slide courtesy of G. Eichmann, Few Body 2018]

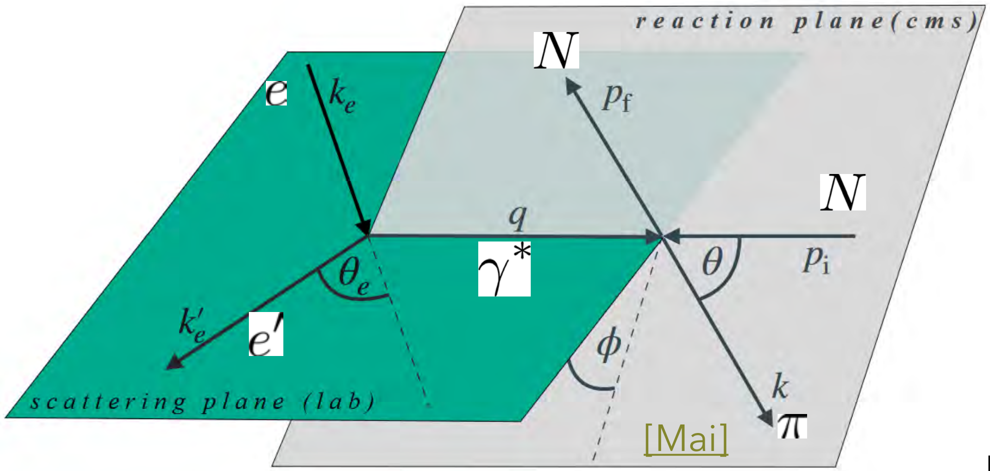


Poincaré-covariant analysis of heavy-quark baryons, Qin, Roberts, Schmidt, PRD (2018)
Spectrum of light- and heavy-baryons, Qin, Roberts, Schmidt, Few Body Syst. 60 (2019)

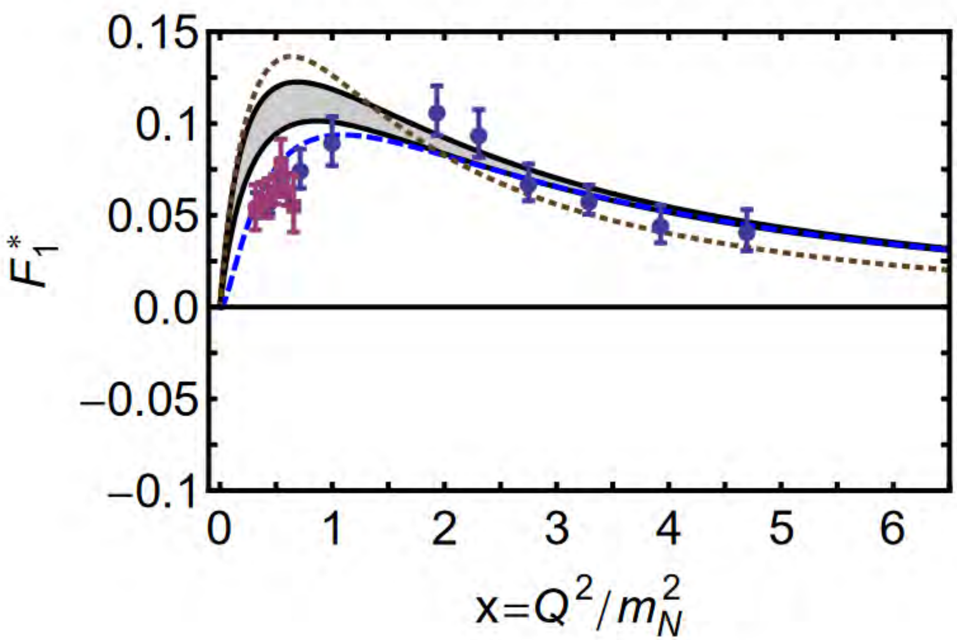




Electroproduction reveals resonance structure



Proton-Roper Transition [Burkert]



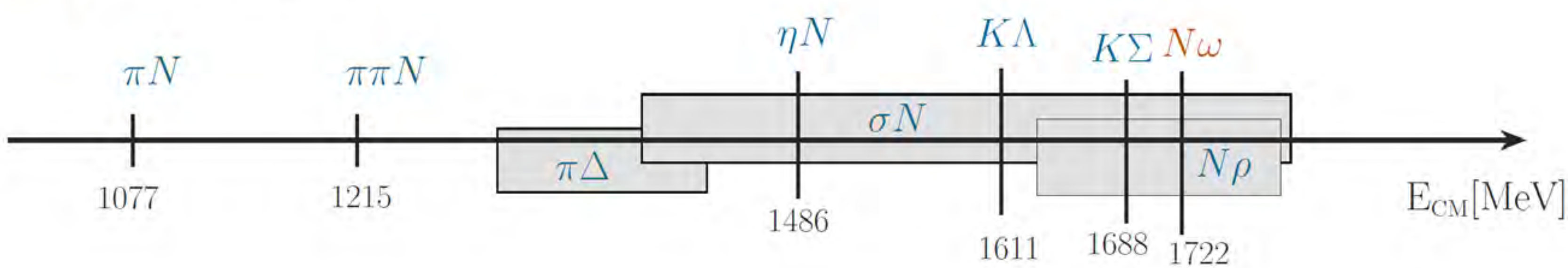
JBW DCC approach (Jülich-Bonn-Washington)

Dynamical coupled-channels (DCC): **simultaneous** analysis of different reactions

The scattering equation in partial-wave basis

$$\langle L'S'p'|T^{IJ}_{\mu\nu}|LSp\rangle = \langle L'S'p'|V^{IJ}_{\mu\nu}|LSp\rangle + \sum_{\gamma,L''S''}\int_0^\infty dq \quad q^2 \quad \langle L'S'p'|V^{IJ}_{\mu\gamma}|L''S''q\rangle \frac{1}{E-E_\gamma(q)+i\epsilon}\langle L''S''q|T^{IJ}_{\gamma\nu}|LSp\rangle$$

- channels ν, μ, γ:



JBW DCC approach (Jülich-Bonn-Washington)

The scattering equation in partial-wave basis

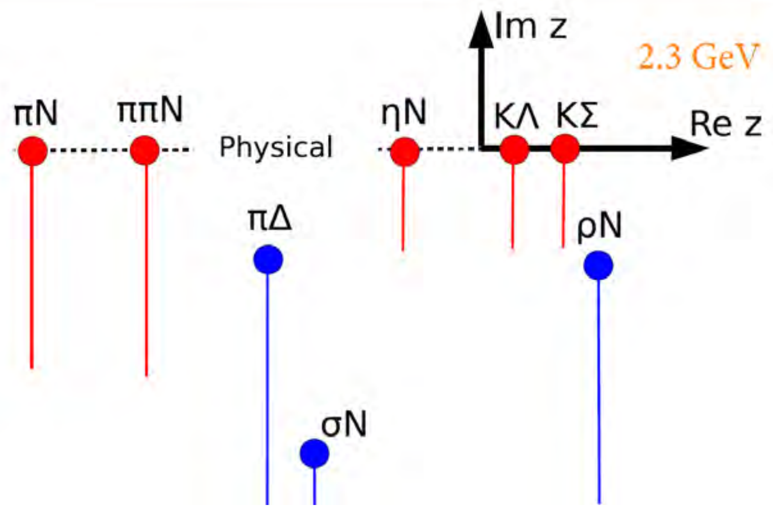
$$\langle L'S'p'|T^{IJ}_{\mu\nu}|LSp\rangle = \langle L'S'p'|V^{IJ}_{\mu\nu}|LSp\rangle +$$
$$\sum_{\gamma,L''S''}\int_0^\infty dq \quad q^2 \quad \langle L'S'p'|V^{IJ}_{\mu\gamma}|L''S''q\rangle \frac{1}{E-E_\gamma(q)+i\epsilon}\langle L''S''q|T^{IJ}_{\gamma\nu}|LSp\rangle$$

3-body $\pi\pi N$ channel:

- parameterized effectively as $\pi\Delta$, σN, ρN
- $\pi N/\pi\pi$ subsystems fit the respective phase shifts

↳ branch points move into complex plane

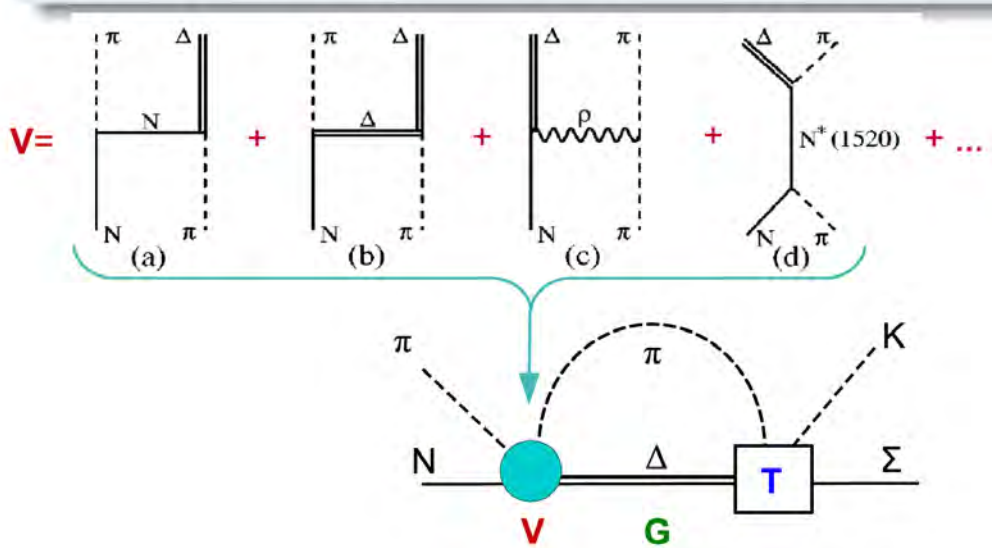
Inclusion of branch points important to avoid false resonance signal!



JBW DCC approach (Jülich-Bonn-Washington)

The scattering equation in partial-wave basis

$$\langle L'S'p'|T^{IJ}_{\mu\nu}|LSp\rangle = \langle L'S'p'|V^{IJ}_{\mu\nu}|LSp\rangle + \sum_{\gamma,L''S''}\int_0^\infty dq \quad q^2 \quad \langle L'S'p'|V^{IJ}_{\mu\gamma}|L''S''q\rangle \frac{1}{E - E_\gamma(q) + i\epsilon} \langle L''S''q|T^{IJ}_{\gamma\nu}|LSp\rangle$$

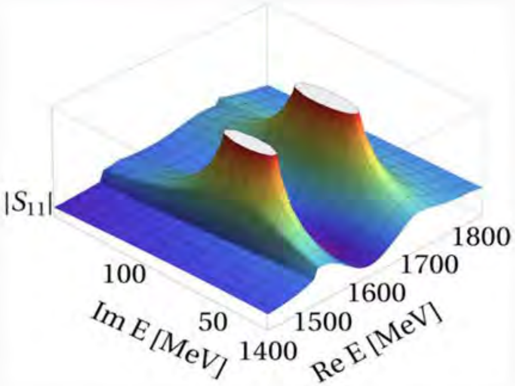


- potentials V constructed from effective $\mathcal{L}$
- s-channel diagrams: T^P
  genuine resonance states
- t- and u-channel: T^{NP}
  dynamical generation of poles
  partial waves strongly correlated
- contact terms

Resonance Couplings

Resonance states: **Poles in the T-matrix** on the 2^{nd} Riemann sheet

[D. Roenchen, M. D., U.-G. Meißner, EPJ A 54, 110 (2018)



- $\mathrm{Re}(E_0)$ = "mass", $-2\mathrm{Im}(E_0)$ = "width"
- elastic πN residue $(|r_{\pi N}|, \theta_{\pi N \to \pi N})$, normalized residues for inelastic channels $(\sqrt{\Gamma_{\pi N}\Gamma_\mu}/\Gamma_{\rm tot}, \theta_{\pi N \to \mu})$
- photocouplings at the pole: $\tilde{A}^h_{pole} = A^h_{pole} e^{i\vartheta^h}$, $h = 1/2,\ 3/2$

Inclusion of $\gamma p \to K^+\Lambda$ in JüBo ("JuBo2017-1"): 3 additional states

	z_0 [MeV]	$\frac{\Gamma_{\pi N}}{\Gamma_{\rm tot}}$	$\frac{\Gamma_{\eta N}}{\Gamma_{\rm tot}}$	$\frac{\Gamma_{K\Lambda}}{\Gamma_{\rm tot}}$
$N(1900)3/2^+$	$1923 - i\,108.4$	1.5 %	0.78 %	2.99 %
$N(2060)5/2^-$	$1924 - i\,100.4$	0.35 %	0.15 %	13.47 %
$\Delta(2190)1/2^+$	$2191 - i\,103.0$	33.12 %		

- $N(1900)3/2^+$: s-channel resonances, seen in many other analyses of kaon photoproduction (BnGa), 3 stars in PDG
- $N(2060)5/2^-$: dynamically generated, 2 stars in PDG, seen e.g. by BnGa
- $\Delta(2190$ 1 2^+: dyn. gen., no equivalent PDG state

Ambiguities & complete experiment

- Does the measurement of a set of observables allow to determine the partial-wave amplitudes (up to one global undetermined phase)?
- Polynomial expansion of cross section: Assume only $\ell = 0, 1$ exist. Then:

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left[\sin^2\delta_0 + 6\sin\delta_0\sin\delta_1\cos(\delta_1-\delta_0)\cos\theta + 9\sin^2\delta_1\cos^2\theta\right] \qquad (*)$$

- Assume experimental cross section is well described by A, B, C, where

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left[A + B\cos\theta + C\cos^2\theta\right]$$

- Sign ambiguity: (*) does not change if signs of both $\delta_0,\ \delta_1$ are changed.
- Generalization to systems with spin (usually, photoproduction):
  - "Complete experiment" (up to a global, energy-dependent phase)
  - "Complete truncated-partial-wave experiment"

2.2 One statistical aspect [Landay]

- How many resonances does one need to describe a given data set?
- Search for a “minimal set” (Occam’s razor)
- Too many hypothesis to test in fits (all combinations of all condidates)
- Automatized methods → Model selection techniques
  - “Least absolute shrinkage and selection operator” (LASSO) creates a whole family of models automatically from smaller to larger complexity
  - Additional criteria help to select the minimal model (usually weighing the chi-square against degrees of freedom)

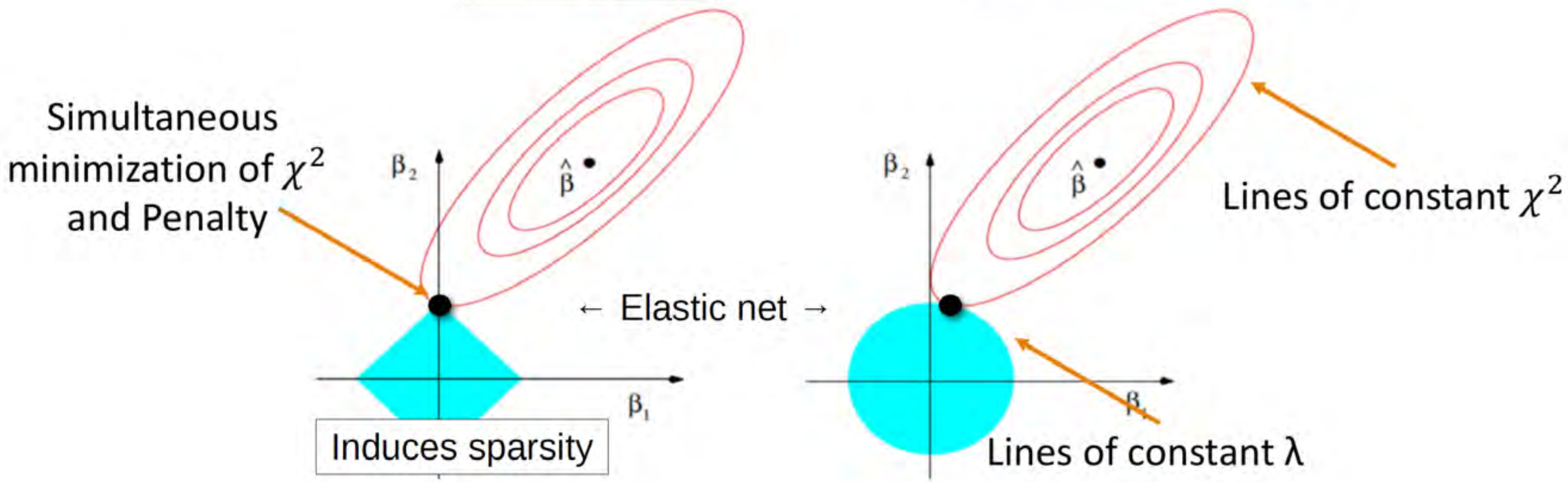
LASSO

$$\underbrace{\sum_{i=1}^{n} \frac{(y_i - f(x_i, \beta_j))^2}{\sigma_i^2}}_{\text{Normal } X^2} + \underbrace{\lambda \sum_{j=1}^{m} |\beta_j|^n}_{\text{Penalty Term}}$$

$\hat{\beta}_i$: Best parameters without penalty
$\beta_i = 0$: Best parameters only penalty

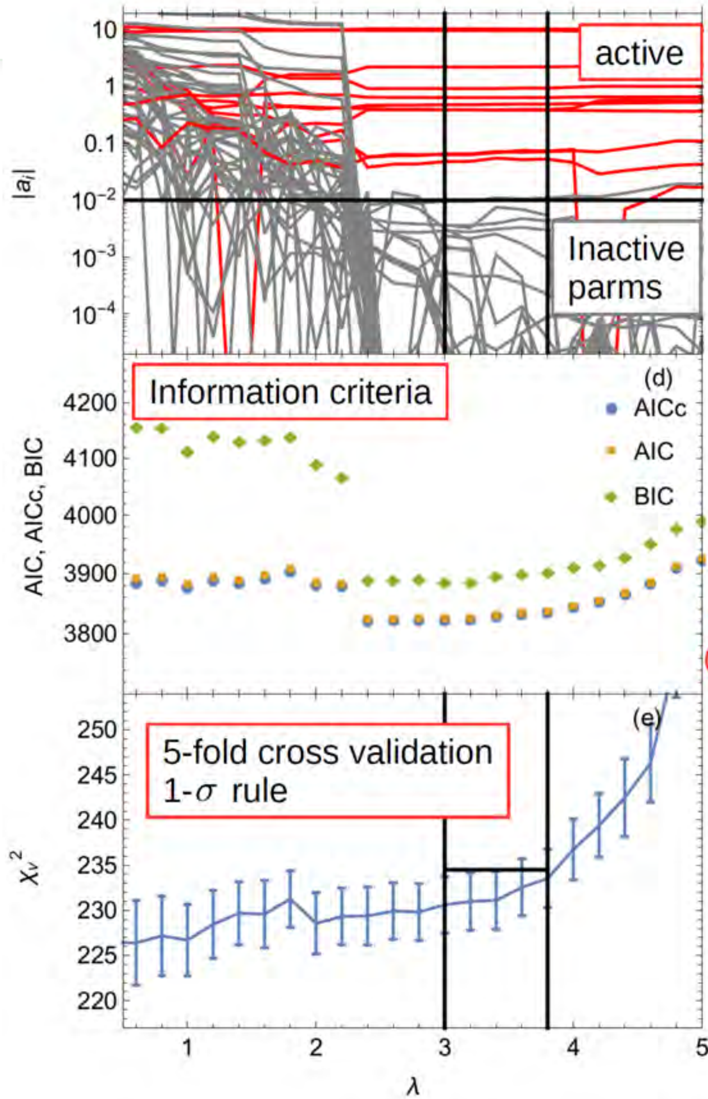
LASSO $(n = 1)$

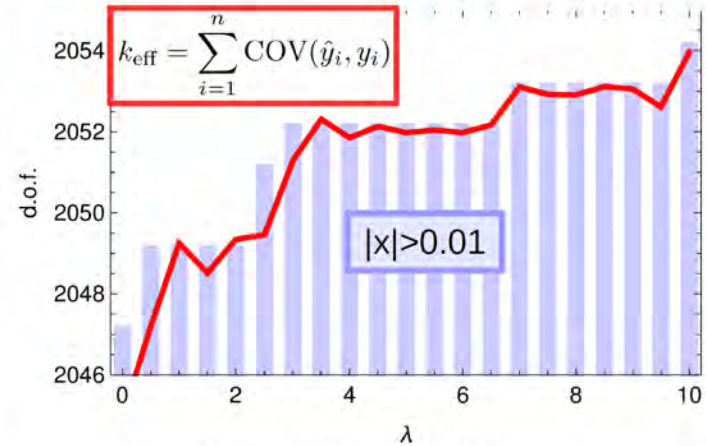
Ridge Regression $(n = 2)$



See, e.g.: *The Elements of Statistical Learning*: Data Mining, Inference, and Prediction, T. Hastie, R. Tibshirani, J. Friedman, Springer 2009 second ed.

Information theory criteria





$k \to k_{\mathrm{eff}}$ subsequently

$$AIC = -2 \max \log(L(\hat{\theta}|data)) + 2k = \chi^2 + 2k$$

$$AIC_c = AIC + \frac{2k(k+1)}{n-k-1}$$

$$BIC = -2 \max \log(L(\hat{\theta}|data)) + 2log(n) = \chi^2 + k\, log(n)$$

Close relation to Bayesian model comparison
(here: $n \gg k$)

See, e.g.: Andrew A. Neath, Joseph E. Cavanaugh, DOI: 10.1002/wics.199

Synthetic data results

- 10 partial waves
- 10 resonance candidates
- Synthetic data with 4 active resonances

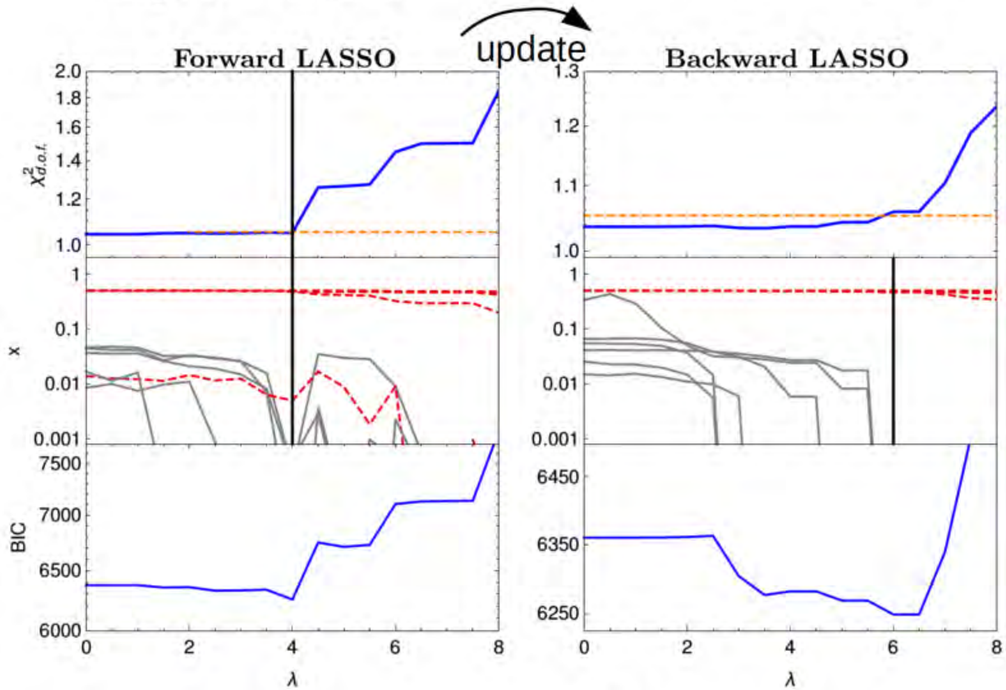
$$W) = e^{i\phi}\left(\frac{k_f(W)}{\Lambda}\right)^{L+1/2} \times \left(a\, e^{-\alpha^2\left(\frac{k_f(W)}{\Lambda}\right)^2} - \boxed{x}\, e^{i\Phi}\frac{\Gamma/2}{W-M+i\Gamma/2}\right) \times$$

- Penalty (group LASSO):

$$P_{gr}(\lambda) = \lambda^4 \sum_{i=1}^{i_{\max}} \sqrt{p_i}\,|x_i|$$

Large λ → small λ

Small λ → large λ



Selected:
4 active resonances
1 inactive resonance

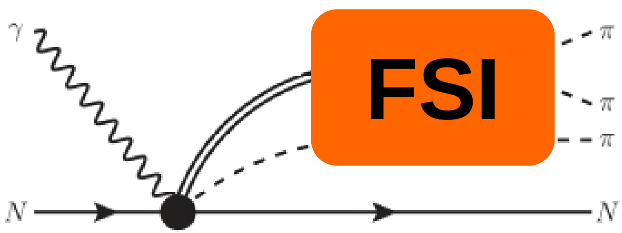
Finds good local minima!

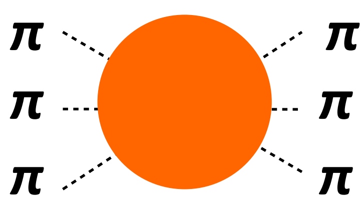
Selected:
4 active resonances
0 inactive resonance

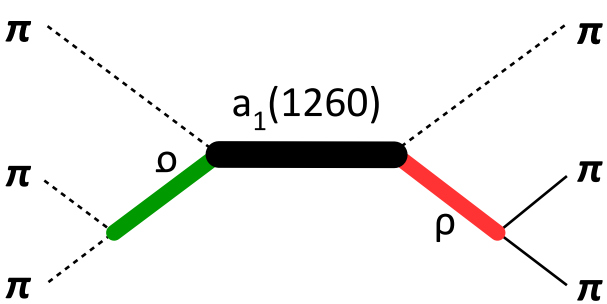
Greediness built in

3. Three-body systems

Light mesons

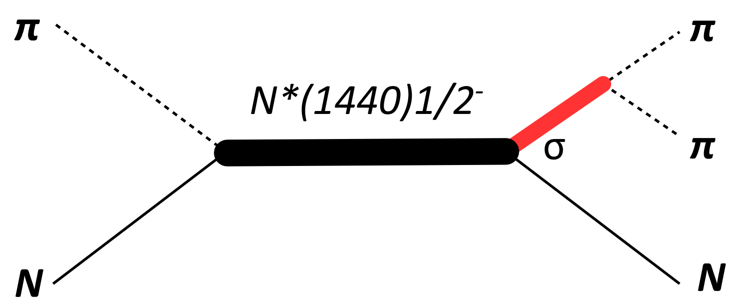






- Important channel in GlueX @ Jlab: hybrids and exotics
- Finite volume spectrum from lattice QCD:
  Lang (2014), Woss [HadronSpectrum] (2018)
  Hörz (2019), Culver (2020), Fischer (2020),
  Hansen (2020),...

Light baryons



- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

3.1 Three-body unitarity with isobars * [Mai]

$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$

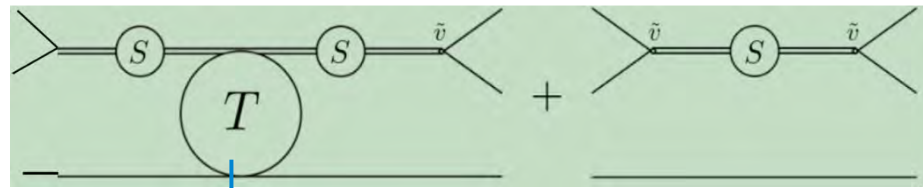
$$\times \prod_{\ell=1}^{3} \left[\frac{d^4 k_\ell}{(2\pi)^4} (2\pi) \boxed{\delta^+(k_\ell^2 - m^2)} \right] (2\pi)^4 \delta^4 \left(P - \sum_{\ell=1}^{3} k_\ell \right)$$

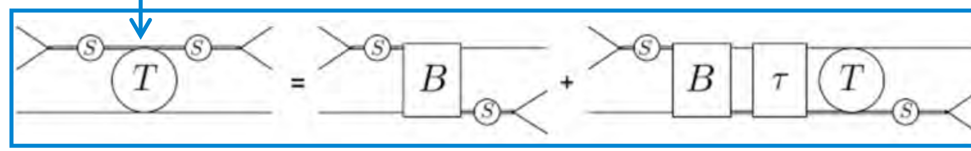
delta function sets all intermediate particles on-shell

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrization of full 2-body amplitude [Bedaque] [Hammer]

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

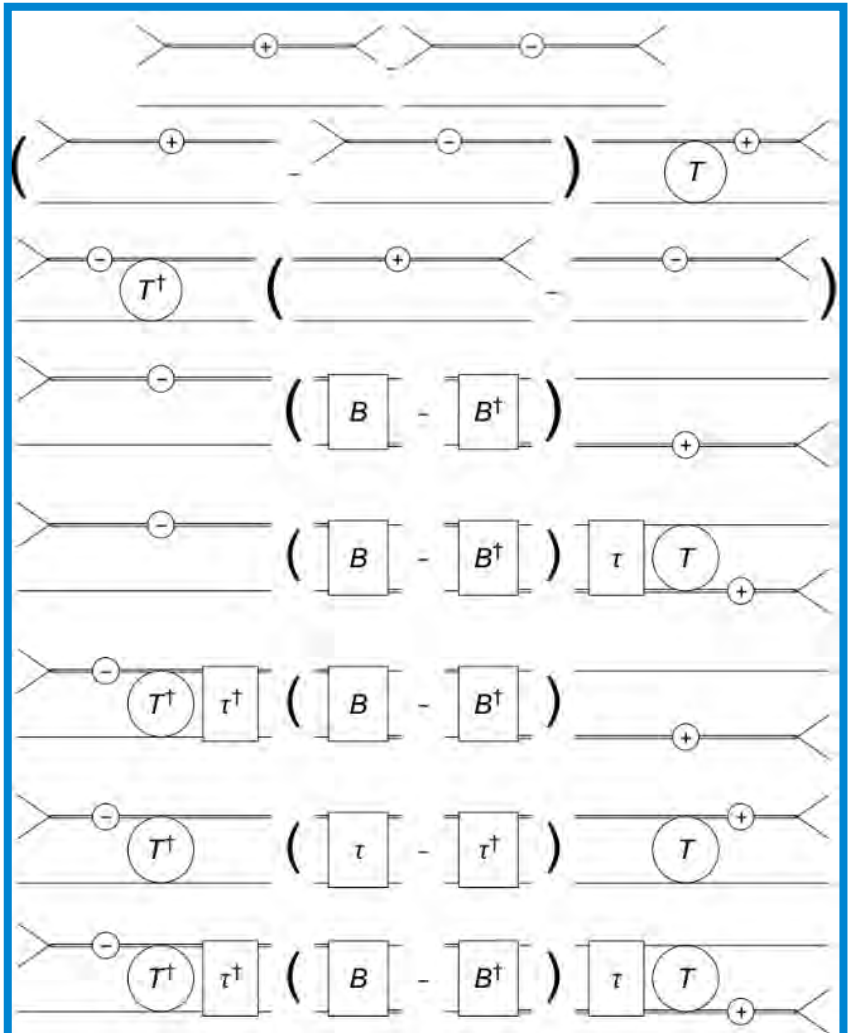




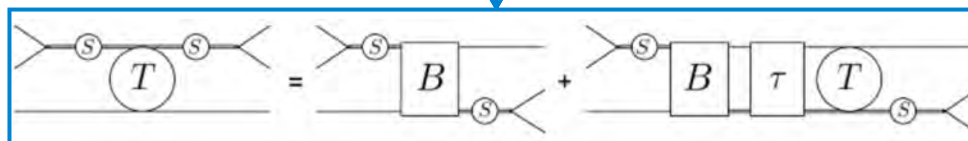
General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \;=\; i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$



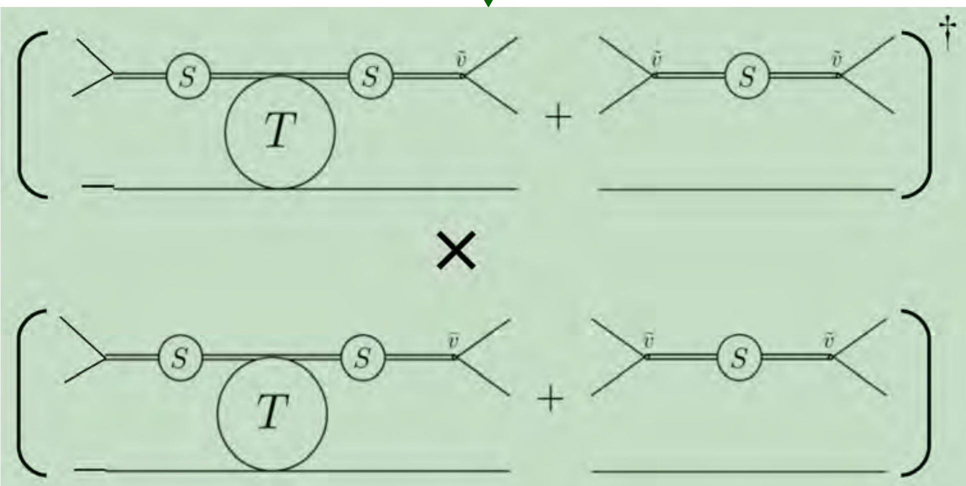
Bethe-Salpeter equation



$$\begin{aligned}\hat{T}-\hat{T}^\dagger &= v(S-S^\dagger)v + vSTSv - vS^\dagger T^\dagger S^\dagger v\\ &= v(S-S^\dagger)v + \left(vS-vS^\dagger\right)TSv + vS^\dagger T^\dagger\left(Sv-S^\dagger v\right) + vS^\dagger(T-T^\dagger)Sv\end{aligned}$$

Three-body unitarity

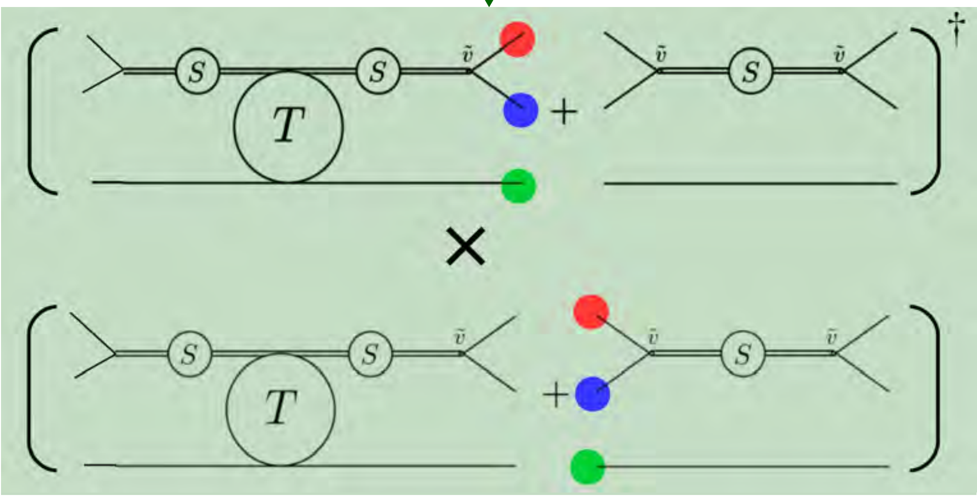
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

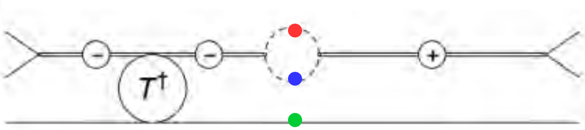


General connected-disconnected structure

Three-body unitarity

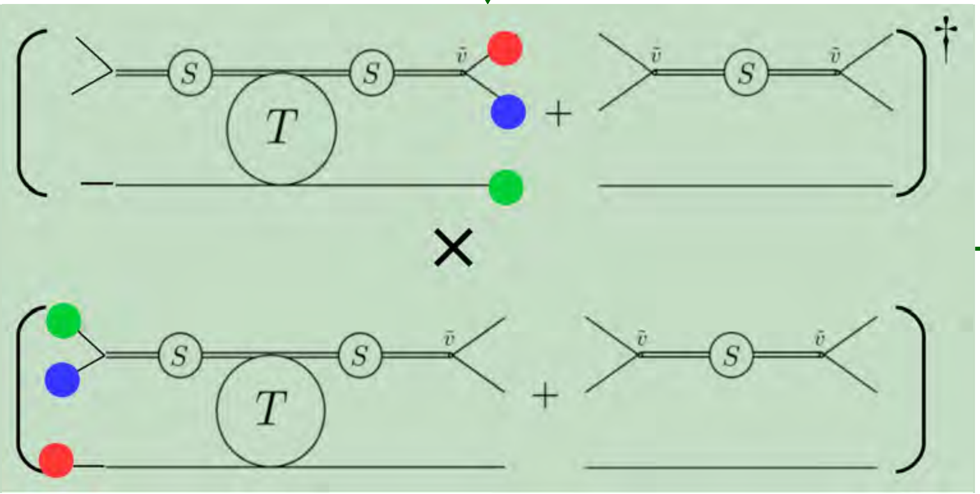
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

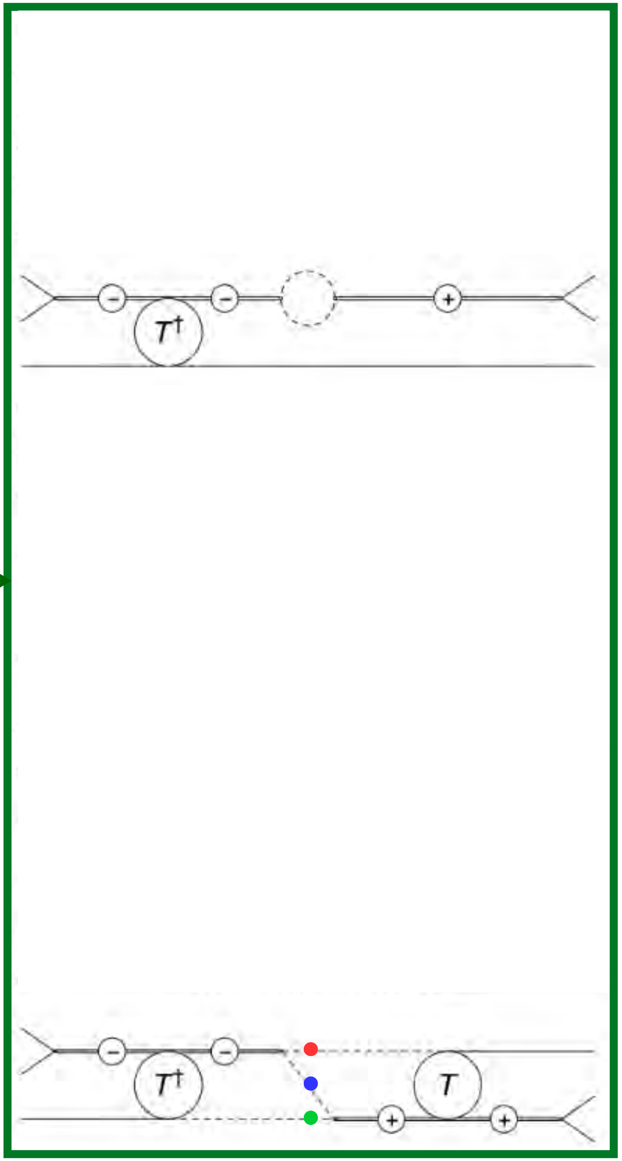




Three-body unitarity

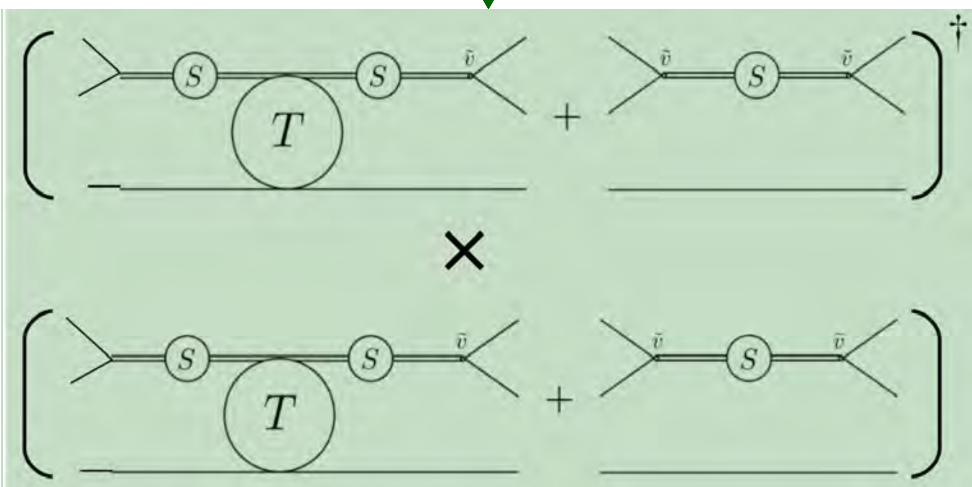
$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



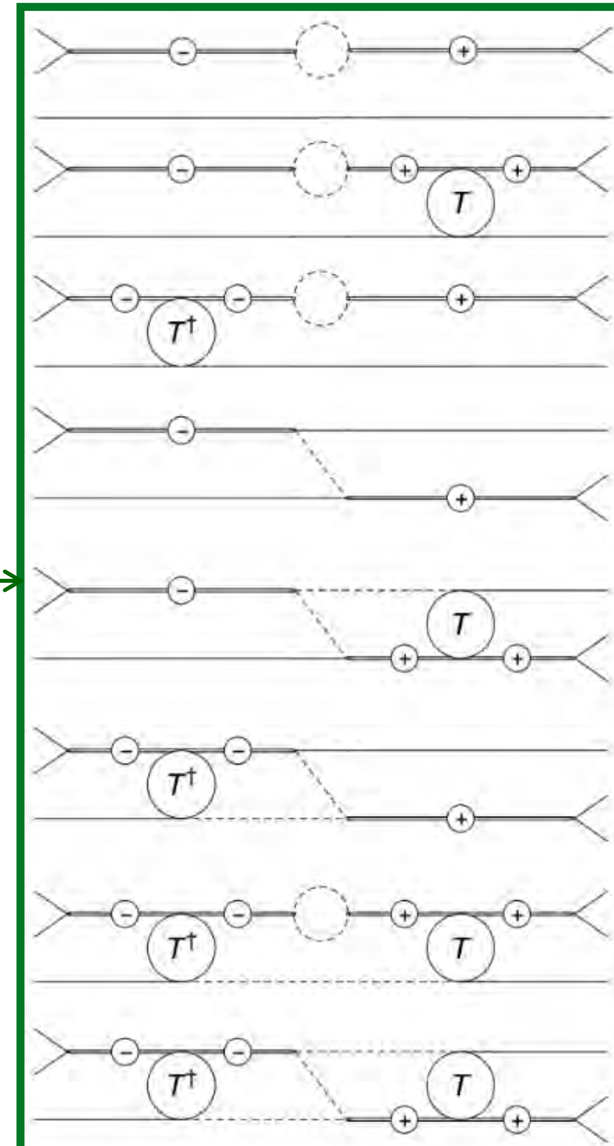


Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

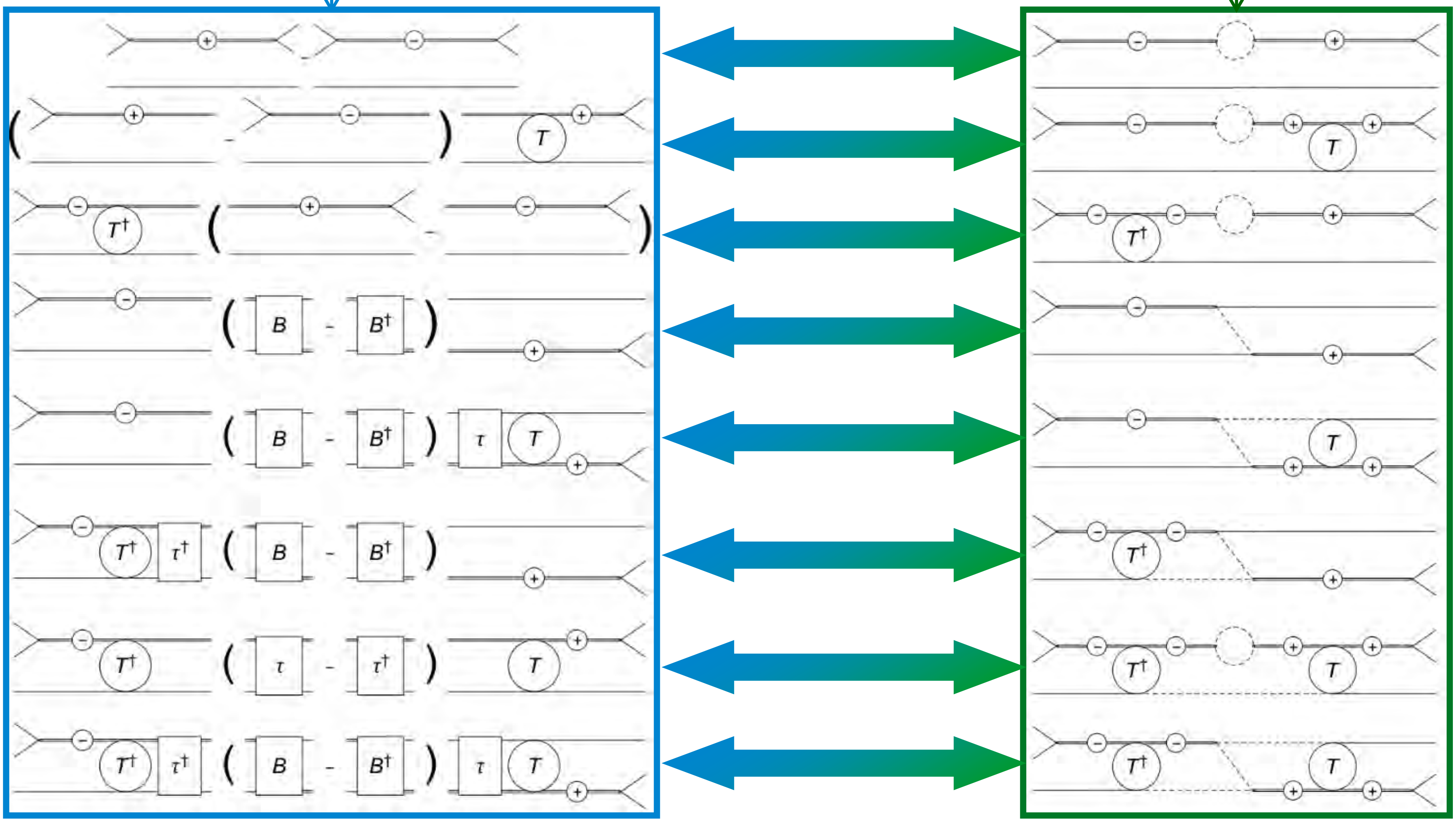


8 topologies



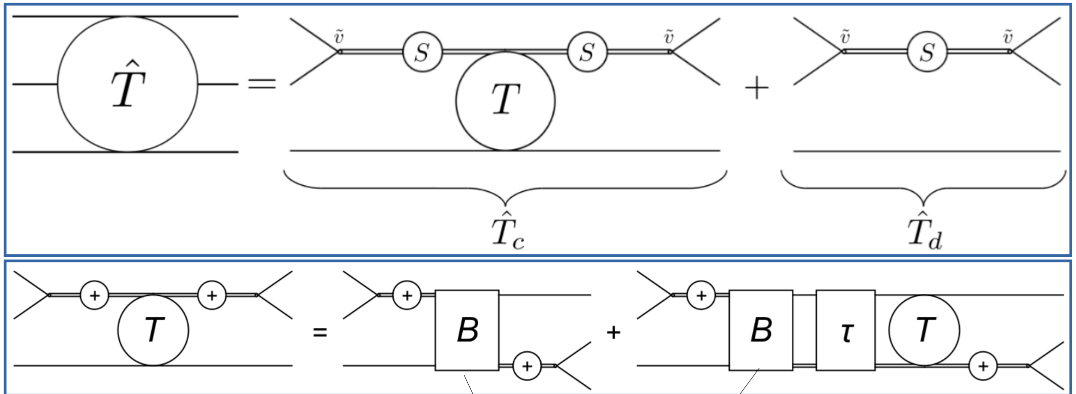
Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$



Scattering amplitude (1)

$3 \rightarrow 3$ scattering amplitude is a 3-dimensional integral equation

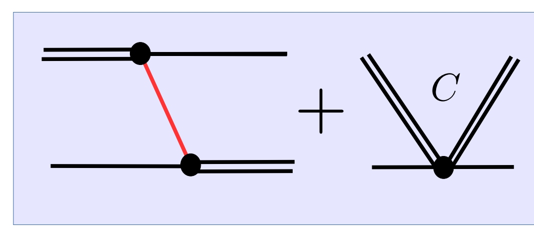


Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

$$\text{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$



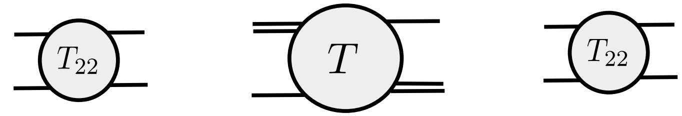
- one-π exchange in TOPT $\rightarrow$ *RESULT, NOT INPUT !*

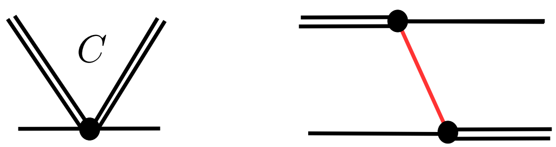
- One can map to field theory but does not have to. Result is a-priori dispersive.

Scattering amplitude (2)

Here: Version in which isobar rewritten in on-shell 2 → 2 scattering amplitude T_{22}

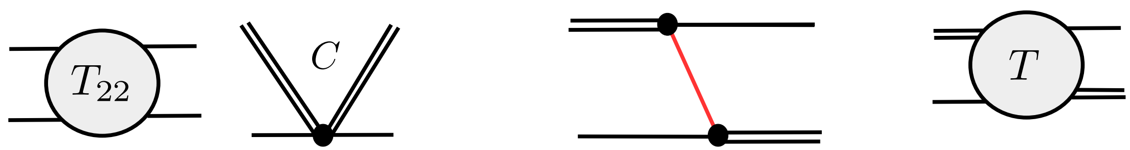
$$\langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle = \frac{1}{3!} \sum_{n=1}^{3} \sum_{m=1}^{3} T_{22}(\sigma(q_n)) \langle q_n | T(s) | p_m \rangle T_{22}(\sigma(p_m))$$





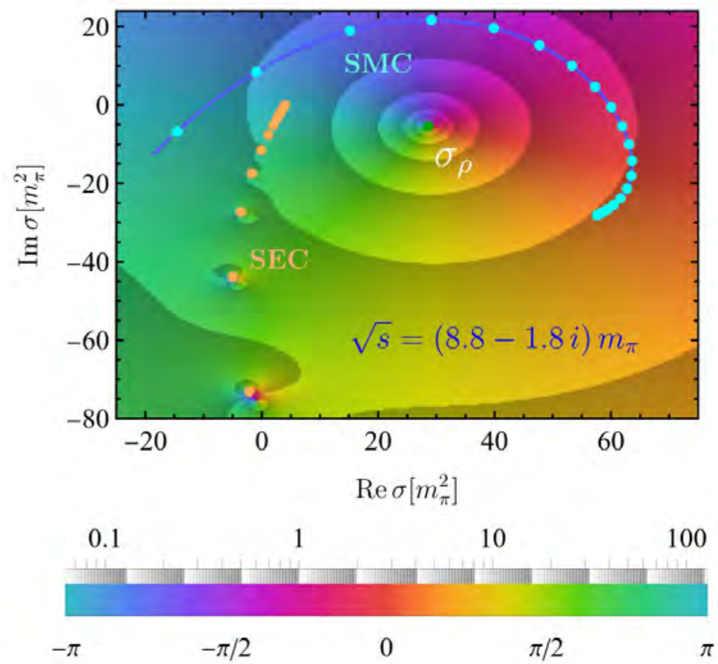
$$\langle q | T(s) | p \rangle = \langle q | C(s) | p \rangle + \frac{1}{m^2 - (P - p - q)^2 - i\epsilon}$$

$$- \int \frac{d^3 \boldsymbol{\ell}}{(2\pi)^3} \frac{1}{2E_\ell} T_{22}(\sigma(\ell)) \left(\langle p | C(s) | \ell \rangle + \frac{1}{m^2 - (P - p - \ell)^2 - i\epsilon} \right) \langle \ell | T(s) | p \rangle$$



(S-wave)

3.2 Analytic cont. 3B (1)



SMC

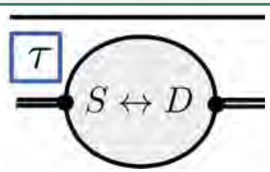
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\boxed{dl}\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right) \tau(\sigma(l)) T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

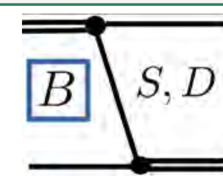
$$\Sigma = \int_0^\infty \frac{\boxed{dk}\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^* \tilde{v}(k)}{\boxed{\sigma - 4E_k^2 + i\epsilon}}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\, v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\boxed{\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}}$$

SEC



Singularities

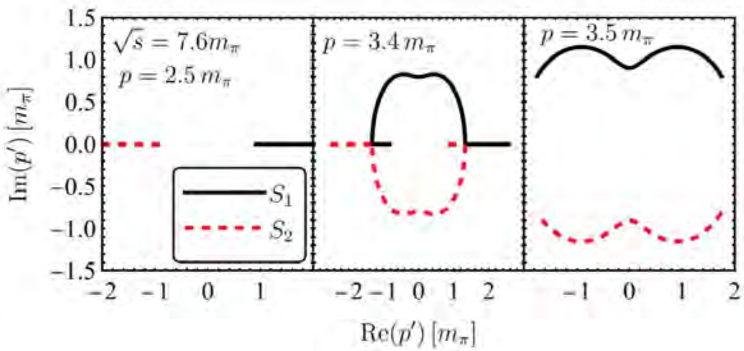


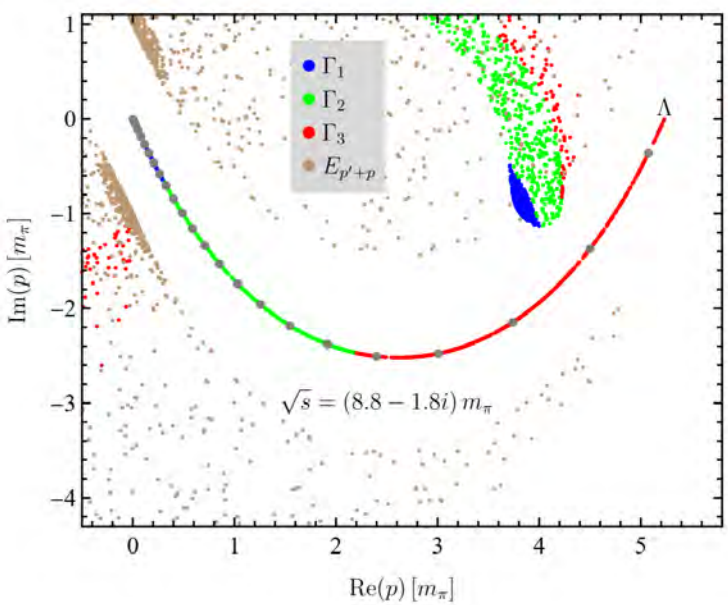
- Two contours (SMC and SEC)
- Deform both "adiabatically" to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

Analytic continuation 3B (2)

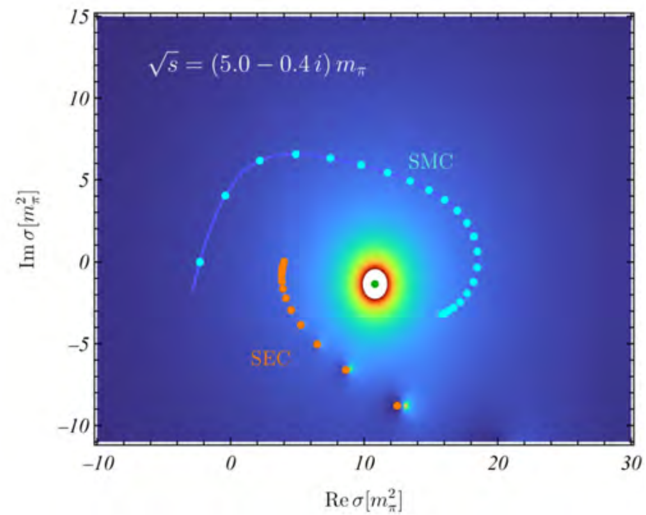
- Three-body cuts

$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$

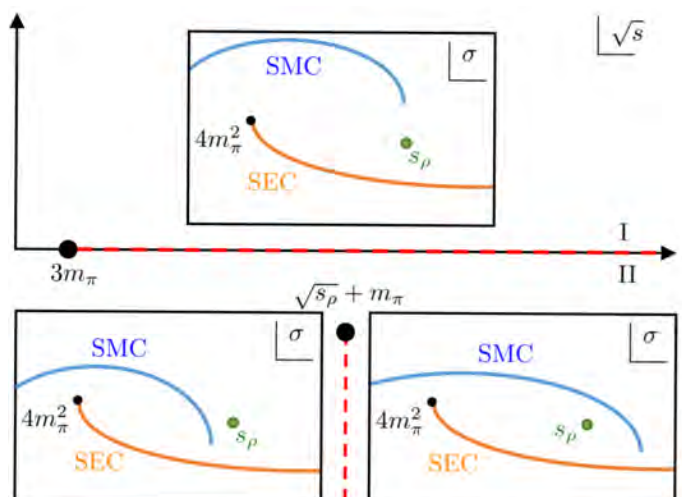




- Complex branch points

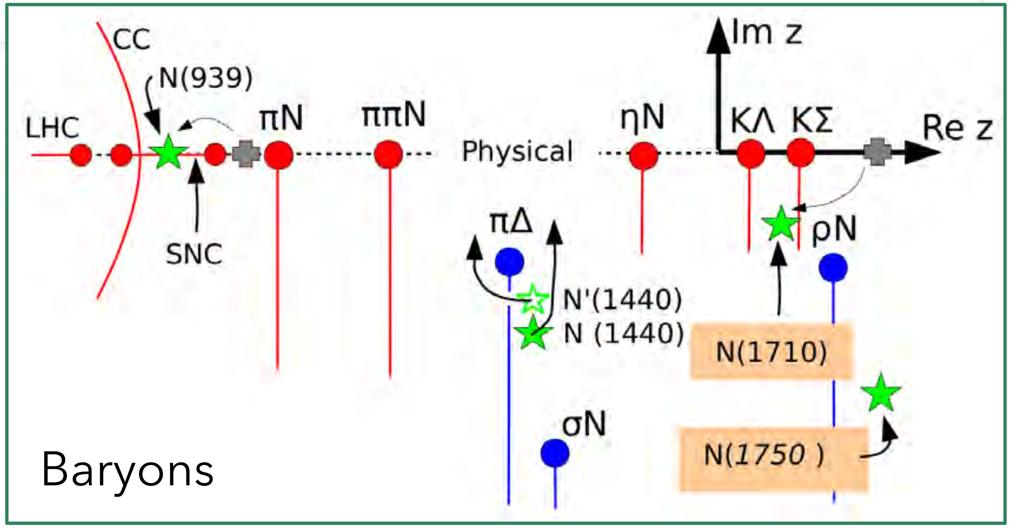


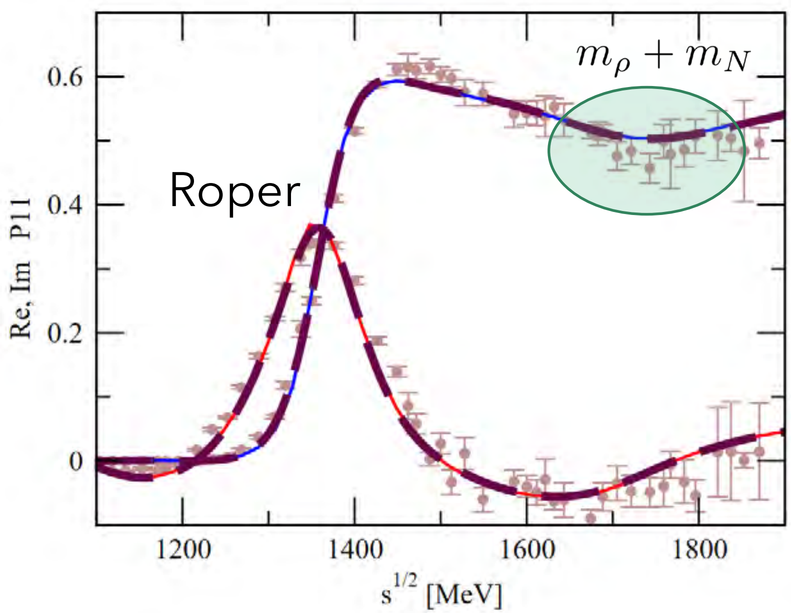
Integration limits at poles induce branch points

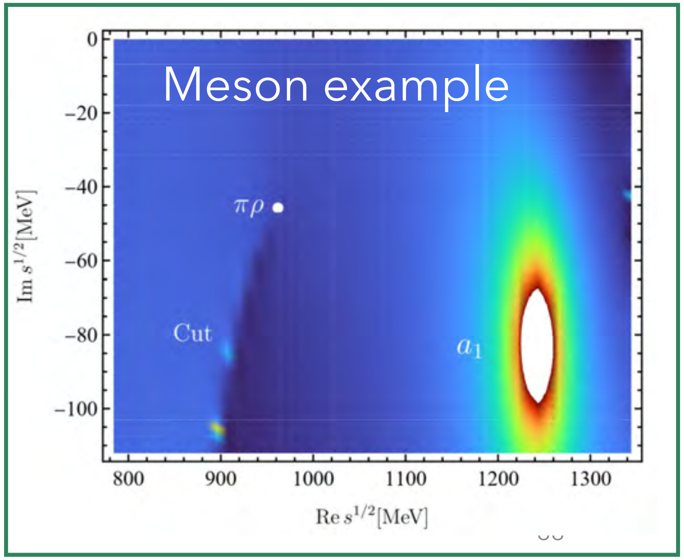


Analytic continuation 3B (3)

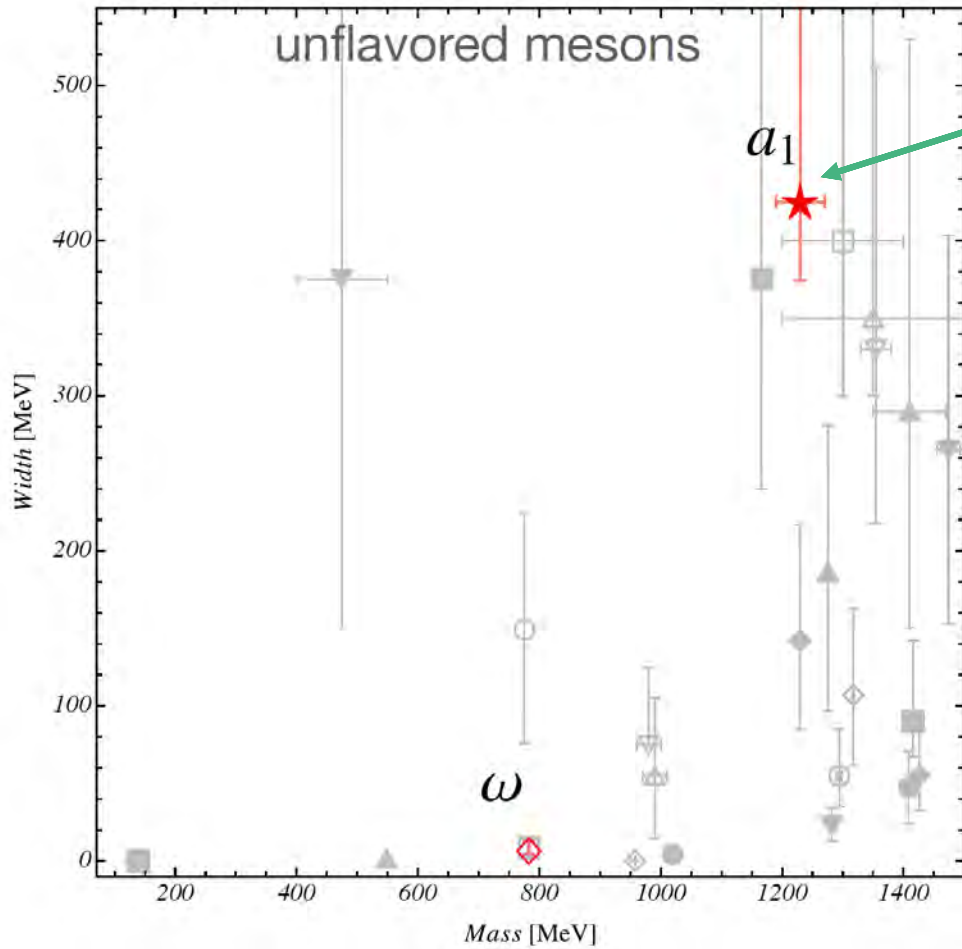
- Real and <u>complex</u> branch points ✓
- Poles appear doubled due to new Riemann sheets due to (complex) thresholds ✓
- Circular cut (CC) and short (nucleon) cut (SNC) exist only in partial waves
- Complex branch points can mimic resonances (ρN) [Ceci]







Light unflavored mesons



We concentrate on this resonance!
(because 3-body)

Huge body of work on 2-body coupled channel resonances from Lattice QCD (HadSpec collaboration, BGR group, Bonn group, …) [Briceno]

Exotic quantum numbers

- A $q\bar{q}$ pair cannot form all possible $(I^G)J^{PC}$ [Meyer]
  - Finding a meson with exotic quantum numbers reveals explicit gluon dynamics at low energies (exp. programs @ COMPASS, GlueX,…)
  - Exp. evidence for $\pi_1(1600)$ rather solid [PDG]
- Which are the allowed forbidden quantum numbers/naming?

Allowed

L	S	J^{PC}	L	S	J^{PC}	L	S	J^{PC}
0	0	0^{-+}	1	0	1^{+-}	2	0	2^{-+}
0	1	1^{--}	1	1	0^{++}	2	1	1^{--}
			1	1	1^{++}	2	1	2^{--}
			1	1	2^{++}	2	1	3^{--}

Some exotics ($J^{PC} = 1^{-+}, \ldots$)

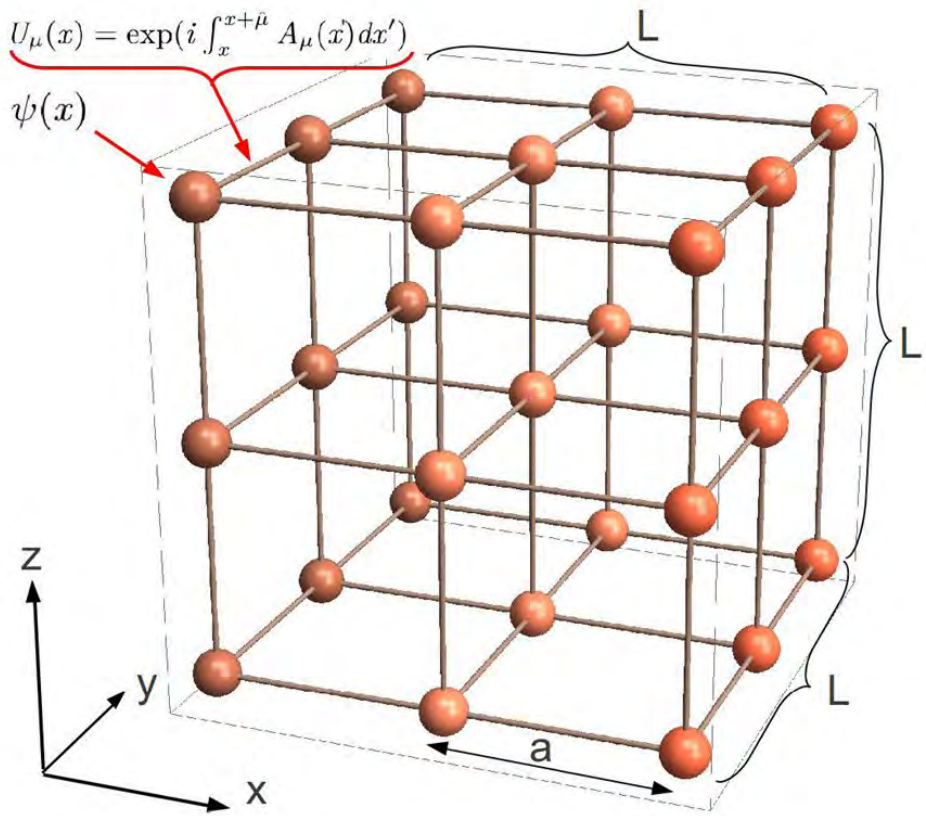
J^P	normal meson name	(I^G)	exotic meson name	(I^G)
0^+	a_0	(1^-)	b_0	(1^+)
1^-	ρ	(1^+)	π_1	(1^-)
2^+	a_2	(1^-)	b_2	(1^+)

[Meyer]

- How can we determine these quantum numbers?

$$P(q\bar{q}) = -(-1)^L \qquad C(q\bar{q}) = (-1)^{L+S} \qquad G(q\bar{q}) = (-1)^{L+S+I}$$

3.3 Scattering on a lattice



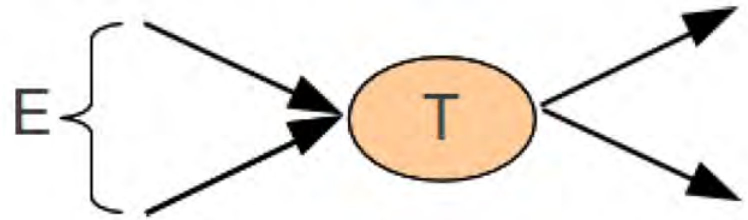
- Side length L,
  periodic boundary conditions
  $\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L)$
  $\rightarrow$ finite volume effects
  $\rightarrow$ Infinite volume $L \rightarrow \infty$
  extrapolation
- Lattice spacing a
  $\rightarrow$ finite size effects
  Modern lattice calculations:
  $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV
  $\rightarrow$ (much) larger than typical
  hadronic scales;

  not considered here.
- Unphysically large
  quark/hadron masses
  $\rightarrow$ (chiral) extrapolation
  required.

3.1 Two-body scattering & Lüscher equation

- Unitarity of the scattering matrix S: $SS^\dagger = \mathbb{1}$ $\qquad [S = \mathbb{1} - i\,\frac{p}{4\pi E}\,T]$.

$$\mathrm{Im}\,T^{-1}(E) = \sigma \equiv \frac{p}{8\pi\,E}$$



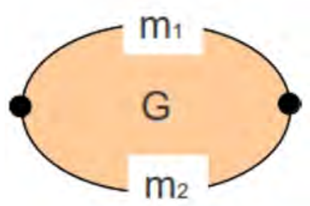
- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the T-matrix:

$$T = V + V\,G\,T \qquad \mathrm{Im}\,G = -\sigma$$

V: (Pseudo)potential, σ: phase space.

- G: Green's function:

$$G = \int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon},$$
$$\omega_{1,2}^2 = m_{1,2}^2 + \vec{q}^{\;2}$$

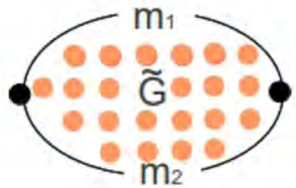


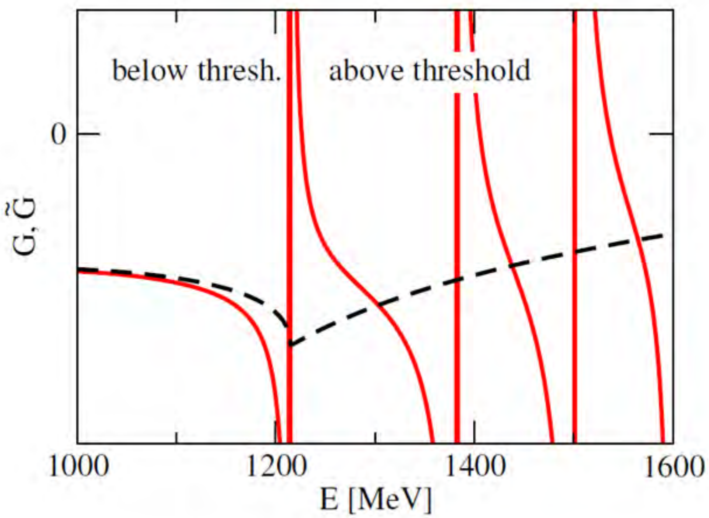
Periodic boundaries and discretization

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(\vec{x}) \implies q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\boxed{\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \to \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\,\vec{n}, \quad \vec{n} \in \mathbb{Z}^3}$$

$$G \to \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}$$





- $E > m_1 + m_2$: $\tilde{G}$ has poles at *free energies in the box*, $E = \omega_1 + \omega_2$
- $E < m_1 + m_2$: $\tilde{G} \to G$ exponentially with L (regular summation theorem).

The Lüscher equation

- Measured eigenvalues of the Hamiltonian (tower of *lattice levels* $E(L)$)
  $\to$ Poles of scattering equation $\tilde{T}$ in the finite volume $\to$ determines V:

$$\tilde{T} = (1 - V\tilde{G})^{-1}V \quad \to \quad V^{-1} - \tilde{G} \stackrel{!}{=} 0 \quad \to \quad V^{-1} = \tilde{G}$$

- The interaction V determines the T-matrix in the infinite volume limit:

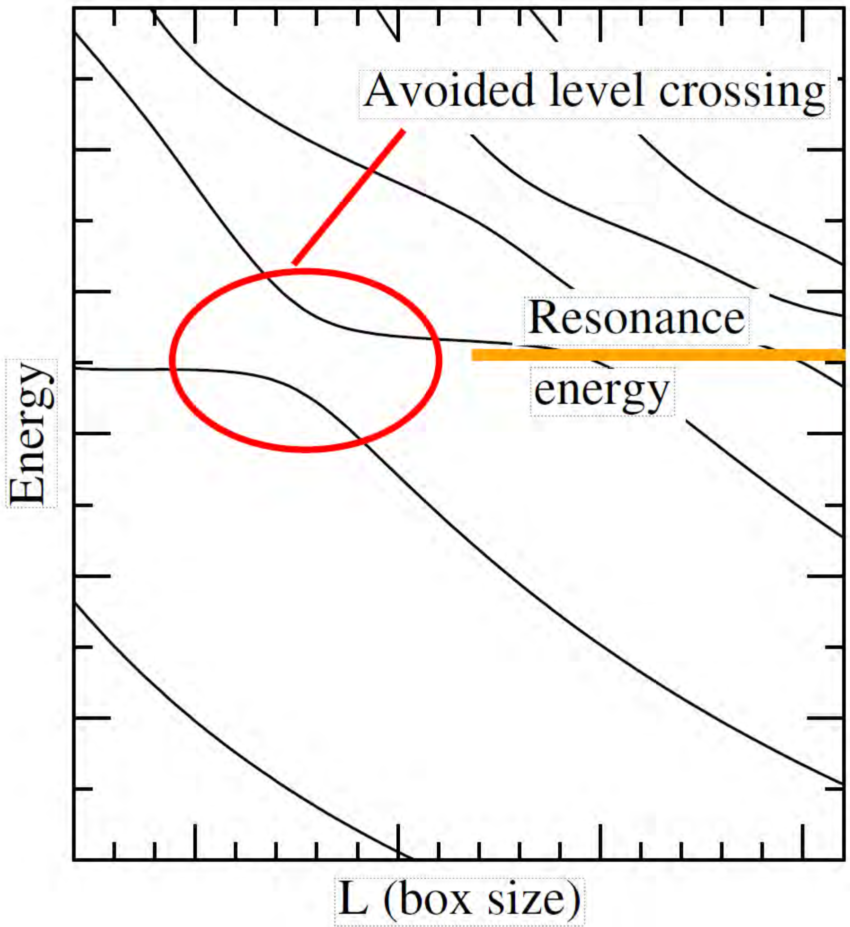
$$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher's equation (T determines the phase shift δ):

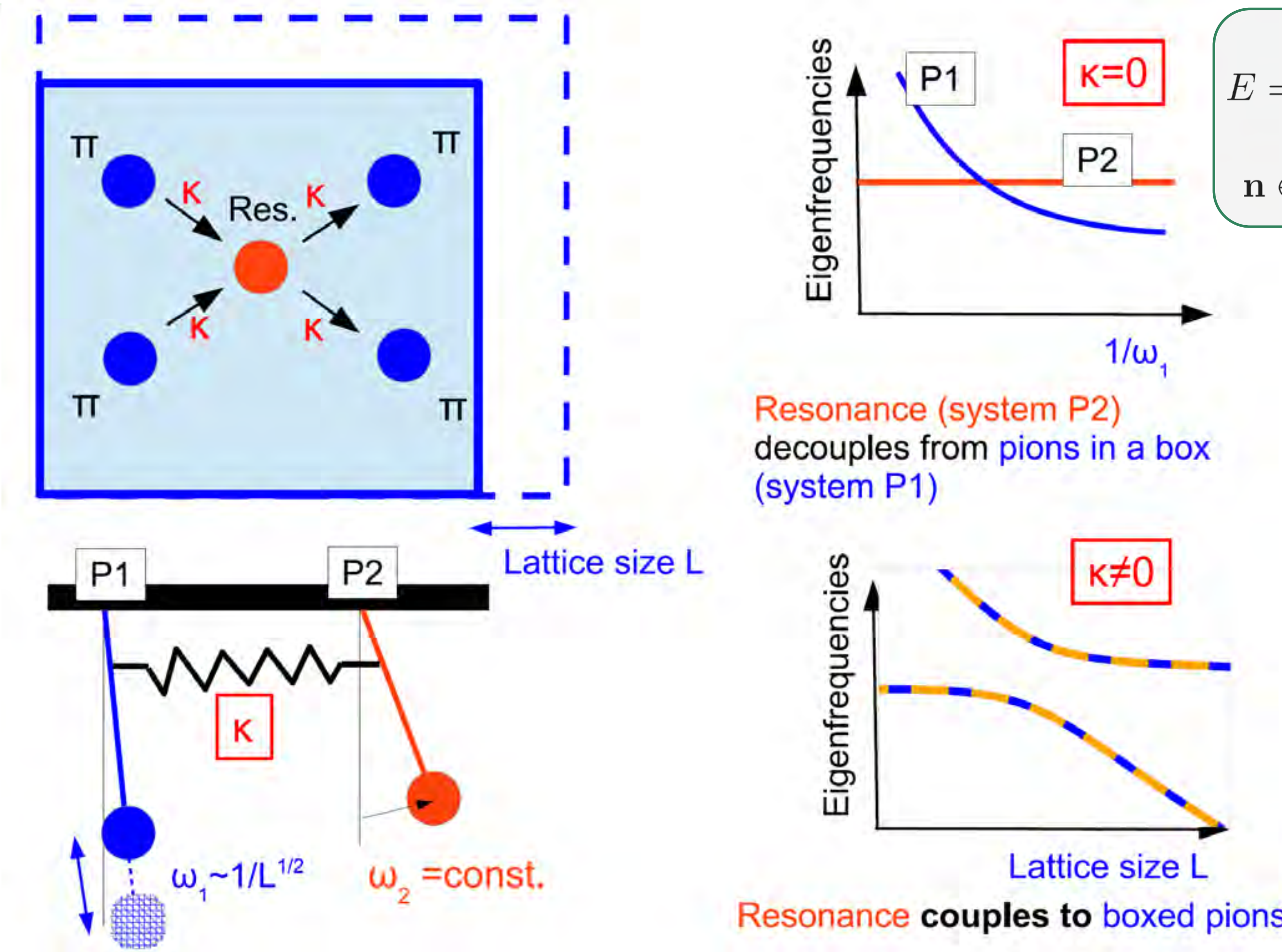
$$\boxed{p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

  - V and dependence on renormalization have disappeared (!)
  - p: c.m. momentum
  - E: scattering energy
  - $\tilde{G} - \mathrm{Re}G$: known kinematical function
    ($\simeq \mathcal{Z}_{00}$ up to exponentially suppressed contributions)
  - One phase at one energy.

2-body resonances in a box



An analogy for avoided level crossing

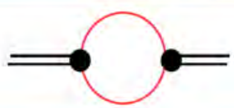


$$E = 2\sqrt{m_\pi^2 + \left(\frac{2\pi\mathbf{n}}{L}\right)^2}$$

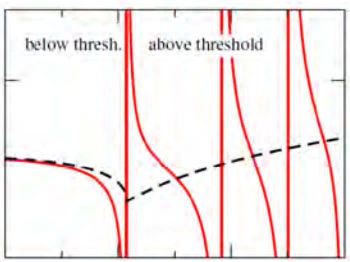
$$\mathbf{n} \in \mathbb{Z}^3$$

Three-body quantization condition

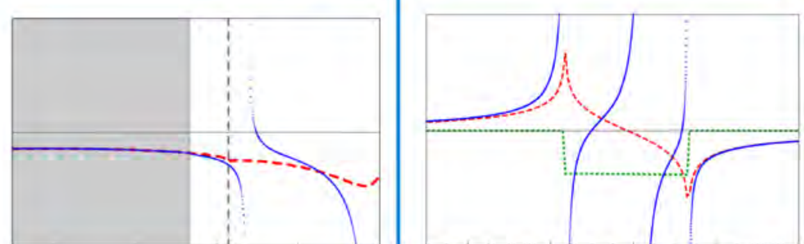
Two-body unitarity	Three-body unitarity
On-shell condition	On-shell condition



Imaginary parts	Imaginary parts



Infinite → Fin. Vol



Power-law fin-vol. effects	Power-law fin-vol. effects

Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

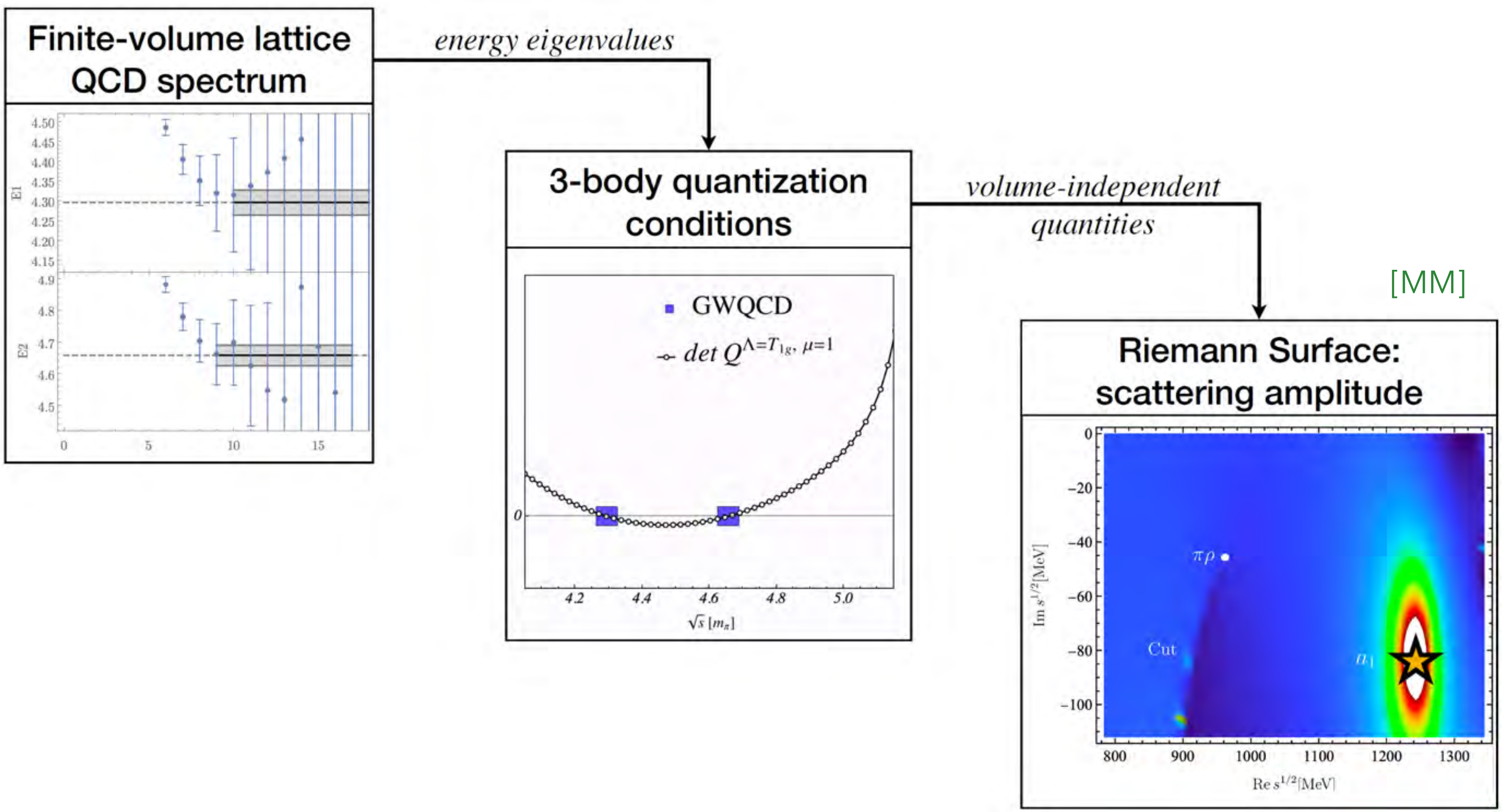
Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

[Review Mai et al.]

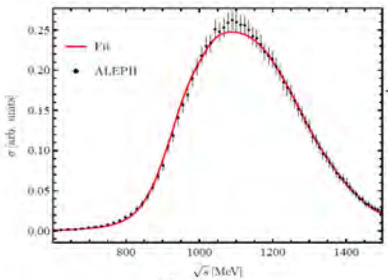
Extraction of $a_1(1260)$ from lQCD [Mai/GWQCD]

- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).

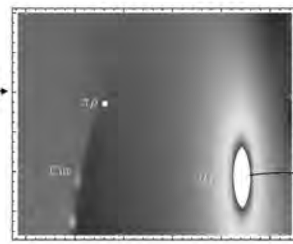


Extraction of $a_1(1260)$ from lQCD

[Mai/GWQCD]

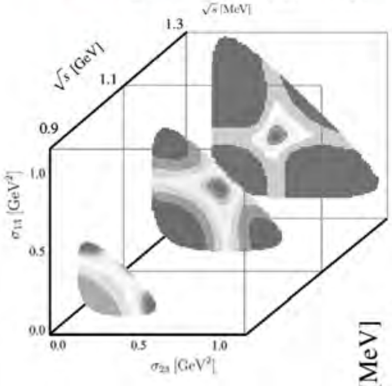


C

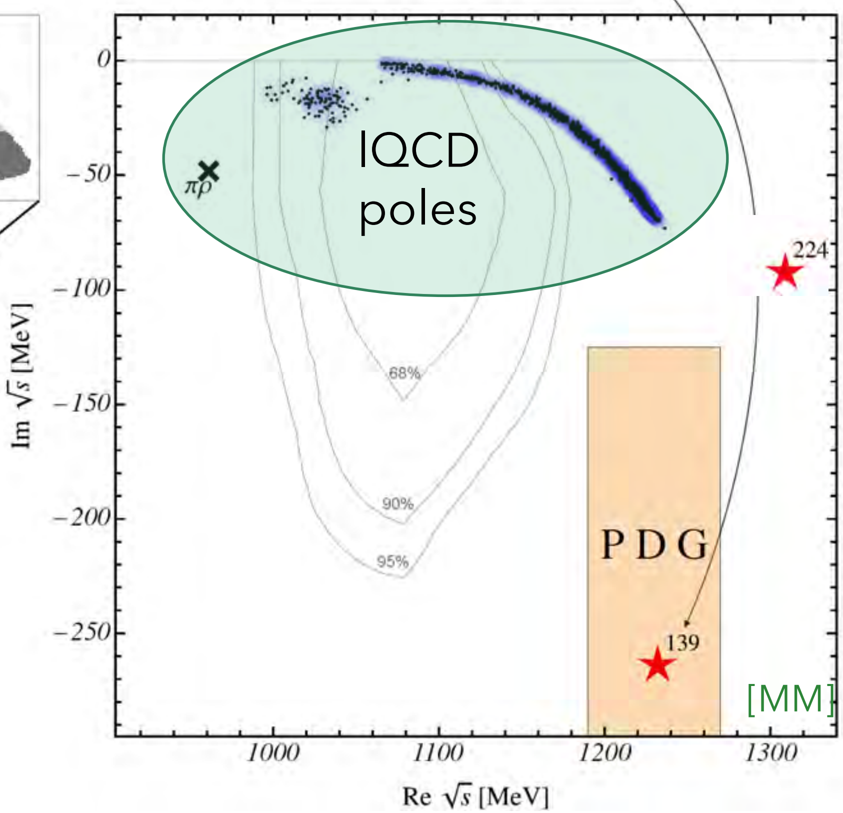


What does phenomenology says?

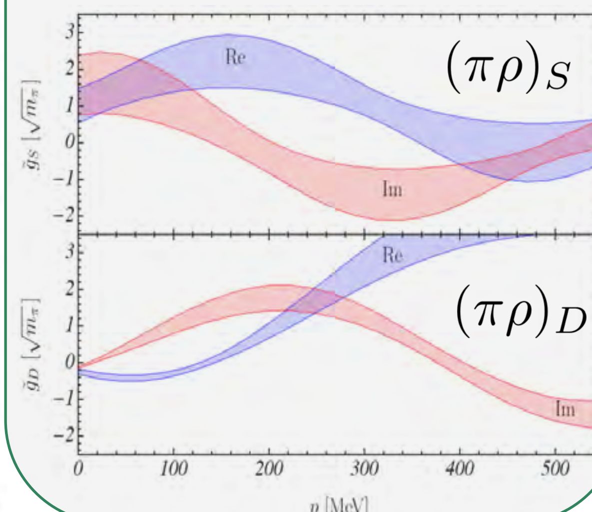
- $\tau \rightarrow (\pi\pi\pi)\nu_\tau$ from ALEPH@CERN
- fit to line shape to fix *C*



[Sadasivan]



"**Branching ratios**" in 3B decays are momentum – dependent, complex pole residues



Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]

Summary

- Resonances are not necessarily bumps in cross sections
- Bumps in cross sections are not necessarily resonances
  - Threshold cusps; complex branch points; triangle singularities, statistical fluctuations
- Quark models and some recent lQCD calculations predict more resonances than found in experiment
  - Large-scale experimental effort at JLab, Elsa, Mami,… together with pheno-analyses found convincing signals of many new states
  - Future: Ongoing efforts; new experiments (BGO-OD); electromagnetic properties through electroproduction reactions (Jlab 12-GeV upgrade)
- Lattice QCD progress in determining phase shifts, coupled-channels amplitudes, three-body non-resonant and resonant systems