Finite-Temperature Many-Body Perturbation Theory

and its

Application in *Nuclear Matter* Calculations

(an interacting Fermi fluid)

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Perturbation theory (PT) is a very common method in theoretical physics, where in general exact solutions are unavailable; PT allows (if applicable) to derive systematically improvable (up to a certain degree) approximations.

PT operates with splitting of Hamiltonian in (solvable 1body part) $\mathcal{T}$ and (2body/n-body part) $\mathcal{V}$; then the matrix elements of a given operator $\mathcal{F}$ give rise to series

$$\langle \Psi_{\mathcal{T}+\mathcal{V}} | \mathcal{F} | \Psi'_{\mathcal{T}+\mathcal{V}} \rangle \sim \langle \Psi_{\mathcal{T}} | \mathcal{F} | \Psi'_{\mathcal{T}} \rangle + \phi^{(1)}[\mathcal{V}\mathcal{F}; \{\Psi^i_{\mathcal{T}}\}] + \phi^{(2)}[\mathcal{V}\mathcal{V}\mathcal{F}; \{\Psi^i_{\mathcal{T}}\}] + \dots$$

where $\Psi^{(\prime)}_{\mathcal{T}}$ is an eigenstate and $\{\Psi^i_{\mathcal{T}}\}$ the spectrum of $\mathcal{T}$, and similar for $\Psi_{\mathcal{T}+\mathcal{V}}$

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MBPT: $\mathcal{N}\Psi \sim \infty\Psi$

One can distinguish:

- thermodynamic MBPT: perturbation theory for “EOS” (thermodynamic potentials)
- time-dependent MBPT: perturbation theory for dynamical quantities, usually considered in Fourier space (at finite T: discrete Fourier space (“Matsubara space”))

Using resummations or **self-concistency methods**, (**MB**)PT can provide also general (qualitative) information.

- Luttinger, PRC 1961: existence of a Fermi surface for (normal) interacting Fermi fluids & the low-lying elementary excitations have quasiparticle form (approximately)

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But: Luttinger, J.Math.Phys. 1963: "An exactly solvable model of a many-fermion system"; model of electrons in a one-dimensional conductor, solution (Mattis&Lieb 1963) generally has no Fermi surface ("Tomonaga-Luttinger liquid").

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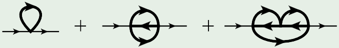
$\Sigma(\zeta_\ell) =$  $+ \ \dots$, $\equiv \mathcal{G}_r(\zeta_{\ell'}) = \int \frac{d\omega}{2\pi} \frac{A_r(\omega)}{\zeta_{\ell'} - \omega}$

Self-consistent perturbation series for the Matsubara self-energy $\Sigma_r(\zeta_\ell)$

Analytic continuation $\rightarrow$ Fourier-space self-energy $\Sigma_r(\omega) = K_r(\omega) + J_R(\omega)$, determines via Fourier transform the Green's function $iG_r^>(t-t') = \langle a_r(t) a_r^\dagger(t') \rangle_{t>t'}$

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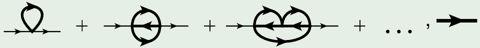
$\Sigma(\zeta_\ell) =$  $\equiv \mathcal{G}_r(\zeta_{\ell'}) = \int \frac{d\omega}{2\pi} \frac{A_r(\omega)}{\zeta_{\ell'} - \omega}$

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Luttinger, PRC 1961: $J_r(\omega)$ vanishes at $\omega = \mu$, where $\mu = \varepsilon_{k_F} + K_{k_F}(\mu)$

$$\Rightarrow \text{for } E_k \simeq E_{k_F}: \quad iG_r^>(t) \sim e^{-i\mathcal{E}_k t}\, e^{-\gamma t}$$

Low-lying excitations are (asymptotically) **quasiparticles** (at short time scales)

$\rightsquigarrow$ Framework of (dynamical) **Landau Fermi-liquid theory**

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Description of Effective Nuclear Interactions

Since 1960-1970: phenomenological models, ad hoc potentials fitted precisely to (scattering) observables

Since 1990-2005: (chiral) **effective field theory (EFT)**
Main Benefits: systematically improvable (to a certain degree), uncertainty quantification

EFT of Nuclear Interactions (Weinberg, van Kolck, Kaplan, Savage, Wise, Epelbaum, Kaiser, Machleidt, Meißner, Hammer, . . .)

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$\mathscr{L}_{\text{EFT}}(N, \pi, \Delta, \ldots)$, infinite hierarchy of interactions orded by **power counting**: $\sim (Q/M)^n$, with Q the low-energy scale ($\sim m_\pi$) and $M \sim 1$ GeV the breakdown scale

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Perturbative EFT: *strict application of power counting*

- **order-by-order renormalization**, **low-energy constants (LECs)** c_i, singular ($\Lambda \to \infty$) terms cancelled, remaining (regular) part of LECs c_i^{reg} scale $\mathcal{O}(1) + \mathcal{O}(1/\Lambda)$
- for a given cutoff Λ and truncation n the **LECs** $\tilde{c}_i^{\mathrm{reg}}(\Lambda)$ are fixed by
  - matching to low-energy expansion of fundamental theory (*not avaiable here*)
  - fitting to (two-, few- and many-body) data: low-energy bias, fit ambiguities!

  c_i^{reg} should be of **natural size** (with respect to power counting), inhibited by tunings!

Problem: large scattering lengths, bound-states require nonperturbative treatment!

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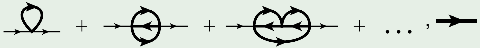
"Potential-EFT": (non)perturbative two-, few- and many-body calculations with nuclear potentials $V_{NN}(n,\Lambda,c_i), V_{3N}(n \geq 3,\Lambda,c_i), V_{4N}(n \geq 4,\Lambda,c_i),\ldots$

- $\boldsymbol{\Lambda \lesssim \Lambda_B}$ **(?)**; low-momentum potentials $\Lambda \lesssim 500$ MeV: $T_{\text{NN,med}}$ perturbative! $\rightsquigarrow$ MBPT
- LECs $c_i(n,\Lambda)$ should be of **natural size**

Chiral have been potentials applied with reasonable success in nuclear many-body physics!

$\Sigma(\zeta_\ell) =$



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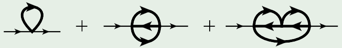
$\Sigma(\zeta_\ell) =$  $+ \ldots$, $\longrightarrow \equiv \mathcal{G}_r(\zeta_{\ell'}) = \int \frac{d\omega}{2\pi} \frac{A_r(\omega)}{\zeta_{\ell'} - \omega}$

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Alternative Approach: use 'bare' propagators, but improve reference point

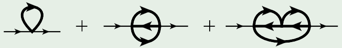
- Re-Partitioning of $\mathcal{H}$: $\quad \mathcal{H} = \mathcal{T}_{\text{kin}} + \mathcal{V}_{\text{int}} = \underbrace{(\mathcal{T}_{\text{kin}} + \mathcal{U})}_{\substack{\text{reference system } \mathcal{T} \\ \text{"mean-field theory"}}} + \underbrace{(\mathcal{V}_{\text{int}} - \mathcal{U})}_{\substack{\text{perturbation } \mathcal{V} \\ \text{"correlations"}}}$

with $\mathcal{U} = \sum_r U_r a_r^\dagger a_r$, where U_r is a self-consistent single-particle potential ("mean field")

- expansion of ensemble-averages in terms $\mathcal{V}'$ using as basis states $\Psi_{\mathcal{U}}$, where $\mathcal{H} = \mathcal{T} + \mathcal{V}'$.

Suitable choice of $\mathcal{U}$ improves perturbation series ('perturbativeness')

Usual choices: $U_r = 0$ or $U_r = \sum_i \bar{V}^{ir,ir} n_i = \Sigma_{1;r}$ (Hartree-Fock)

$\Sigma(\zeta_\ell) =$  $+ \ \ldots \ , \longrightarrow \equiv \mathcal{G}_r(\zeta_{\ell'}) = \int \frac{d\omega}{2\pi} \frac{A_r(\omega)}{\zeta_{\ell'} - \omega}$

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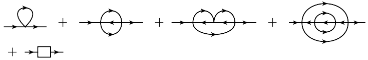
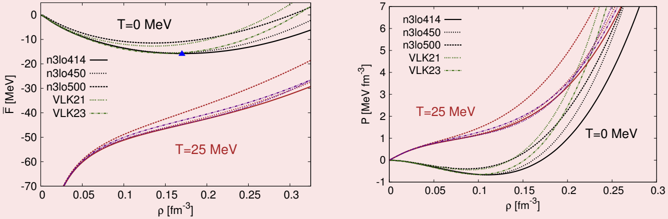
Homogeneous fluid (nuclear matter): eigenstates of $\mathcal{T}_{\text{kin}}$, $\mathcal{T}$: plane waves in a box L^3 in the limit $L \to \infty$, self-consistent equation for single-particle energies:

$$\boxed{\varepsilon_r = \frac{k^2}{2M} + U_r[n_r(\varepsilon_r)]}$$

$\Sigma(\zeta_\ell) =$  $+ \ \ldots \ , \longrightarrow \equiv \mathcal{G}^0_r(\zeta_{\ell'}) = \frac{1}{\zeta_{\ell'} - \omega}$

MBPT Application: Thermodynamics of Nuclear Matter

Isospin-symmetric nuclear matter: $\delta := (\rho_n - \rho_p)/\rho = 0$, $Y := \rho_p/\rho = 1/2$



- "empirical" saturation point: n3lo414, n3lo450, ~~n3lo500~~, ~~VLK21~~, VLK23
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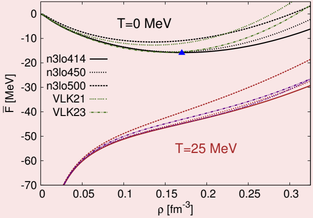

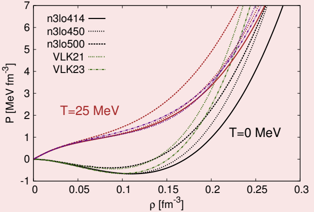

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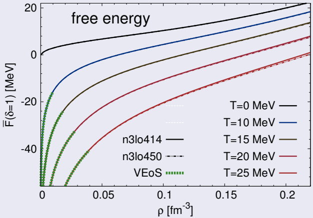
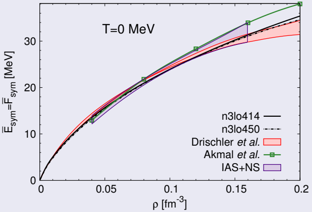
Pure neutron matter ($\delta = 1$, $Y = 0$) $\quad \bar{E}_{\text{sym}} := \bar{E}(\delta = 1) - \bar{E}(\delta = 0)$





Wellenhofer, Holt, Kaiser, Weise; PRC 89 (2014)

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MBPT: General Self-Consistent Single-Particle (SP) Potential

Self-consistent equation for single-particle energies

$$\boxed{\varepsilon_r = \frac{k^2}{2M} + U_r[n_r(\varepsilon_r)]}$$

Usual choices: $U_r = 0$ or $U_r = \sum_i \bar{V}^{ir,ir} n_i = \Sigma_{1;r}$ (Hartree-Fock)

[In the nuclear physics case, $\Sigma_{1;r}$ is sizeable, and the change from $U_r = 0$ or $U_r = \Sigma_{1;r}$ leads to considerable changes in the perturbation series (this feature is more pronounced in nuclear structure, cf. A. Tichai et al., Phys.Lett.B 756 (2016)).]

Generalization of self-consistent SP potential? $\rightarrow U_r := \sum_n U_{r;n}$

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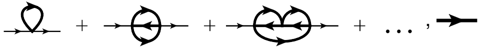
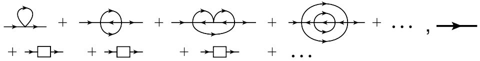
$\Sigma(\zeta_\ell) =$  $, \equiv \mathcal{G}_r(\zeta_{\ell'}) = \int \frac{d\omega}{2\pi} \frac{A_r(\omega)}{\zeta_{\ell'} - \omega}$

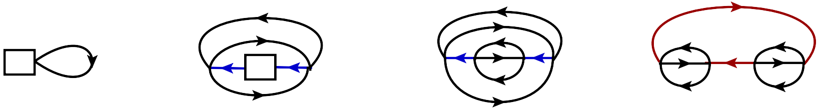
$\rightsquigarrow$ Dynamical quasiparticles at the Fermi surface

$\Sigma(\zeta_\ell) =$  $, \equiv \mathcal{G}^0_r(\zeta_{\ell'}) = \frac{1}{\zeta_{\ell'} - \omega}$

Keep dynamical quasiparticles at the Fermi surface: $\Rightarrow U_{n;k_F} \stackrel{!}{=} \tilde{K}_{n;k_F}(\mu)$ at $T = 0$, where $\tilde{K}_{n;k}$ is the diagrammatic contribution to the real part of the self-energy

MBPT: Anomalous Contributions, ‘Statistical’ Choice for $\mathcal{U}$

At fourth order and beyond, the perturbation series for the free energy F involves so-called ‘anomalous diagrams’:



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- $U^{(I)}_{n;r} = \mathrm{Re}\Big[\Sigma_{n;r}(\varepsilon_r - i\eta)\Big] = \dfrac{\delta \tilde{F}_n}{\delta n_r}\Big|_{r \notin \{\text{articulation lines}\}}$ satisfies $U_{n;k_F} \overset{!}{=} \tilde{K}_{n;k_F}(\mu)$

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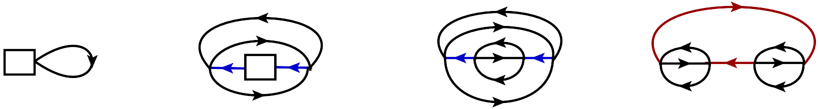

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- $U^{(I)}_{n;r} = \mathrm{Re}\big[\Sigma_{n;r}(\varepsilon_r - i\eta)\big] = \frac{\delta \tilde{F}_n}{\delta n_r}\Big|_{r \notin \{\text{articulation lines}\}}$ satisfies $U_{n;k_F} \overset{!}{=} \tilde{K}_{n;k_F}(\mu)$
- $U^{(II)}_{n;r} = \frac{\delta \mathscr{D}_n}{\delta n_r}$ does not satisfy $U_{n;k_F} \overset{!}{=} \tilde{K}_{n;k_F}(\mu)$

where $\mathscr{D}$ given by (reduced & disentangled & regularized) normal part of $\tilde{F}$

Elimination of Anomalous Diagrams via $U^{(II)}_{n;r}$:

$$F = F_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \tilde{F}_4 + \ldots + \tilde{F}_N - \sum_{n=1}^{N} \sum_r U_{n;r} n_r$$

$$\downarrow U_{n;r} = U^{(II)}_{n;r}$$

$$= \underbrace{F_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \tilde{F}_{4,\text{normal}} + \ldots + \tilde{F}_{N,\text{normal}}}_{\mu N + \Omega' + \mathscr{D}} - \sum_{n=1}^{N} \sum_r U^{(II)}_{n;r} n_r$$

Elimination of Anomalous Diagrams:

- first studied by Balian, Bloch, de Dominicis (var., 1958-1971), Kohn, Luttinger, Ward (PR 118 (1960)) as well as Horwitz, Brout, Englert (PR 120 (1961), PR 130 (1963))

  in particular: $\underbrace{\text{cancellation of anomalous contributions to } F \text{ as } T \to 0}_{\text{('Kohn-Luttinger-Ward theorem')}}$ ($\Rightarrow F \xrightarrow{T\to 0} E_0$)

MBPT: Dynamical vs. Statistical Quasiparticles

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- with $U^{(II)}_{n;r}$, the perturbation series truncated at order n leads to **statistical quasiparticle** relations for N, S and $\partial E/\partial n$:

$$N = \sum_r n_r \qquad S = -\sum_r n_r \ln n_r + \bar{n}_r \ln \bar{n}_r$$

$$E = F + TS = \sum_r \varepsilon_r^{\text{free}} n_r + \mathscr{D} \qquad \frac{\delta E}{\delta n_r} = \varepsilon_r^{\text{free}} + U_r^{(II)}$$

→ Sommerfeld expansion for interacting case (Constantinou, Muccioli, Prakash, Lattimer; Ann. Phys. 363 (2015))

Two aspects of Landau Fermi-Liquid theory: dynamical ($T = 0$) vs statistical ($\forall T$), with $\varepsilon_r^{\text{dynamical}} \neq \varepsilon_r^{\text{statistical}}$ for $n \geq 4$

cf. Carneiro & Pethick; PRB (1975), Pethick & Baym; "Landau Fermi-Liquid Theory"

MBPT Binary System: Asymmetry Expansion

Explicit parametrization via expansion about $\delta = 0$, where $\delta = (\rho_n - \rho_p)/\rho$

$$F(\delta) \sim \overbrace{F(\delta = 0) + A_2\,\delta^2}^{\gtrsim 99\% \text{ of literature}} + A_4\,\delta^4 + A_6\,\delta^6 + \ldots$$

'Usual' δ^2 approximation is good, but higher-order terms not negligible for neutron-rich systems ($\rightsquigarrow$ neutron stars)
$\rightarrow$ compute higher-order coefficients A_4, A_6

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Explicit parametrization via expansion about $\delta = 0$, where $\delta = (\rho_n - \rho_p)/\rho$

$$F(\delta) \sim \overbrace{F(\delta = 0) + A_2\,\delta^2}^{\gtrsim 99\% \text{ of literature}} + A_4\,\delta^4 + A_6\,\delta^6 + \ldots$$

'Usual' δ^2 approximation is good, but higher-order terms not negligible for neutron-rich systems ($\leadsto$ neutron stars)
$\rightarrow$ compute higher-order coefficients A_4, A_6

Higher-order coefficients A_4, A_6 are singular at zero temperature!

$$F_2(T = 0, \rho, \delta) = A_0(0, \rho) + A_2(0, \rho)\,\delta^2 + \sum_{n=2}^{\infty} A_{2n,\text{reg}}(\rho)\,\delta^{2n} + \sum_{n=2}^{\infty} A_{2n,\log}(\rho)\,\delta^{2n} \ln|\delta|$$

Kaiser; PRC 92 (2015)

Logarithmic terms also when ladders are resummed to all orders!

Kaiser; EPJA 48 (2014), Wellenhofer; arXiv:1707.09222

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What is the origin of the logarithmic terms at $T = 0$? What happens at finite T?

$\rightarrow$ energy denominators in contributions beyond first order, e.g.,

$$E_{0;2} = -\frac{1}{4}\sum_{ijab} \bar{V}_{\mathrm{NN}}^{ij,ab}\,\bar{V}_{\mathrm{NN}}^{ab,ij}\,\frac{\Theta_i^-\Theta_j^-\Theta_a^+\Theta_b^+}{\varepsilon_a+\varepsilon_b-\varepsilon_i-\varepsilon_j}$$

integrand diverges at integral boundary

$\rightsquigarrow E_{0;2} \in C^3$

$$F_2 = -\frac{1}{8}\sum_{ijab} \bar{V}_{\mathrm{NN}}^{ij,ab}\,\bar{V}_{\mathrm{NN}}^{ab,ij}\,\frac{\tilde{f}_i^-\tilde{f}_j^-\tilde{f}_a^+\tilde{f}_b^+ - \tilde{f}_i^+\tilde{f}_j^+\tilde{f}_a^-\tilde{f}_b^-}{\varepsilon_a+\varepsilon_b-\varepsilon_i-\varepsilon_j}$$

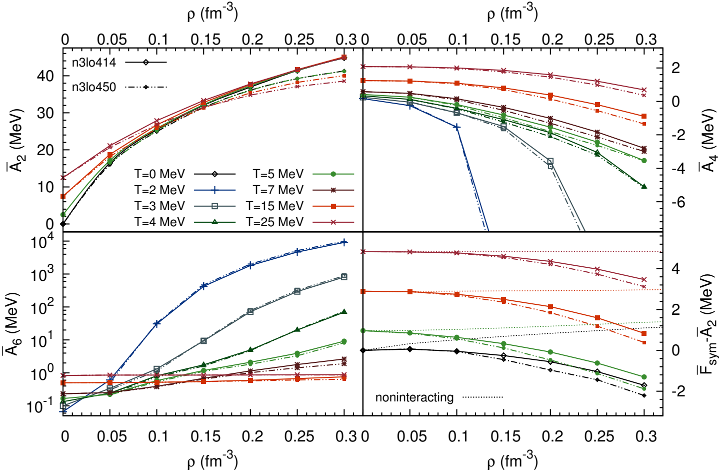
smooth integrand

$\rightsquigarrow F_2 \in C^\infty$, but not analytic ($C^\omega$) at low T/μ

Analytic structure and zero-temperature limit do not commute!

MBPT Binary System: Asymmetry Expansion (Nuclear Matter Results)

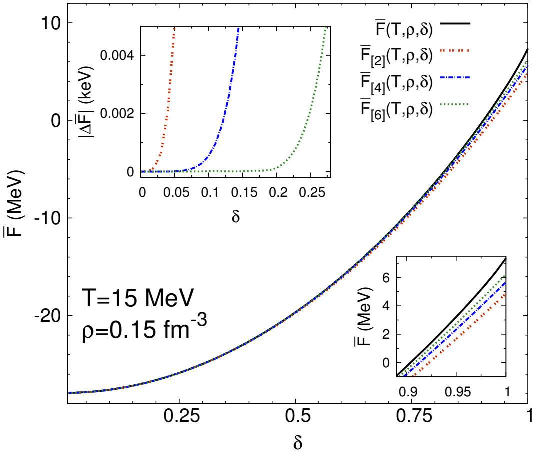
- $A_2 > A_4 > A_6 > \ldots$ at high $T\mu$, $A_2 \ll A_4 \ll A_6 \ll \ldots$ at low $T\mu$ ($A_{2n\geq 4} \xrightarrow{T\to 0} \pm\infty$)



Wellenhofer, Holt, Kaiser; PRC 93 (2016)

- **bottom-right:** accuracy of quadratic approximation governed by $F_{sym} - A_2$

MBPT Binary System: Asymmetry Expansion (**High** Temperature)

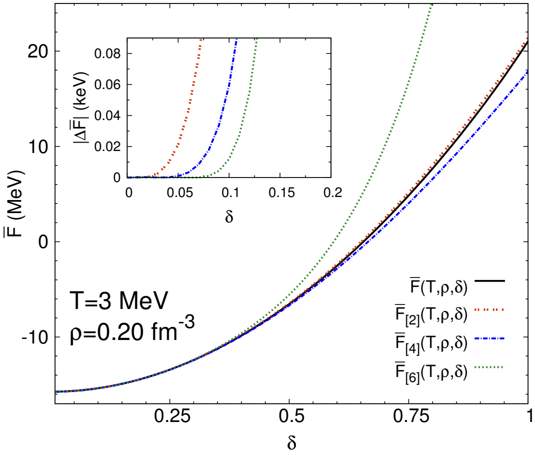


Wellenhofer, Holt, Kaiser; PRC 93 (2016)

Main Plot: Exact $F(T,\rho,\delta)$ vs different orders in the expansion $F_{2,4,6}(T,\rho,\delta)$

Insets: Deviation $\Delta F = F - F_{2,4,6}$

MBPT Binary System: Asymmetry Expansion (**Low** Temperature)



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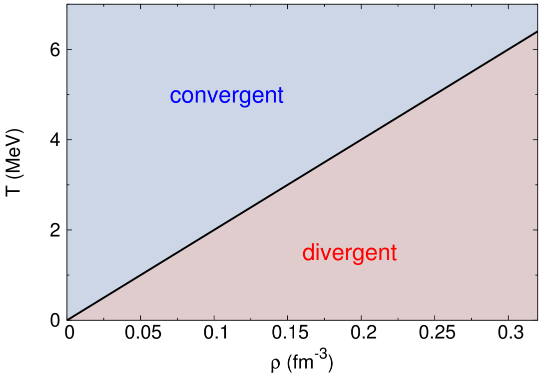
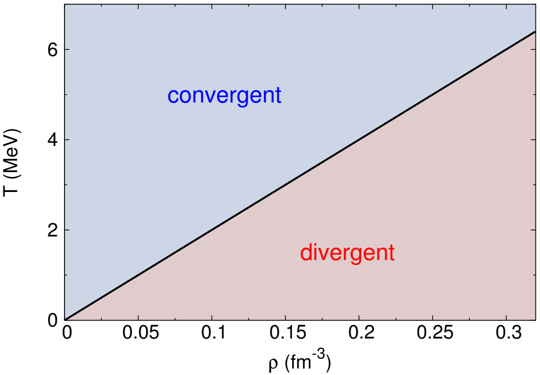
MBPT Binary System: Asymmetry Expansion (Summary)



Wellenhofer, Holt, Kaiser; PRC 93 (2016)

Question remains: is the nonanalyticity of the δ dependence at low T/μ a genuine feature of the EOS or **only a feature of MBPT?**

MBPT Binary System: Asymmetry Expansion (Summary)



Wellenhofer, Holt, Kaiser; PRC 93 (2016)

Question remains: is the nonanalyticity of the δ dependence at low T/μ a genuine feature of the EOS or **only a feature of MBPT?**

Notable: analytic structure (sign of certain derivatives) of δ dependence and $T \rightarrow 0$ limit do not commute

For the Y dependence, already in the EOS of a free Fermi gas: entropy-of-mixing term, in classical limit: $TY \ln Y$;

$T = 0$: $\sim Y^{1-2/D}$, for $D = 1, 2$ the EOS is even analytic in Y!

Summary: Finite-Temperature MBPT

A. Analytic Structure of MBPT and Expansion in Asymmetry δ

- in MBPT, the free energy is a **nonanalytic smooth** function of δ at low T/μ

Question: is this only an artifact of MBPT?

- analytic structure (behavior of higher-order derivatives) changed at $T = 0$

Summary: Finite-Temperature MBPT

A. Analytic Structure of MBPT and Expansion in Asymmetry δ

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B. Fermi-Liquid Theory and Self-Consistent SP Potential Beyond Hartree-Fock

General MBPT partitioning: $$\mathcal{H} = \mathcal{T}_{\text{kin}} + \mathcal{V} = \underbrace{(\mathcal{T}_{\text{kin}} + \mathcal{U})}_{\substack{\text{reference system} \\ \text{“mean-field theory”}}} + \underbrace{(\mathcal{V} - \mathcal{U})}_{\substack{\text{perturbation} \\ \text{“correlations”}}}$$

- Dynamical-Quasiparticle Constraint $\Rightarrow \mathcal{U} = \mathcal{U}^{(I)}$
- Statistical Quasiparticles for $\mathcal{U} = \mathcal{U}^{(II)}$

Corollary to B. Grand-Canonical vs. Canonical($\ni$zero-temperature) MBPT

- “ensemble equivalence” only if $\mathcal{U}^{(II)}$ same order as perturbation series
- **in general:** $\Omega(T,\mu) \not\rightleftarrows F(T,\rho)$ (asymptotic series)
  canonical series has better convergence properties! (for nuclear matter, grand-canonical MBPT fails qualitatively for $U_r = 0$)

A. Grand-Canonical vs. Canonical Formulation

MBPT truncated at order n

- Ensemble equivalence $\Leftrightarrow$ $\mathcal{U} = \mathcal{U}_m^{(II)}$ with $m = n$
- What happens for $m \neq n$?

Grand-Canonical Case: $\Omega(T,\mu)$

- $\mathcal{U} = 0$: $\Omega(T,\mu)$ is single-valued for $\mathcal{U} = 0 \Rightarrow$ no liquid-gas instability!
- $\mathcal{U} = 0 \rightarrow \mathcal{U} = \mathcal{U}_m^{(II)}$: renormalization of $\{\varepsilon_r\}$ in distribution functions;
  M^* approximation: $k^2/(2M) \rightarrow k^2/(2M^*) + \Delta\varepsilon$

$$f_k(T,\mu)^{-1} = 1 + \exp\left[T\Big(\frac{k^2}{2M} - \mu\Big)\right] \rightarrow 1 + \exp\left[T\Big(\frac{k^2}{2M^*} + \Delta\varepsilon - \mu\Big)\right]$$

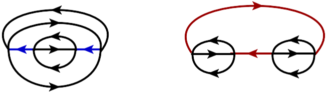
Canonical Case: $F(T,\rho)$

- ensemble averages evaluated via Legendre transform: $F(T,\rho) \rightarrow F(T,\tilde{\mu})$, where $\sum_r f_r(T,\tilde{\mu}) = \rho$
- $\mathcal{U} = 0 \rightarrow \mathcal{U} = \mathcal{U}_m^{(II)}$: $k^2/(2M) \rightarrow k^2/(2M^*) + \Delta\varepsilon$ **and** $\tilde{\mu} \rightarrow \tilde{\mu}' + \Delta\varepsilon$

$$f_r(T,\tilde{\mu})^{-1} = 1 + \exp\left[T\Big(\frac{k^2}{2M} - \tilde{\mu}\Big)\right] \rightarrow 1 + \exp\left[T\Big(\frac{k^2}{2M^*} - \tilde{\mu}'\Big)\right]$$

Effect of 'renormalization' of $\mathcal{T}$ has reduced effect in canonical case $\rightarrow$ better convergence properties!

B. “Reduced & Disentangled & Regularized”



(a) (b)

These two diagrams are “entangled” (→ cyclic permutations):

$$F^{\rm cyclic}_{a+b} = -\frac{1}{4} \sum_{xjaklmn} n_{xxja}\,\bar{n}_{klmn}\,[V] \left\{ \frac{1}{\varepsilon_1^2(\varepsilon_1+\varepsilon_2)} - \frac{e^{-(\varepsilon_1+\varepsilon_2)/T}}{\varepsilon_2^2(\varepsilon_1+\varepsilon_2)} + \frac{e^{-\varepsilon_1/T}(-\beta\varepsilon_1\varepsilon_2+\varepsilon_1-\varepsilon_2)}{\varepsilon_1^2\varepsilon_2^2} \right\}$$

No energy-denominator poles (*good!*), “double-indices” mix normal-anomalous (*not so good*)

- **“reduced”**: normal diagrams similar to $T = 0$ formalism (note: $F_a^{\rm reduced} = \infty$ for $T \neq 0$)

$$F_a^{\rm reduced} = -\frac{1}{4} \sum_{xjaklmn} n_{xxja}\,\bar{n}_{klmn}\,[V] \left\{ \frac{1}{\varepsilon_1^2(\varepsilon_1+\varepsilon_2)} \right\} \xrightarrow{T\to 0} E_a$$

- **“disentangled”**: normal without “double-indices”, anomalous factorized!!
  (but still $F_a^{\rm reduced,disentangled} = \infty$ for $T \neq 0$)

$$F_a^{\rm reduced,disentangled} = -\frac{1}{4} \sum_{xjaklmn} n_{xja}\,\bar{n}_{klmn}\,[V] \left\{ \frac{1}{\varepsilon_1^2(\varepsilon_1+\varepsilon_2)} \right\} \xrightarrow{T\to 0 \text{ formally}} E_a$$

$$F_b^{\rm reduced,disentangled} = -\frac{1}{4} \sum_{xjaklmn} n_{xakl}\,\bar{n}_{xjmn}\,[V] \left\{ \frac{-\beta}{\varepsilon_1\varepsilon_2} \right\} = -\beta \sum_x \frac{\delta F_2}{\delta n_x}\, n_x \bar{n}_x \underbrace{\frac{\delta F_2}{\delta n_x}}_{U_{2,x}}$$

- **“regularized”**: finite part $\mathcal{P}$ **plus** cyclic permutations of integration order (poles!)

$$F_a^{\rm reduced,disentangled,regularized} = -\frac{1}{4} \frac{1}{|C[xjaklmn]|} \sum_{C[xjaklmn]} n_{xja}\,\bar{n}_{klmn}\,[V] \left\{ \frac{\mathcal{P}}{\varepsilon_1^2(\varepsilon_1+\varepsilon_2)} \right\} \xrightarrow{T\to 0} E_a$$