# Charge Densities from Electron Scattering

achievements for stable nuclei, challenges for unstable nuclei

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## Purpose of talk

discuss relation data $\leftrightarrow$ densities
address limitations
particularly with respect to low-L measurements

## Start with historical perspective on extracting $\rho(r)$

### 1. Fifties

first low-$q$ experiments
interpreted in terms of PWBA

$$\sigma(E,\theta) = \sigma_{Mott}\ F^2(q)$$

momentum transfer $q \sim 2\ E\ sin(\theta/2)$

$\sigma_{Mott}$ = cross section for point-nucleus

$$F(q) = \int_0^\infty \rho(r)\ \frac{sin(qr)}{qr}\ r^2 dr$$

Form factor = Fourier-transform of density

Soon first calculations of $\sigma$ via solution of Dirac equation

accounts for distortion of e-waves

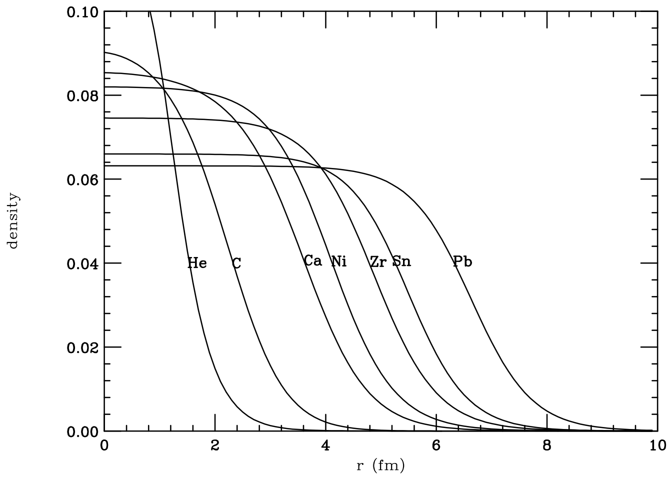
makes relation $\sigma \leftrightarrow \rho(r)$ less transparent

Determination of model densities

3PF $\rho(r) = (1 + w(r/c)^2)/(1 + e^{(r-c)/z})$

3PG $\rho(r) = (1 + w(r/c)^2)/(1 + e^{(r^2-c^2)/z^2})$

Establish general evolution of $\rho(r)$ with Z, A



## Integral properties of interest

– half-density radius $c$

strongly correlated with rms-radius $R$

– surface thickness $z$

not too correlated with $c$

– central depression $w$

problematic, discuss below

## Knowhow for determination of $R$

largely lost since

re-activated + improved with recent problems with proton $R$

new insights (see below)

## 2. Sixties

higher-$q$ data (mainly from HEPL)

more complicated shapes of $\rho$

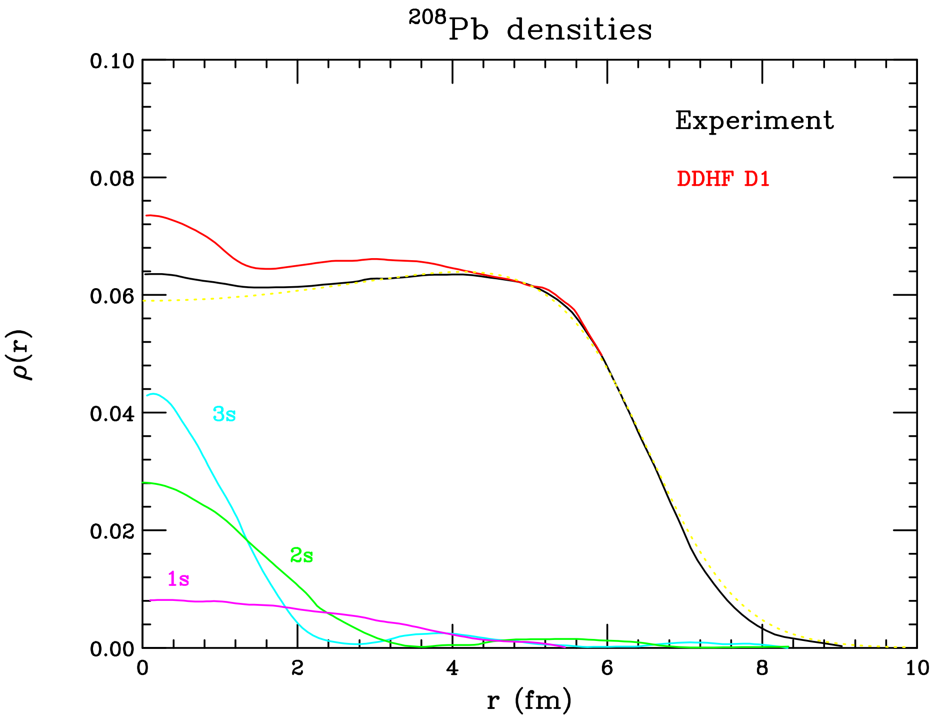
Fermi/Gauss + modifications

not always clear if given feature significant

or very model-dependent

example: $w$

## Example: $^{208}Pb$: central peak/depression [1]




**3PG parameterization yields large $w \sim 0.33$**
**$\rightarrow$ central depression?**

**SOG (model-independent density) does not really show it**

## 3. Seventies

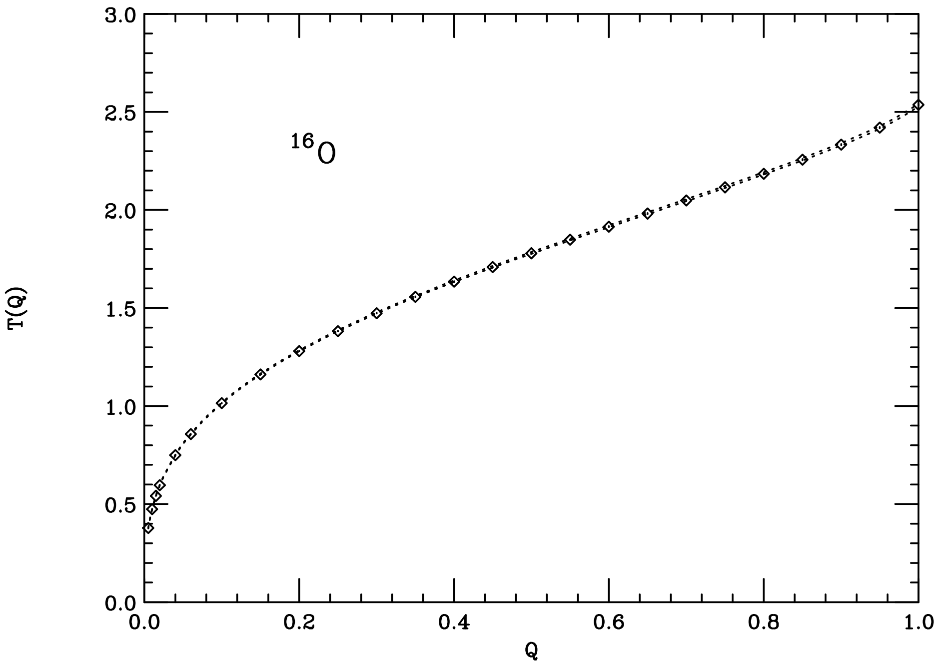
**Extraction of model-independent information**

possible only for *integral* quantities due to finite $q_{max}$

**Partial moment function (F. Lenz [2])**

$T(Q) = \int_0^Q r(Q')dQ'$ with $Q$=fraction of charge out to radius $r$

$T(Q)/Q$ = linear moment averaged over charge $Q$




**Strictly model-independent, welldefined $\delta T$, not intuitive, never caught on**

## Less model-dependent densities

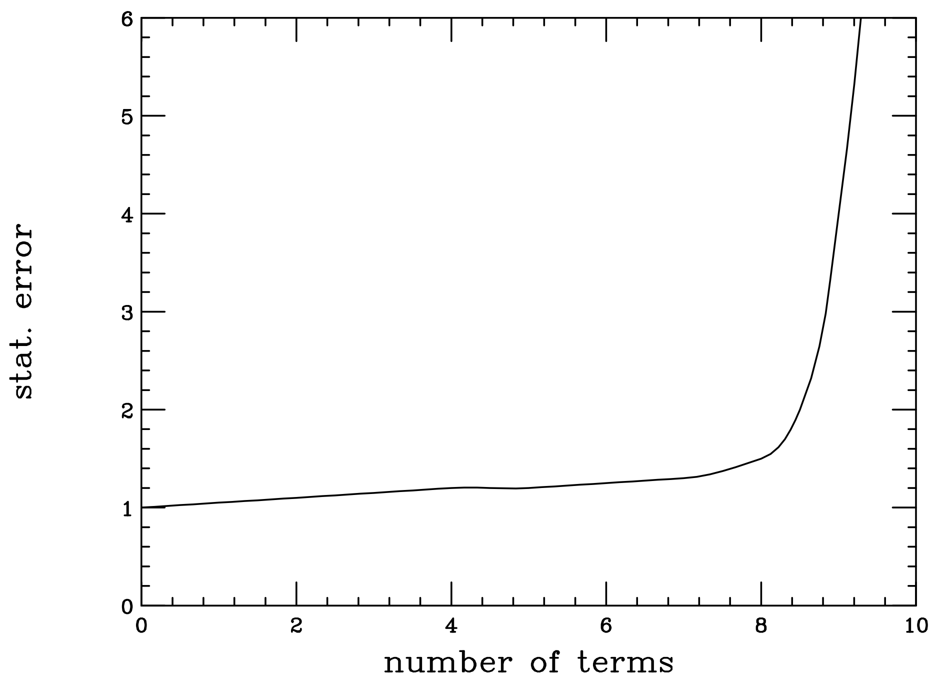
attempts to overcome model dependence

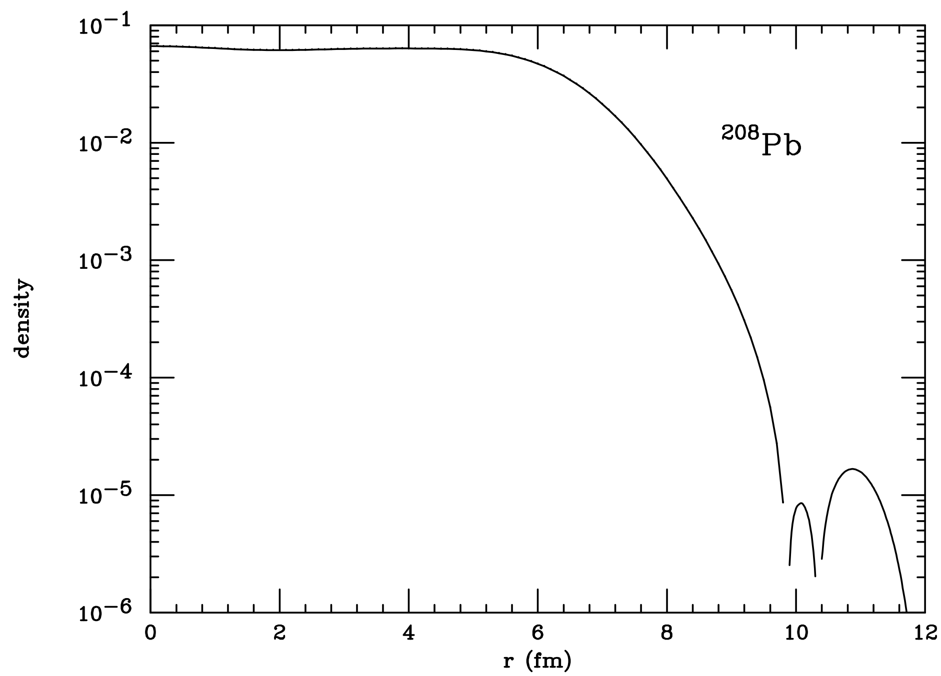
practical once $q_{max} \sim 3fm^{-1}$

expansion of $\rho(r)$ in complete basis

typically Fourier-Bessel

but: $\delta\rho(r)$ arbitrary (function of N) unless constrain $F(q > q_{max})$




to get significant $\delta\rho(r)$ must introduce physics constraint

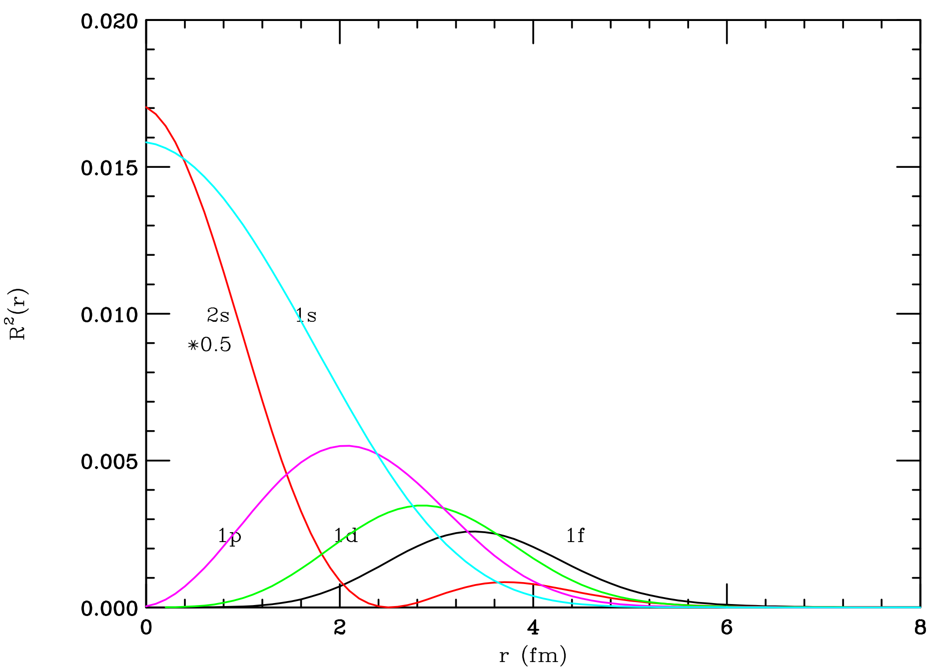
FB misbehaves at $r \sim r_{max}$, detrimental for determination of R (see below)

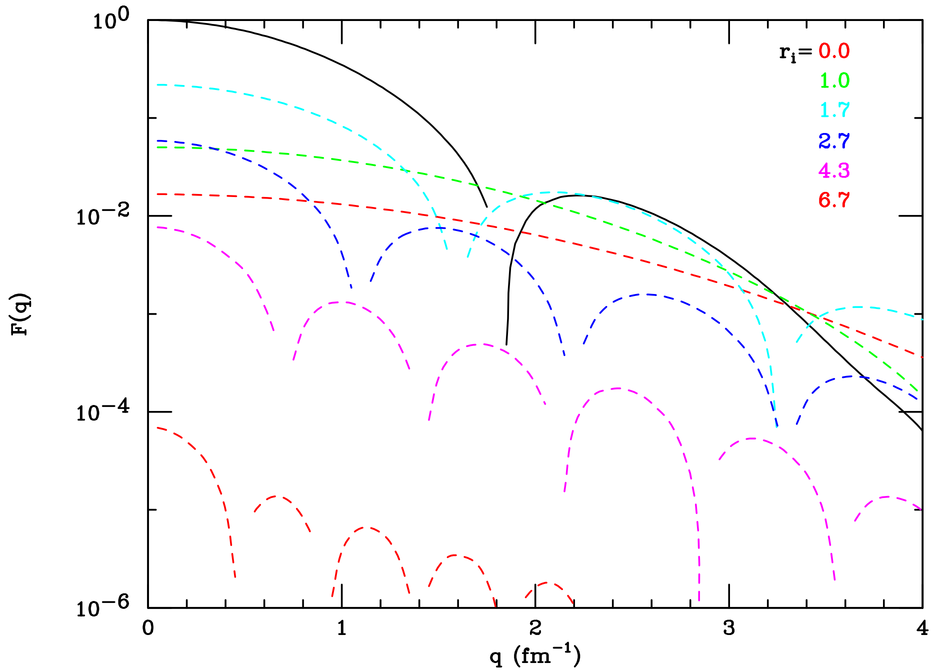
## Physics constraint: minimal width $\Gamma$ of peaks in $\rho$ [3]

smoothly varying $R()$, peaks of minimal width $\Gamma$
property of solutions of Schrödinger equation: minimal curvature, $\langle E \rangle$

Parameterize $\rho$ as sum of *many* gaussians SOG with width $\Gamma$, placed at radii $r_i$

## Example: $^{12}C$

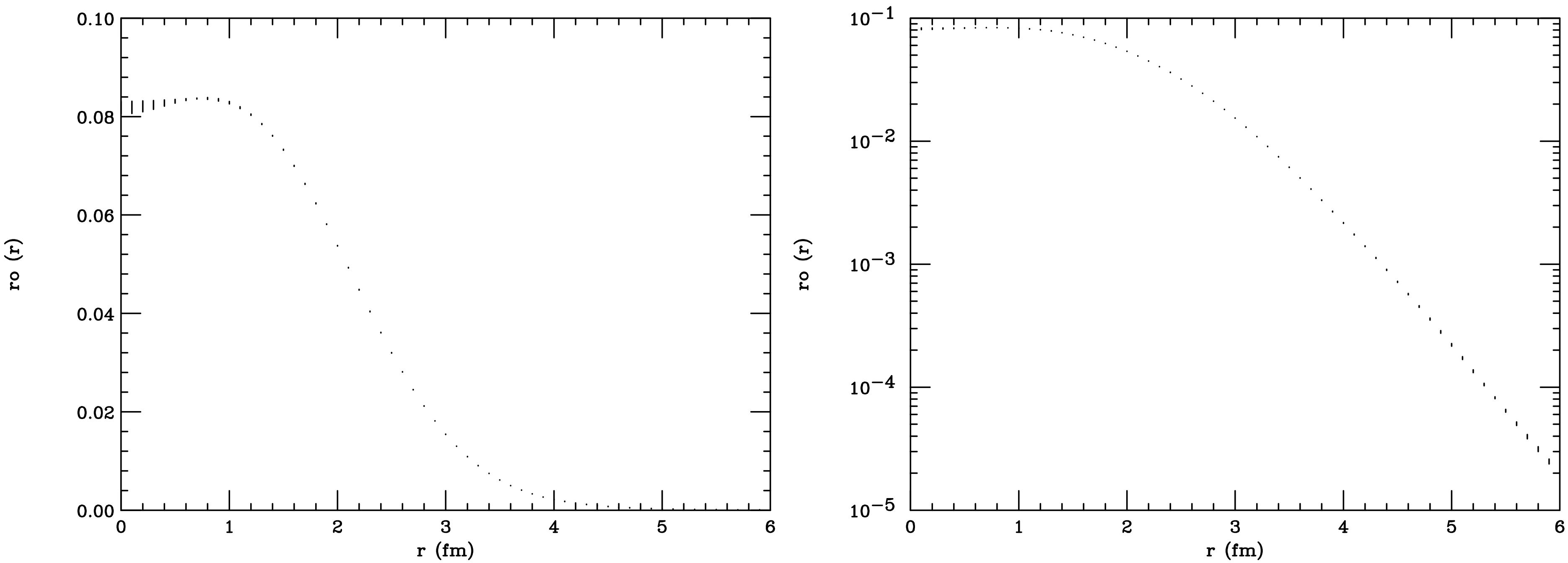



**With physics constraint get significant $\delta\rho(r)$**

note: large $r \leftrightarrow$ high-frequency oscillations in $q$

## Get density with significant error bar: $^{12}C$




## Note (for later)

**$\delta\rho$ in tail small (accurate extensive data)**

**displays typical exponential behavior**

$$R(r) \sim e^{-\kappa r}/r$$

## 4. Eighties: most transparent extraction of $\rho$: DFT [4]

form factor $F(q)$ = Fourier transform of $\rho(\mathrm{r})$

density $\rho(\mathrm{r})$ = Fourier transform of $F(q)$

$$\rho(r) = ... \int_0^\infty F(q)\frac{sin(qr)}{qr}q^2 dq$$

Can exploit? Potential difficulties:

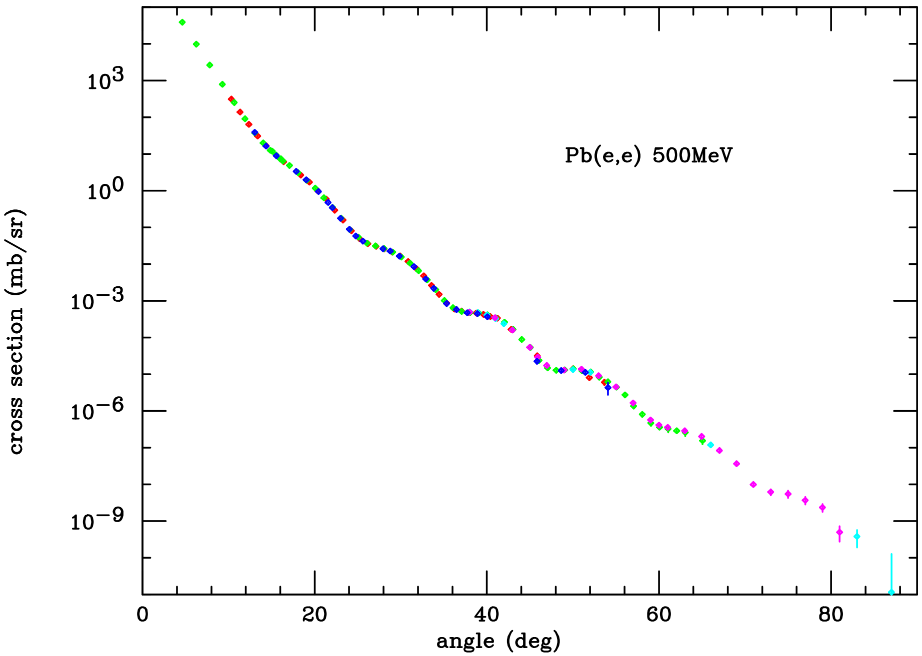
- $F(q)$ not available, only cross sections
  Coulomb distortion!
- $q_{max}$ not infinite

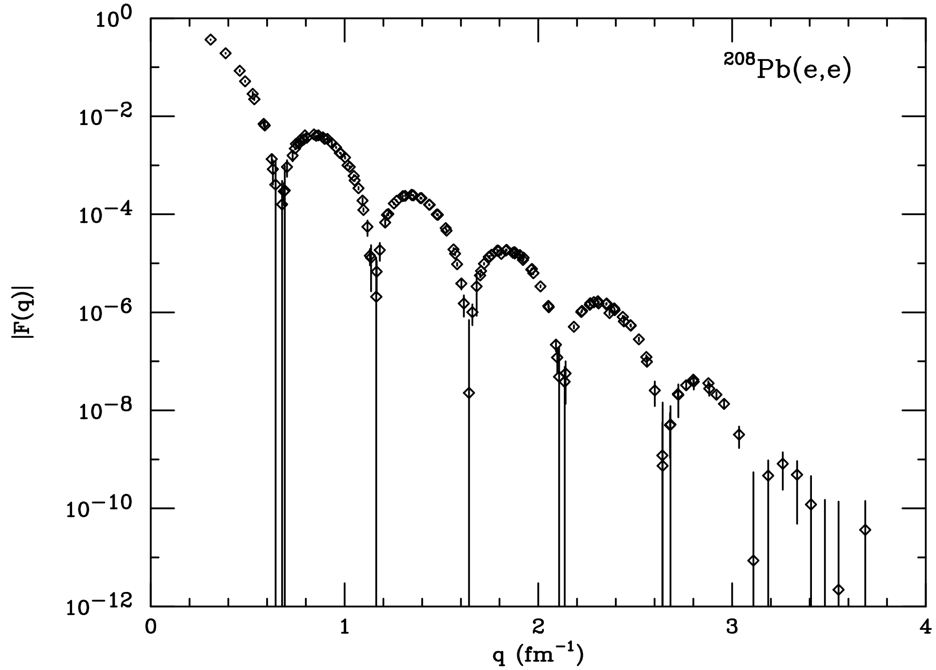
Advantage of electrons

Coulomb distortion relatively small
only function of Z (known)
can remove to get $F(q) \pm \delta F(q)$ via DW code

$$F^2_{exp} = F^2_{model}\frac{\sigma_{exp}}{\sigma^{DW}_{model}}, \qquad \delta F^2_{exp} = \delta\sigma_{exp}/\sigma_{point}$$

Example: cross sections for Pb




experimental F(q) contain same information as $\sigma$

To deal with finite $q_{max}$: study $\int$ as function of upper integration limit

$$\rho(r, q_{max}) = ... \int_0^{q_{max}} F(q) \frac{sin(qr)}{qr} q^2 dq$$

$\rho(r, q_{max})$ = damped oscillatory function of $q_{max}$

extrema at every zero of $F(q)$, $sin(qr)$

in limit $q_{max} = \infty$ equals $\rho(r)$

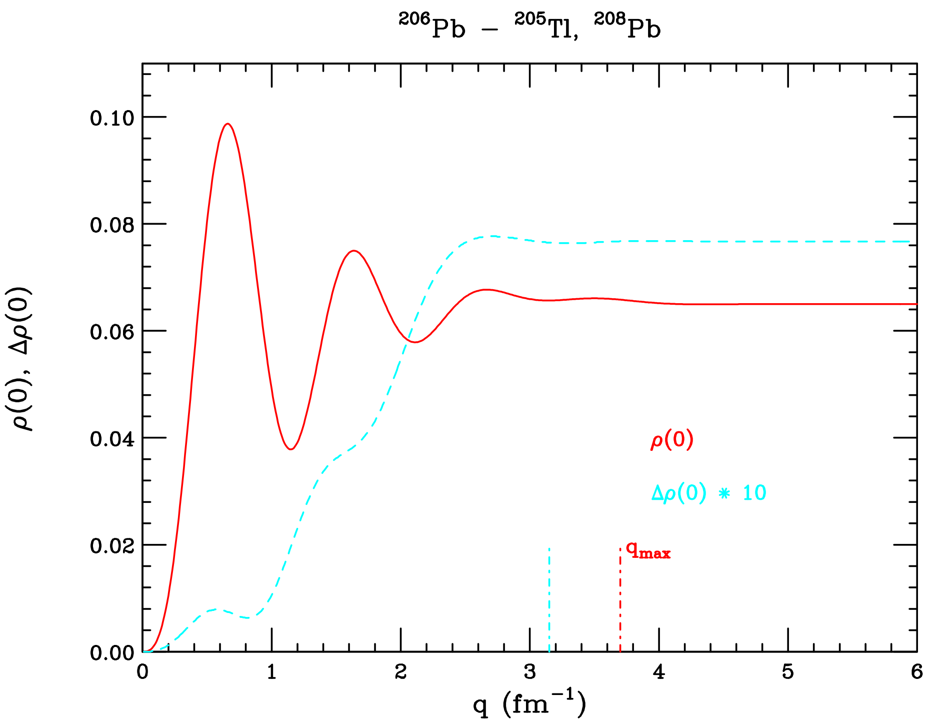
difference last minimum/maximum = uncertainty

$\rightarrow$ model-independent estimate of $\delta\rho$ $\rightarrow$ benchmark for $\delta\rho$

Illustration: $r = 0$ (most difficult case)

$$\rho(0, q_{max}) = ... \int_0^{q_{max}} F(q)\ q^2 dq$$

Example: Pb (red curve)




difference last mini/maxi <1%

Major assets of Direct Fourier Transform

bias-free determination of $\delta\rho$, transparent relation data - $\rho$, benchmark $\delta\rho$

$F(q)$ most useful to get insight, only in the end need solution Dirac eq.

Data of low $q_{max}$ for exotic nuclei: DFT not possible

must rely on model densities
must aim at *integral* properties such as $rms$, $c$, $z$, ...

Discuss first: rms-radius $R = \langle r^2 \rangle^{1/2}$

*the* integral quantity of interest
most often quoted
can compare to radii from electronic/muonic atoms
needed to convert isotope shifts to radii
particularly relevant for unstable nuclei
discuss partly in PWBA (more transparent), in practice use DWBA

R strongly correlated with half-density radius $c$, discuss only $R$

Determination of $R$ is deceptively easy:

$$F(q) = 1 - q^2 \langle r^2 \rangle / 6 + q^4 \langle r^4 \rangle / 120 - ...$$

with $F(q)$ obtained as in DFT

apparently model-independent way to get $R$
$\pm$ everybody uses it (at least conceptually)
but it does not work!!

## Insights (in part not so recent [5])

1. Expansion of $F(q)$ in $q^2$ has very small convergence radius, $\sim 6/R^2$

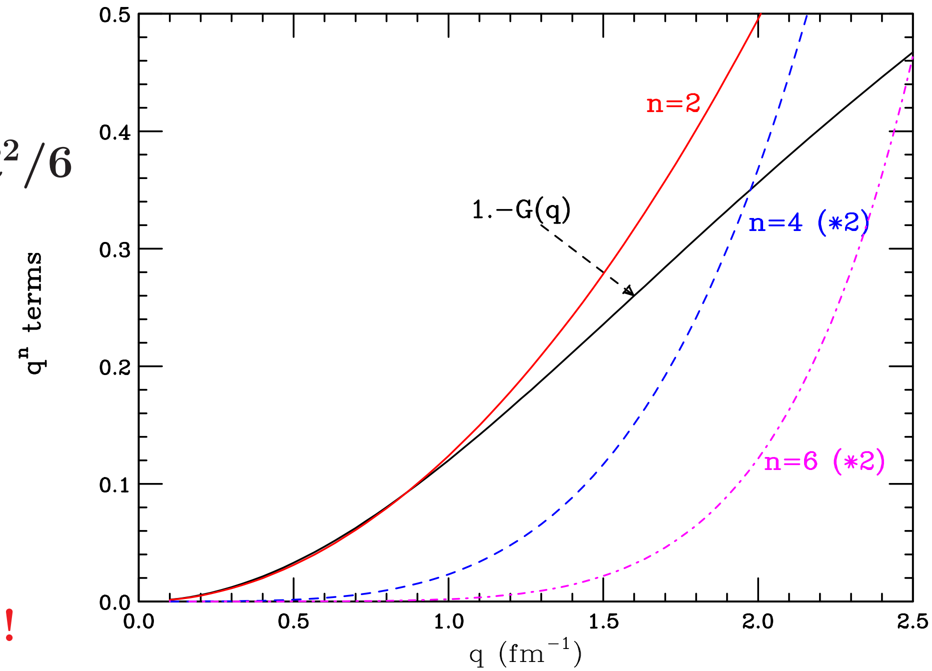
2. $F(q)$ does not correspond to a density, Fourier transform diverges

3. Higher moments always contribute

if go to small enough $q_{max}$:
then finite size effect too small
measure only the "1" in $1 - q^2R^2/6$

if $q_{max}$ is larger for small $\delta R$:
then $\langle r^4 \rangle$ important
but $\delta\langle r^4 \rangle$ affects $\delta R$

get smaller $\delta\langle r^4 \rangle$ through higher $q_{max}$
get same problem with $\langle r^6 \rangle$, ....

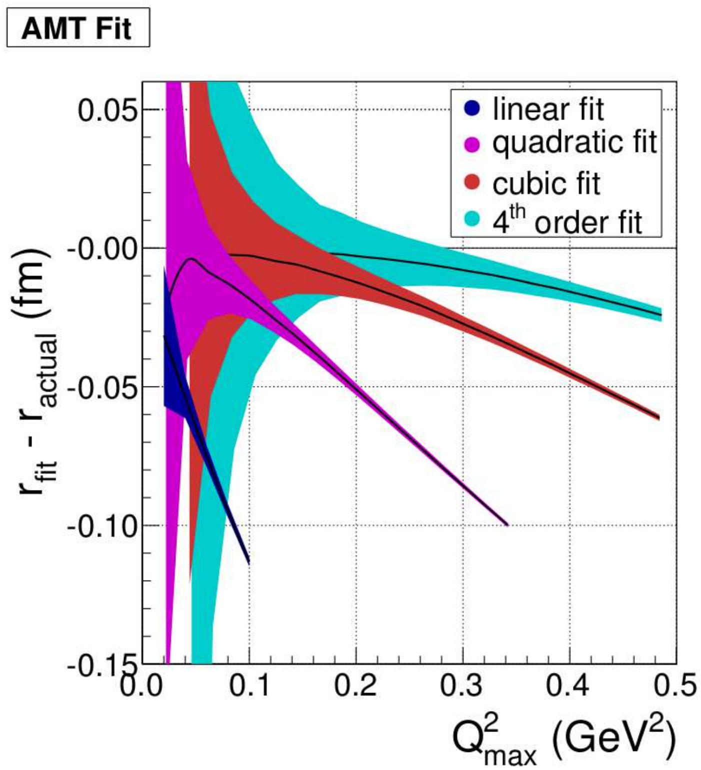



known since a decade for p, but ignored!

Problems with proton rms-radius have illustrated this point

## Quantitative demonstration: Kraus *et al.* [6]

generate $F(q)$-data from known density, extract $R$ from fit of $F(q)$
find systematic differences input-$R \leftrightarrow$ output-$R$




**Expansion in powers of $q^2$ useless!**

**Consequence: need model densities/physics to (implicitly) fix higher moments**

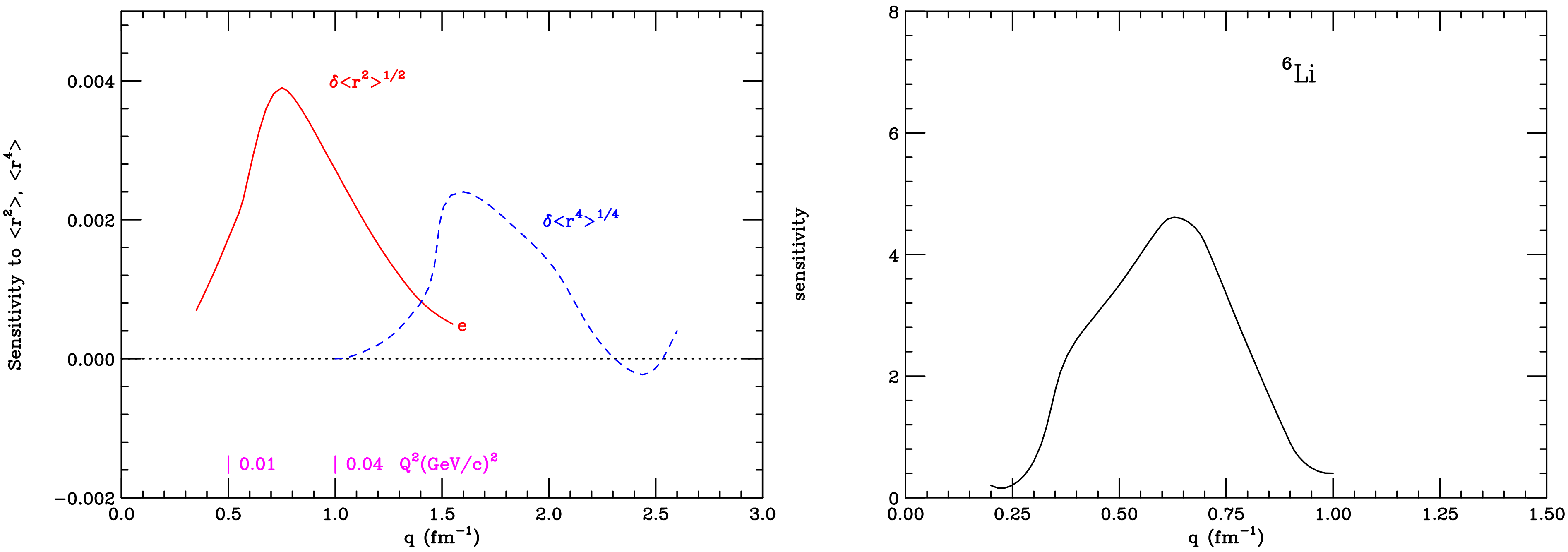
for resulting problems (model dependence) see below

## Important consideration: which $q$'s are important?

most often ignored
standard answer "small $q$" not good enough

## Quantitative study: notch test [7]

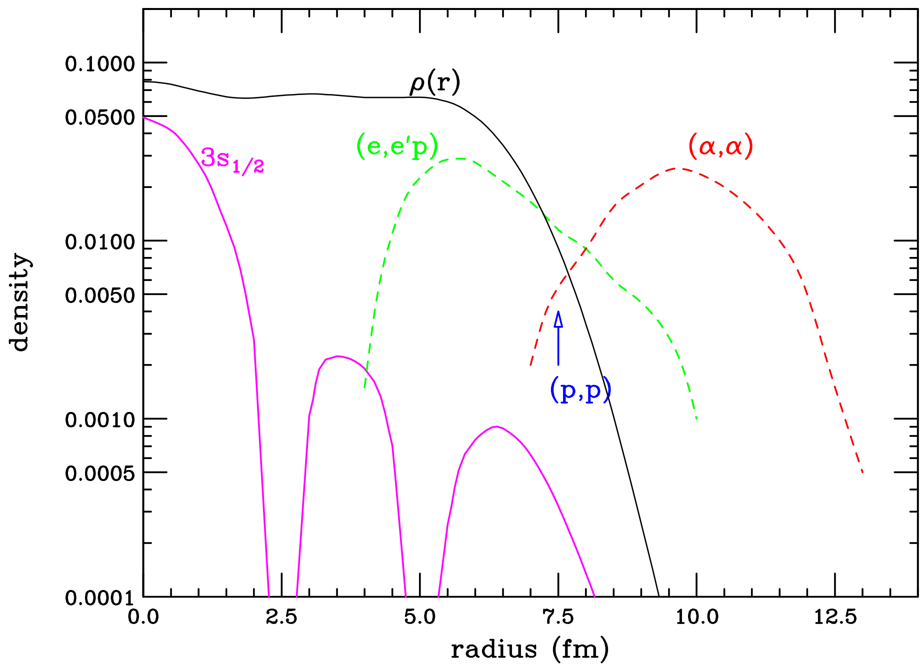
change data by (say) 1% in narrow region around $q_0$
refit data
plot changes of $R$ as function of $q_0$




For nuclei heavier than proton:
maximum sensitivity to $R$ at $q$ where $F(q) \sim 0.7 - 0.8$

## Important consideration: which $r$ do matter?

mainly relevant for comparison with *other* probes aiming at $\rho$'s and $R$'s
all too often ignored




**(e,e) = only probe sensitive to *all r***

strongly interacting probes = strongly absorbed probes
only sensitive to very large $r$
can never determine an $rms$-radius!
but are regularly used to do so

**Not everything simple with (e,e) either: $R$ dominated by large $r$, $\int ..r^4\,dr$**

## Importance of large-$r$ tail [8]

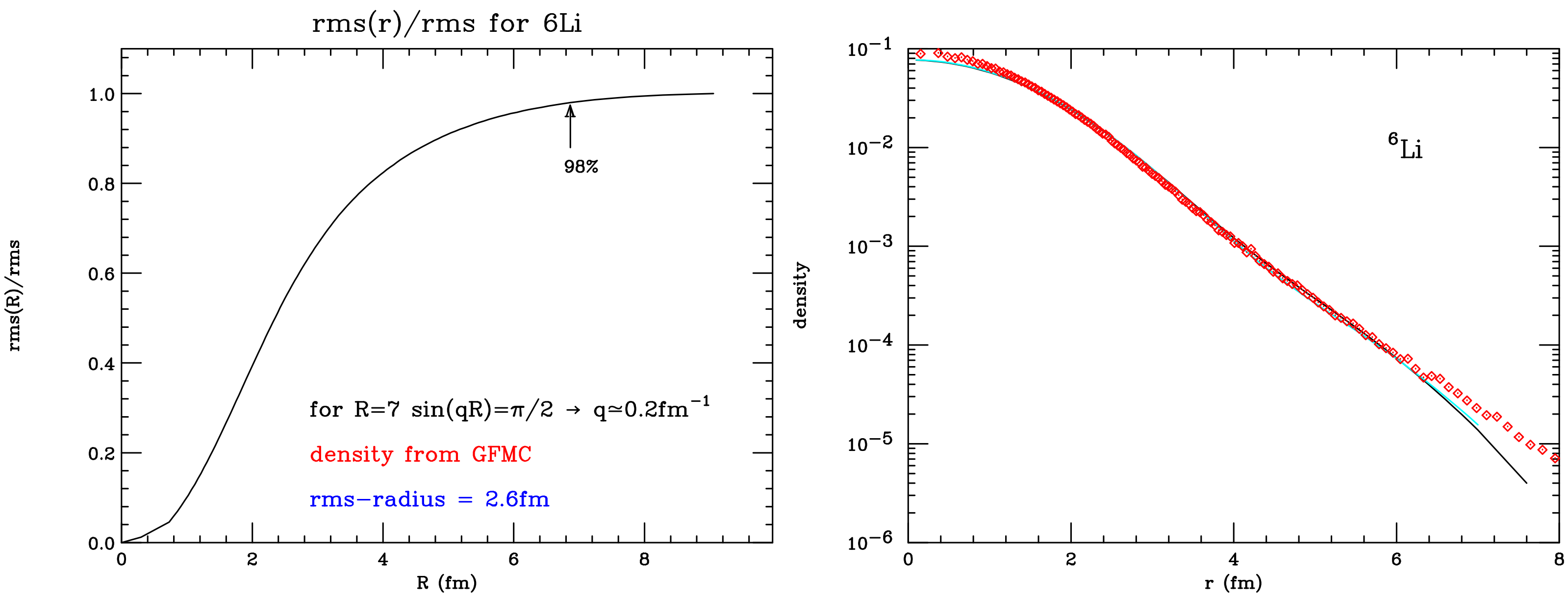
most often ignored in analyses using models
perhaps even worse in 'model-independent' analyses (see above)
parameterizations allow for too much flexibility of $\rho$ at large $r$

## Large-r particularly important for rms-radius and low SE!

## Example: $^6$Li

study $[\int_0^{r_{cut}} \rho(r)\ r^4 dr / \int_0^{\infty} \rho(r)\ r^4 dr]^{1/2}$ as function of cutoff $r_{cut}$

note: $rms$-radius of $^6$Li $= 2.6 fm$

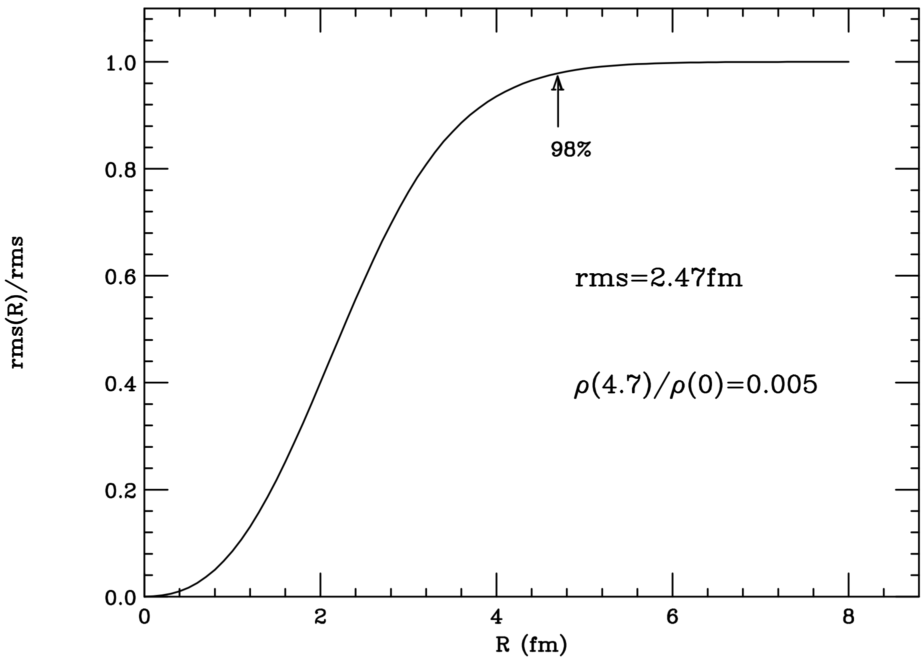


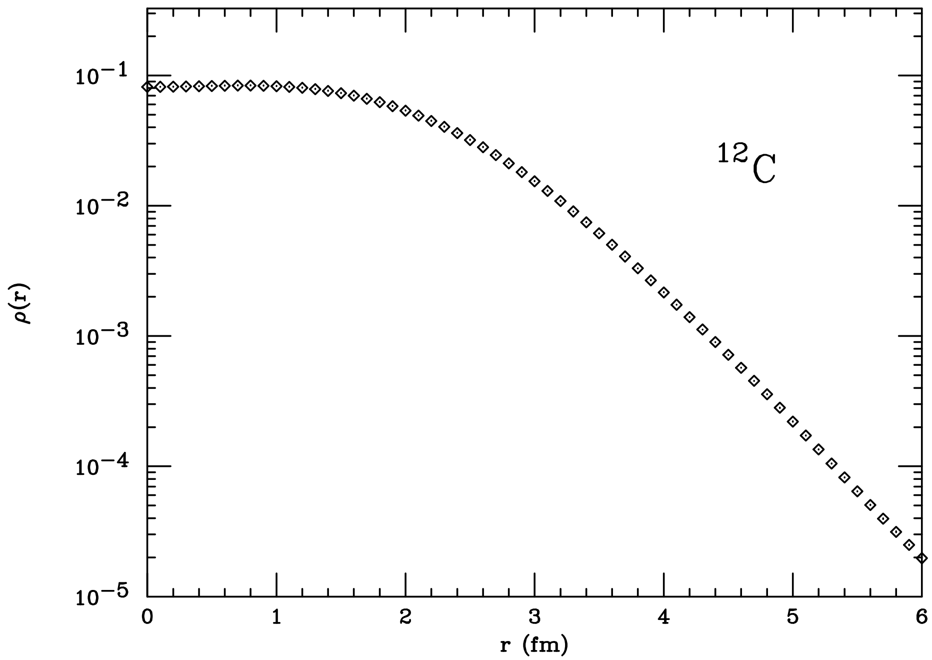

if want R to <2% must know $\rho$ at $r > 7fm$ where $\rho/\rho(0) \sim 10^{-4}$!!

## Only a problem for $^6$Li with long tail?

**similar, though less pronounced for $^{12}C$**

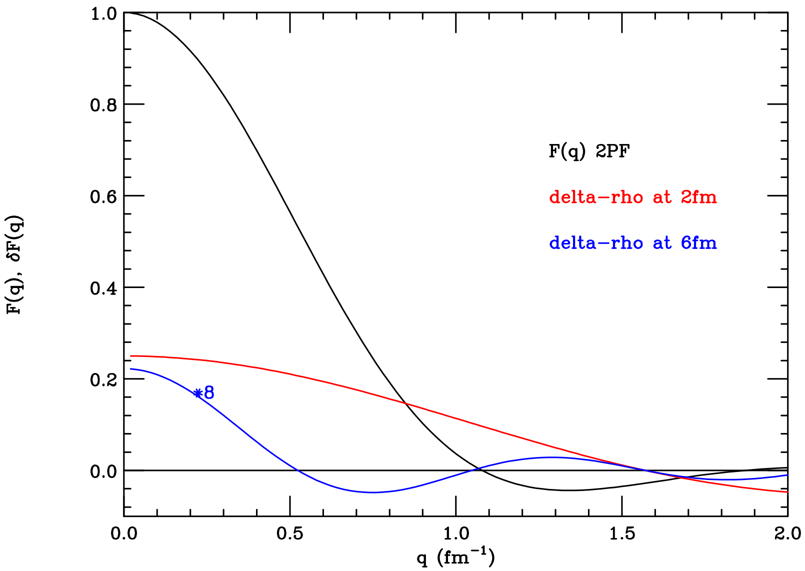
**SE=16MeV!**



## Qualitative demonstration large $r \leftrightarrow$ low $q$

density = 2-parameter Fermi, 1/2-density radius $c = 4fm$
add local change of $\rho(2fm)$, or
add local change of $\rho(6fm)$, same rms-radius change



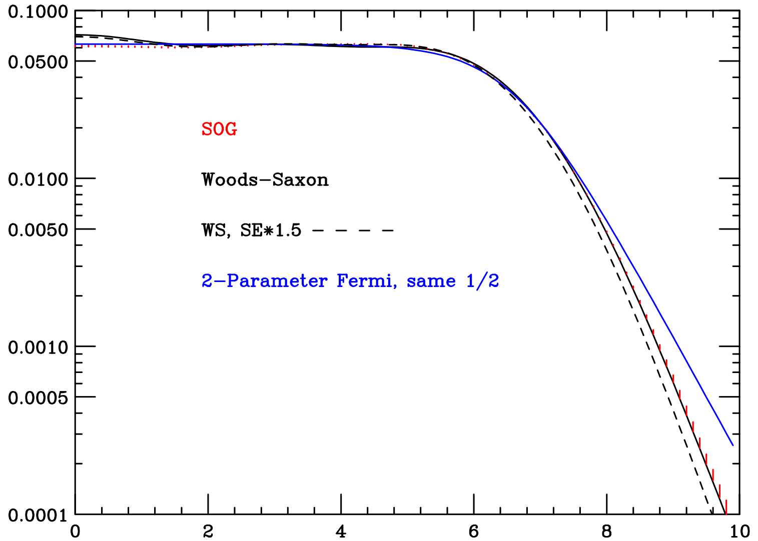

large $r$ produce curvature of $F(q)$ at very low $q$
affects extrapolated slope $F(q=0)$ which yields rms-radius

**Consequence: must be VERY careful about $\rho$(large $r$)!**

parameterization $\longrightarrow$ implicit assumption on $\rho$(large $r$)

## Example: compare tails for $Pb$

experimental SOG density
Woods-Saxon density with exp. separation energy
Woods-Saxon with increased separation energy
2-parameter Fermi with same 1/2-density radius




tail of 2PF leads to 2% larger rms-radius

## Lesson for analysis using model-$\rho$'s:

must constrain large-$r$ behavior using physics knowledge

fall-off of $\rho(r)$ dominated by least-bound protons

**What know on $\rho(r)$ at large $r$?**

dominated by least-bound proton, radial wave function = Whittaker function

$$R(r) \sim W_{-\eta,l+1/2}(2\kappa r)/r \text{ with } \kappa^2 = 2\mu E/\hbar^2,\ \eta = (Z-1)e^2/\hbar\sqrt{\mu/2E}$$

depends only on removal energy

presumably available from accurate mass-measurements

easy to calculate

overall normalization (asymptotic normalization) generally not known →shape

**Shape automatically given if calculate $R(r)$ using Woods-Saxon potential**

solution of Schrödinger equation for sensible potential

→ physical shape of asymptotic $R(r)$

**Note: Harmonic oscillator has *wrong* asymptotic shape**

has been used often in past

has caused many problems (*e.g.* Darmstadt rms-radii)

keep away from it

**Important to use physics information on large-$r$ behavior !**

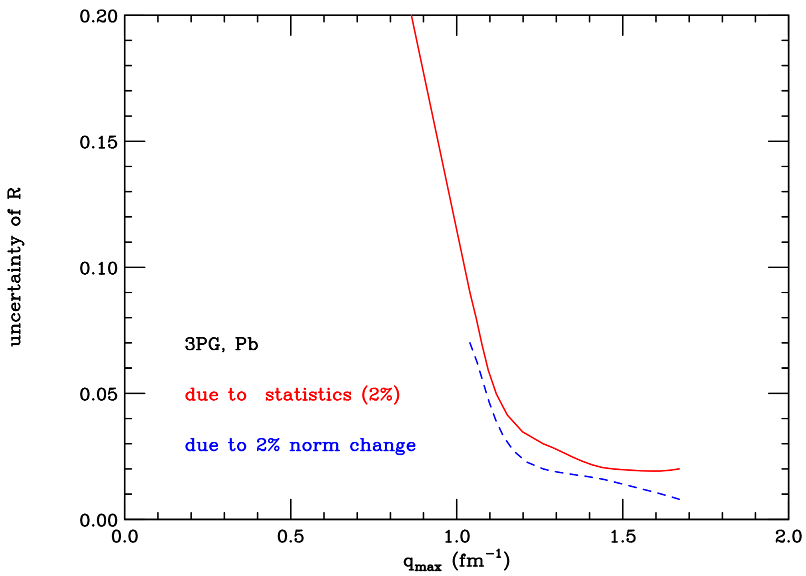
**Should have been done in past analyses of data for stable nuclei**

## Essential insight for determination of R: role of normalization

generate cross sections for Pb using 3PG, $\pm 2\%$
  in realistic $q$-range

analyze as function of $q_{max}$ using 3PG

study effect of normalization change




## Absolute normalization of cross sections very important

must make great effort to determine

## Replace absolute normalization by floating of data?

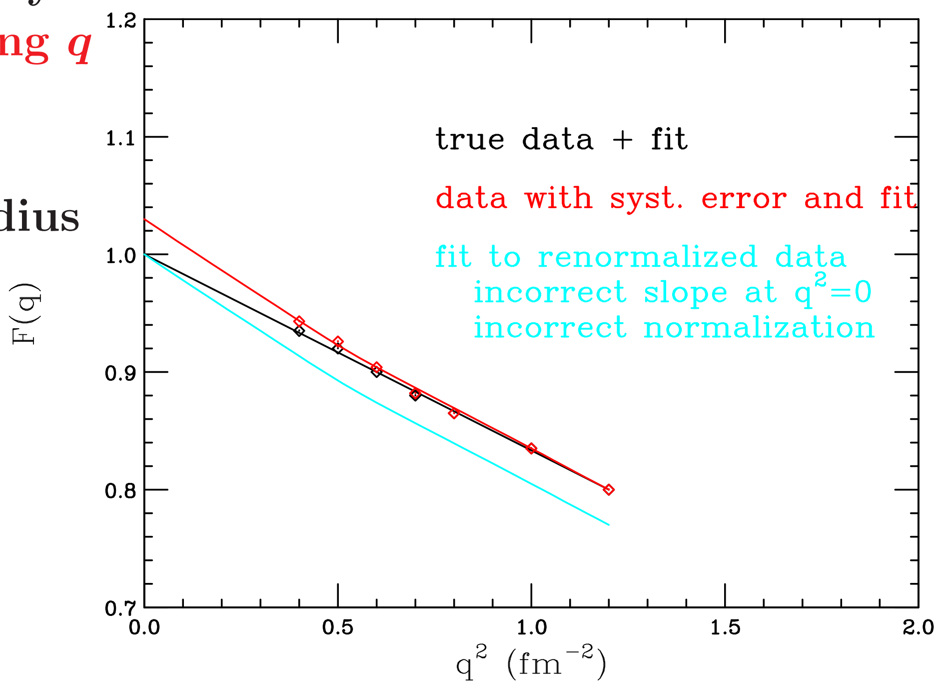
traditional: people float data, as absolute norm difficult to determine
main purpose: get good-looking $\chi^2$

**Problems:**

1. Ignores $\sim 50\%$ of effort of experimentalists
   often great effort made to get ***absolute*** normalization, dont want to throw away

2. When floating ***extrapolate*** via fit to $q = 0$
   become very sensitive to $q$-dependent systematic errors
   errors clearly increasing with decreasing $q$
   - otherwise data taken at lower $q$
   - extrapolation increases effect

   affects in particular extracted $rms$-radius




$\rightarrow$ High premium on ***absolute*** $\boldsymbol{\sigma}$'s

## Alternative look at accuracy needed

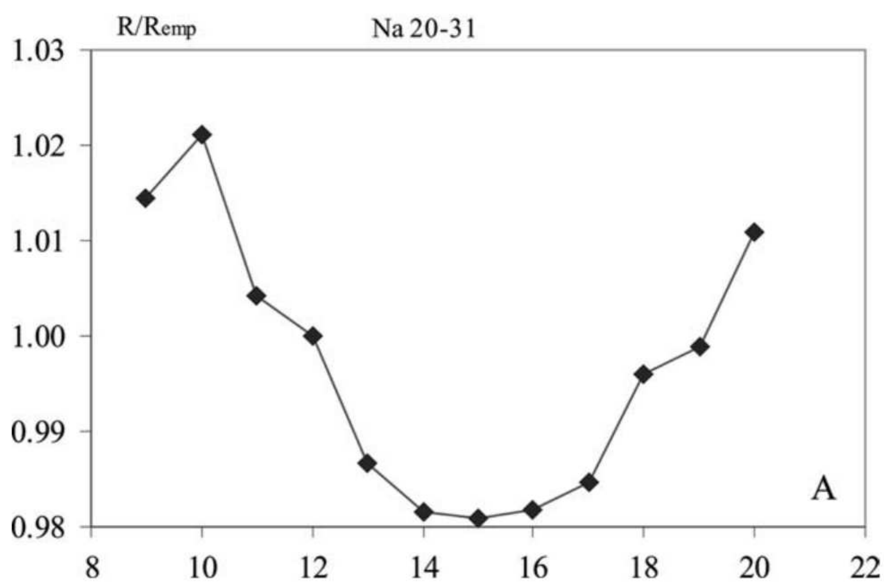
can estimate from sensitivity-curves discussed above
no need for detailed quantitative study

**maximum sensitivity to R at $q$ where $F(q) \sim 0.75$**

$\rightarrow$ finite-size effect in $\sigma \sim 50\%$

**If measure $\sigma$ to $\pm 1\%$**

measure $R^2$ to $\sim 2\%$
measure $R$ to $1\%$
then can see evolution of $R$ of interest



**Absolute $\sigma$'s to $\pm 1\%$ for stored ions??**

already difficult enough for stable isotopes
already there dominating experimental error
presumably not possible

**For stored ions**

measure cross section *ratios* to stable isotope
$\pm$straightforward as these much more abundant
  modest increase of time

**Requirement**

accurate (relative) luminosity monitor
same response for different isotopes
should be feasible
- elastic scattering at very low q (recoils near $90^\circ$)
- or exploiting atomic electrons (?), nuclear Bremsstrahlung

**But remember**

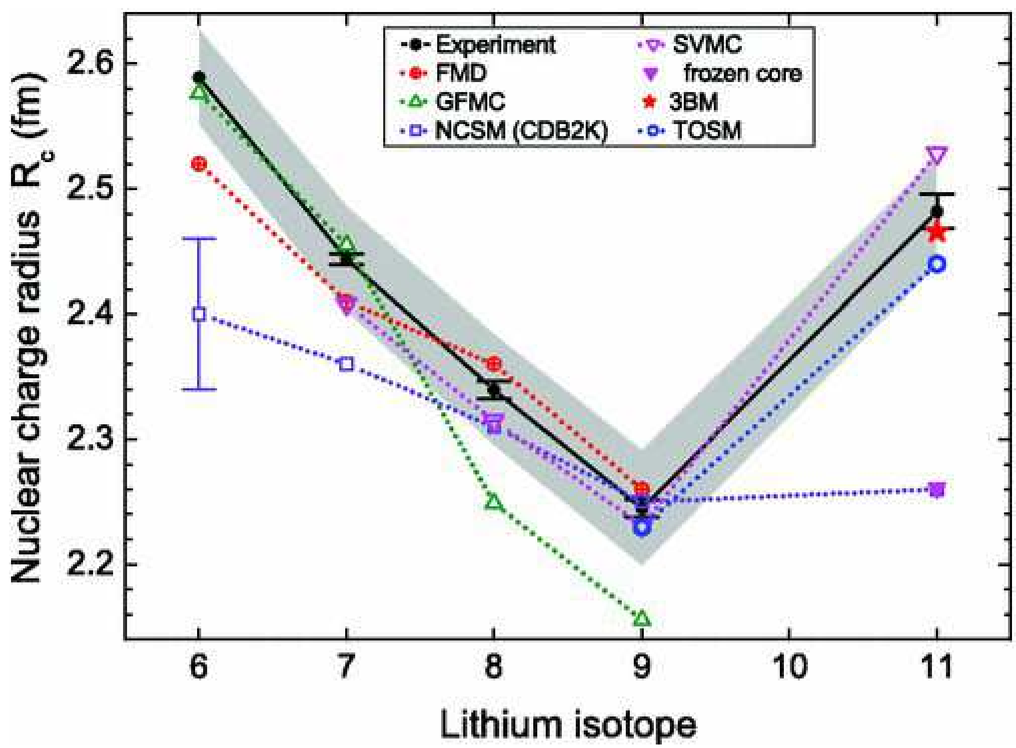
results only as good as result for reference nucleus
(re-)analyze reference nucleus data respecting above insights

**Conditio sine qua non: precise *ratio*-measurements!**

Special difficulty for unstable isotopes: halo, for small SE

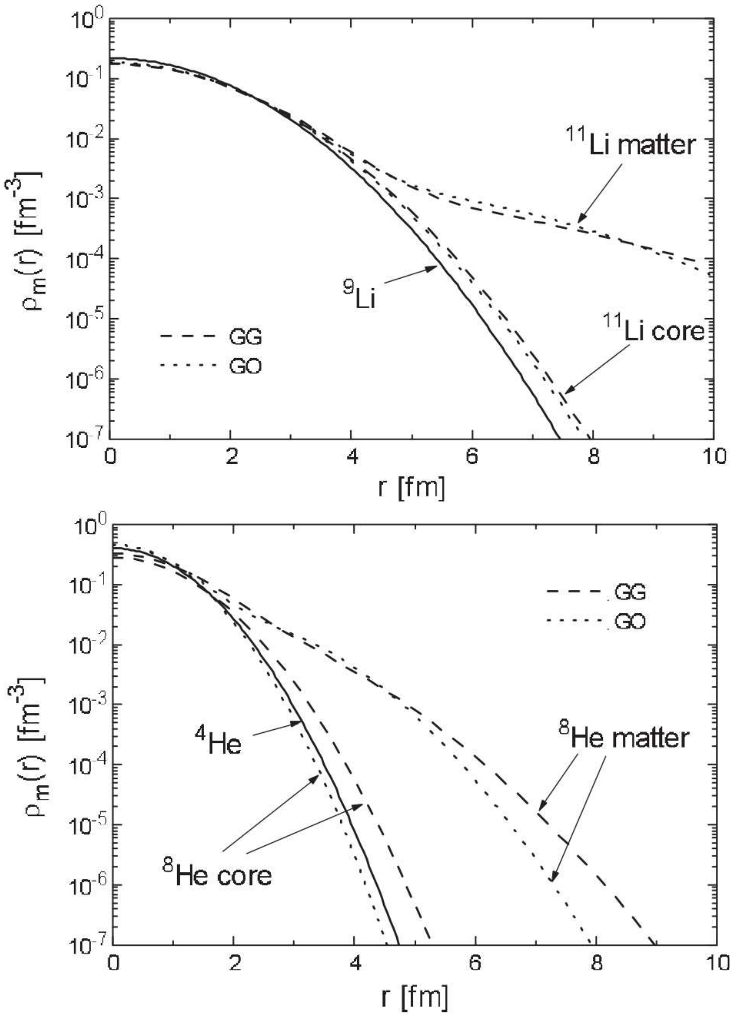
Example: isotope shifts for Li

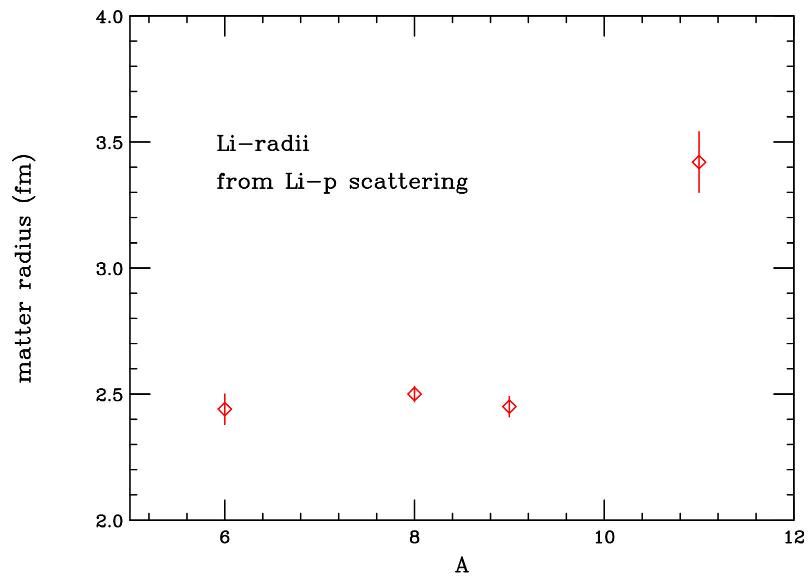
measured by laser spectroscopy with stored ions
Nörtershäuser *et al.*[9]
$^{11}$Li = Borromean nucleus (2n, $^{10}$Li unbound)
2n-separation energy only 369KeV




Jump in $R$ due to large-$r$ tail from outermost proton

## Extreme example of tail-importance: matter radii [10]




## Densities from GeV/N p-nucleus scattering in inverse kinematics

## How to proceed in presence of such a halo?

standard analysis with model for $\rho$ not promising
density has too 'complicated' a shape
large-$r$ tail affects R way too much

## More promising approach

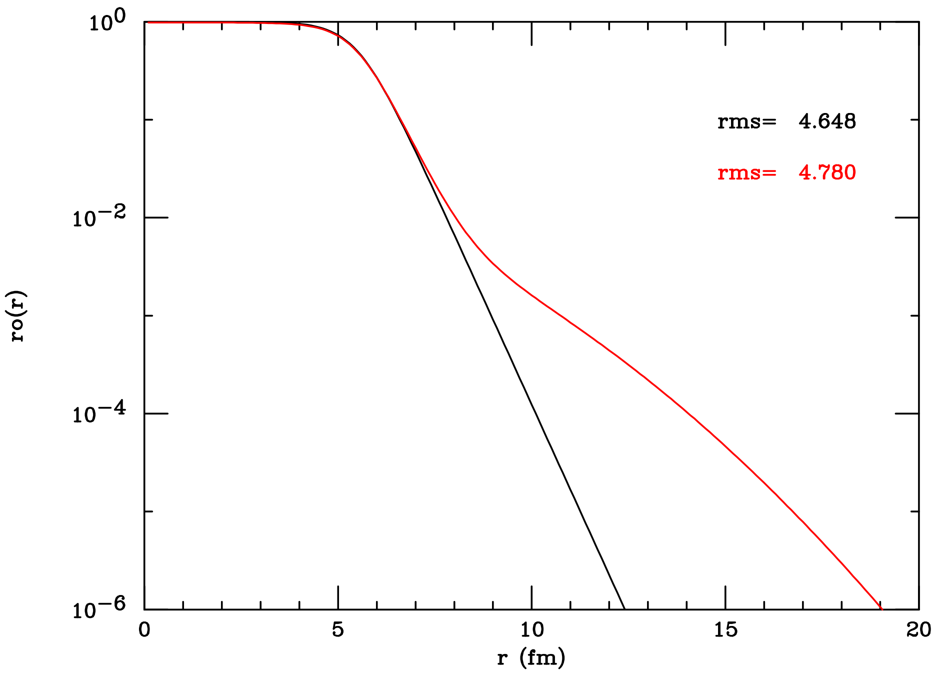
tail due to *one* shell with low separation energy
core in general more 'normal'

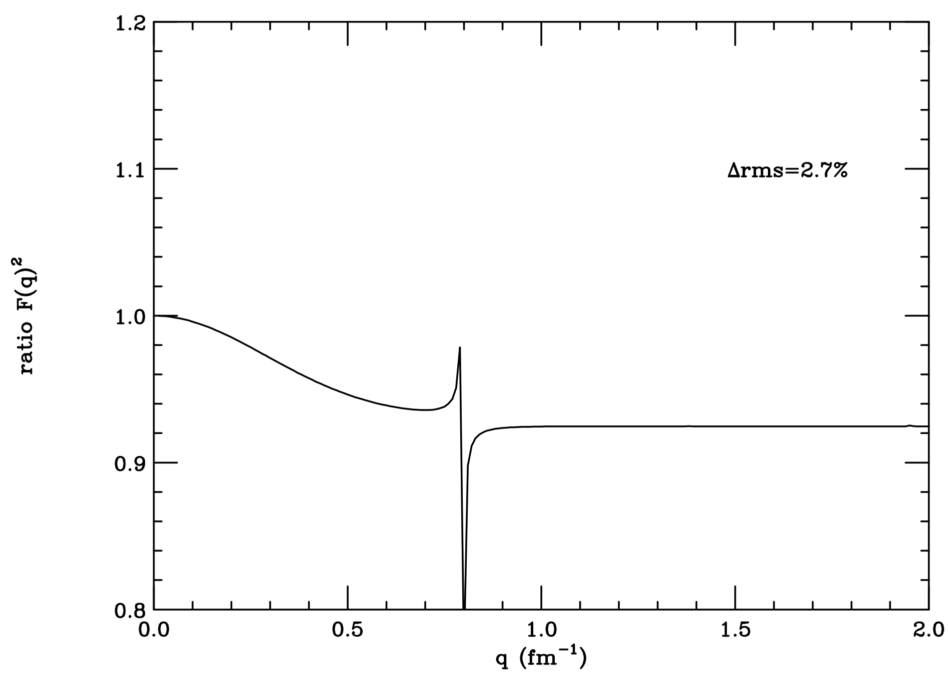
## Parameterize $\rho$ as model+tail

model = standard 2PF, 2PG, ...
tail = calculated in WS potential using experimental separation energy

## Example

add tail to Pb density
similar to findings from above example Li
compute change in cross section




## Find (as expected)

main effect of halo at *very* low *q*
need accurate data at these low q's
- feasible from ratio measurements
- main issue: precision, *not* luminosity

# Determination of $z$

## Lessons from the "early days"

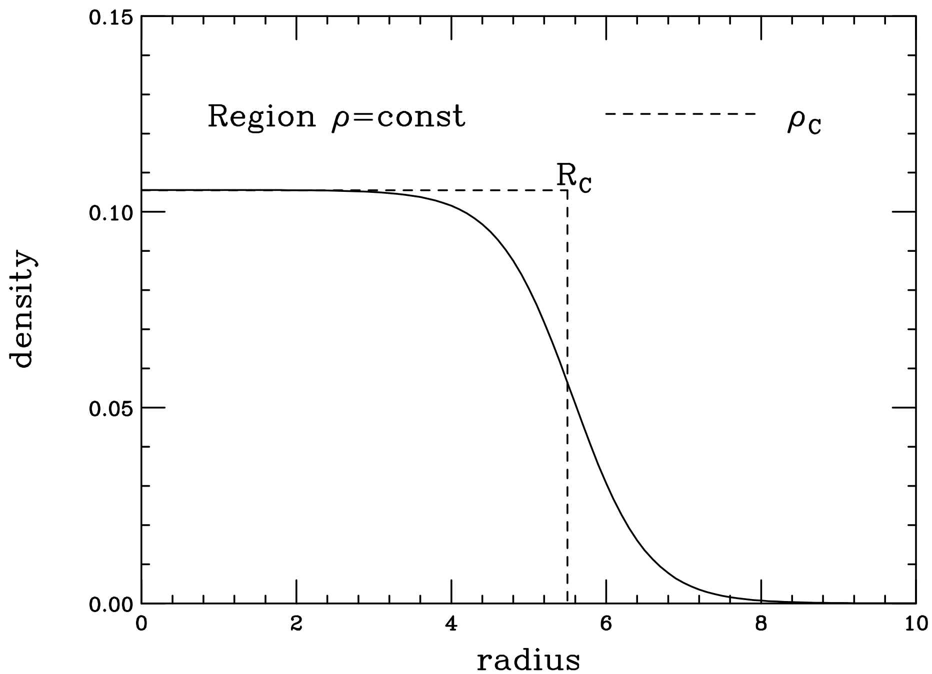
$z$ obtained from $F(q)$ in 2. diffraction maximum

## Understanding in PWBA:

consider $\rho(r)$ as result of folding $*$

fold uniform density $\rho_c$ with gaussian G(r-r') of width z

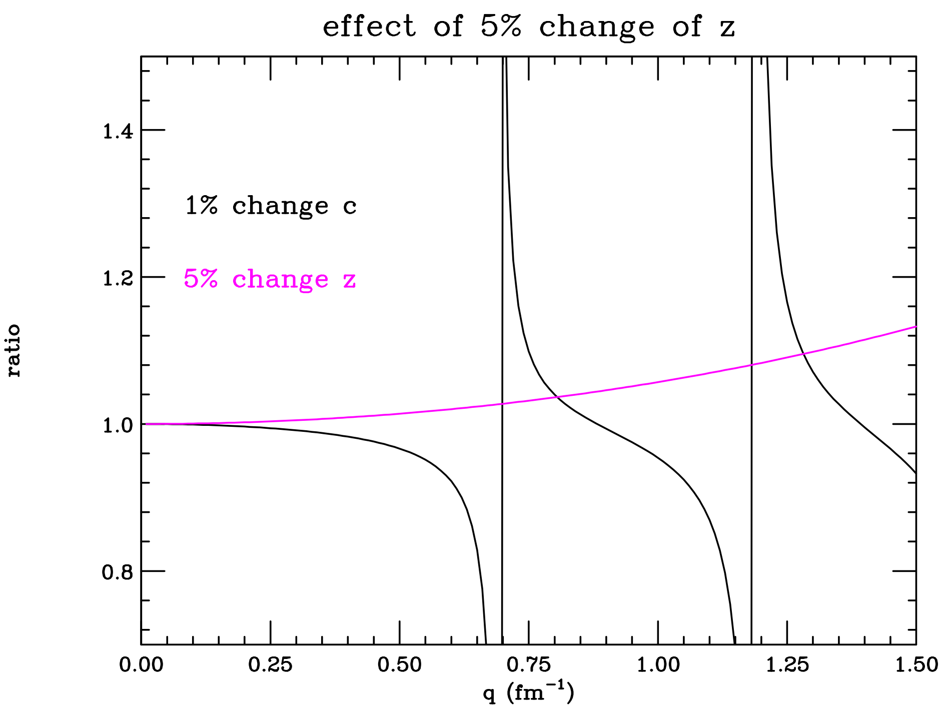
$\rho(r) = \rho_c(r) * G(r)$



## Result for Fourier transform FT

$$F(q) = FT(\rho_c) \cdot FT(G)$$

**Effect of $z$ ~ multiplication of $F_c(q)$ with FT of $G$ *i.e.* $e^{-q^2 z^2}$**




region of 2. diffraction maximum sensitive to $z$
shape of $F(q)$ for $\Delta c$ and $\Delta z$ different

but: effect not large
third diffraction maxi would be better

Particular interest in $z$ for unstable nuclei

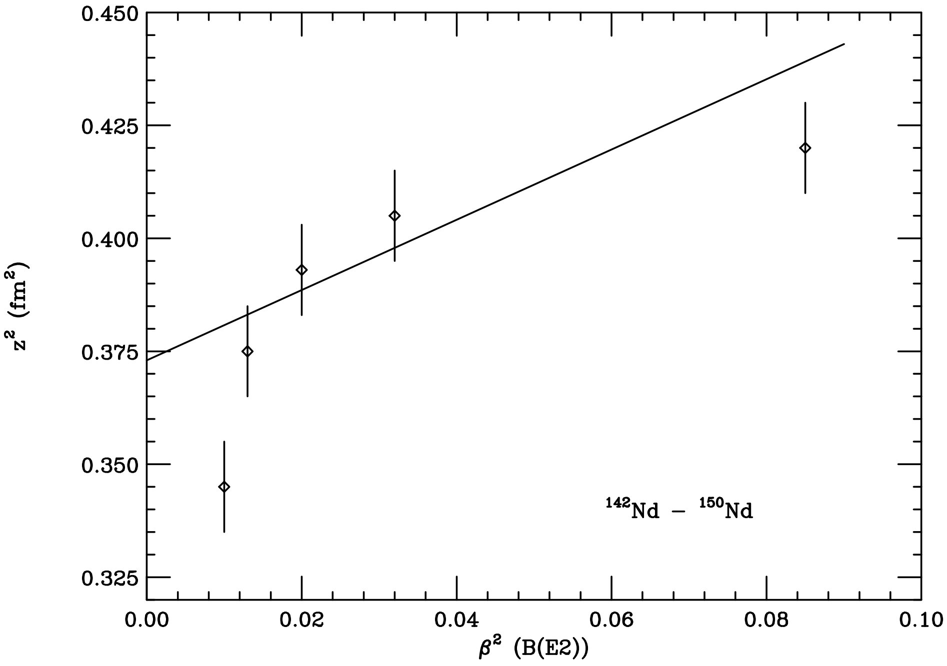
larger for halo-nuclei?

large for deformed nuclei

$$z_{eff}^2 = z_0^2 + \frac{3}{4\pi^3}\beta^2 c^2 + O(\beta^3)$$

Particularly useful for series of isotopes

Example: Nd-isotopes [1]




could be best way to get at deformation, is only way for I=0 nuclei

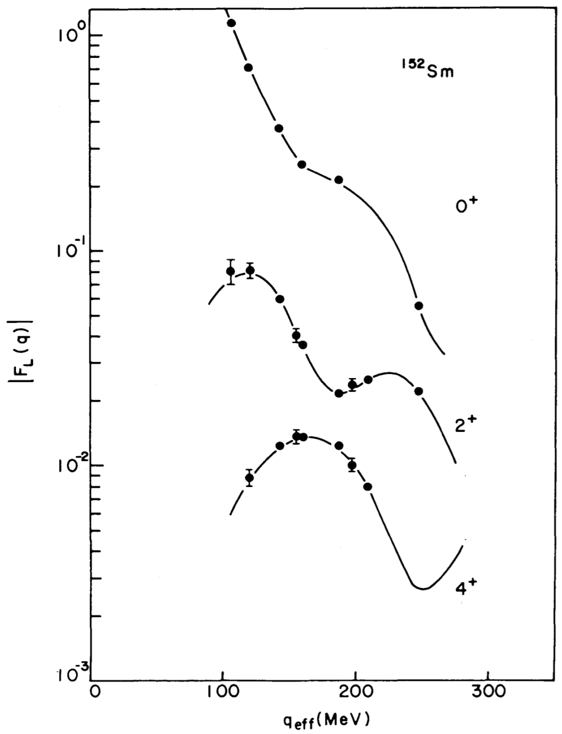
note: need rather *precise* $z$'s

## Challenge beyond L: energy-resolution

near 1. minimum, inelastic F's peak
  have typically several 10% of probability
must separate elastic/inelastic scattering
exotic nuclei typically far from magic
  low excitation energies
also needed for data on transition form factors

## Problem: long targets

required to get high L
makes small $\Delta E/E$ difficult




## Advantage of fixed-target experiments

resolution needed $\sim E^*$
$\Delta E/E \quad \sim 10$ times easier
  than for collider experiments

## Most realistic solution

magnetic spectrometer
deflection $\perp$ to scattering plane
decouples coordinate and $E$

## Additional considerations $\rightarrow$ precise $z$

a priori correlation $c \leftrightarrow z$ not too strong
nevertheless: more precise $c$ (or $R$) allows for more precise $z$
independent knowledge on $R$-type observable $\rightarrow$ smaller $\delta z$

## For stable (reference)-isotope

add muonic X-ray information
available for *many* nuclei
yields *very* precise moments of $\rho$
for larger Z also yield higher moments of type $\langle r^4 \rangle$, $\langle r^6 \rangle$
(Barrett moments, to be precise)

## For precise isotope shifts of $R$-type observables

include also electronic isotope shifts
occasionally easier to measure than (e,e) cross sections
particularly for exotic nuclei
need to know e-wave function at origin
$\Psi(r = 0)$ calibrated using stable isotopes

## General rule: add as much independent $R$-type info as possible

also helps to mitigate problems with absolute normalization of $\sigma$

Central depression $w$

large for heavy nuclei away from valley? Bubble nuclei??

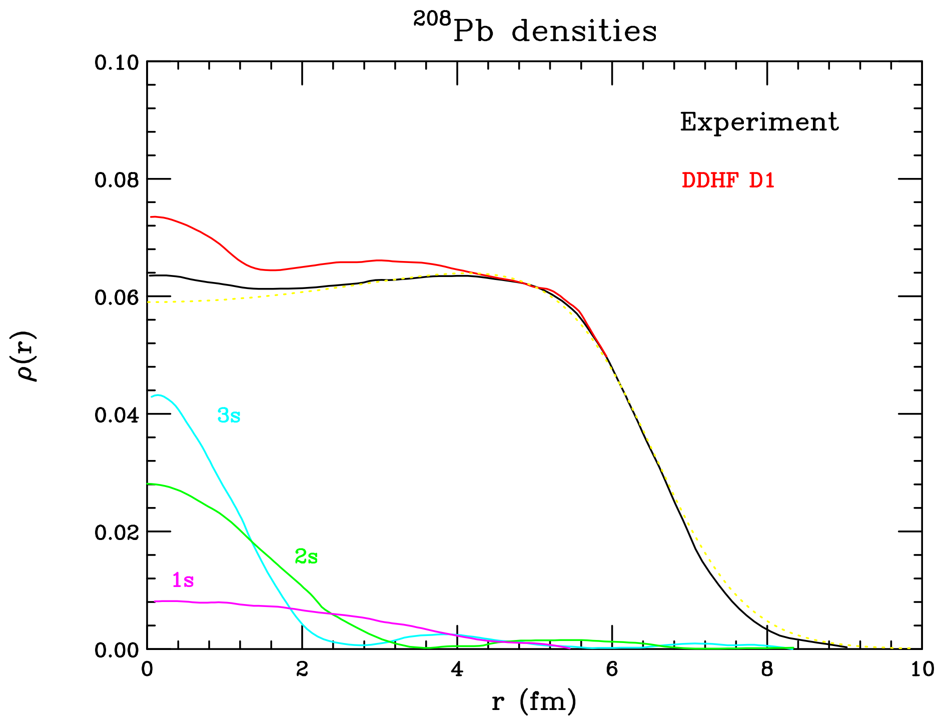
predicted for super-heavy, proton-rich nuclei

Lesson from 'early days'

not clear if $w$ describes central depression

biggest effect of $(1 + wr^2/c^2)$ occurs in tail of $\rho(r)$ at *large* $r$

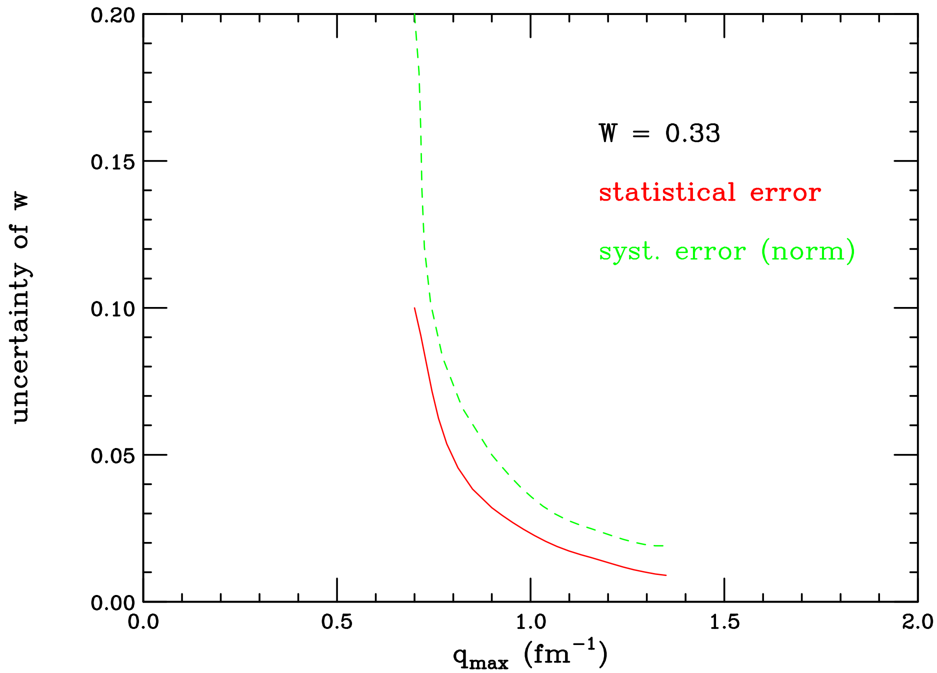
Example: $^{208}Pb$: $w = 0.33 \rightarrow$ depression?? Not there in modelindependent $\rho$ [11]




$w$ essentially compensates too fast fall-off of Gaussian distribution

## Sensitivity to $w$ with 3PG (phase-shift calculation)

use data with 2% stat. errors, up to $q_{max}$
fit with 3PG
derive uncertainly of $w$, syst. error due to norm change



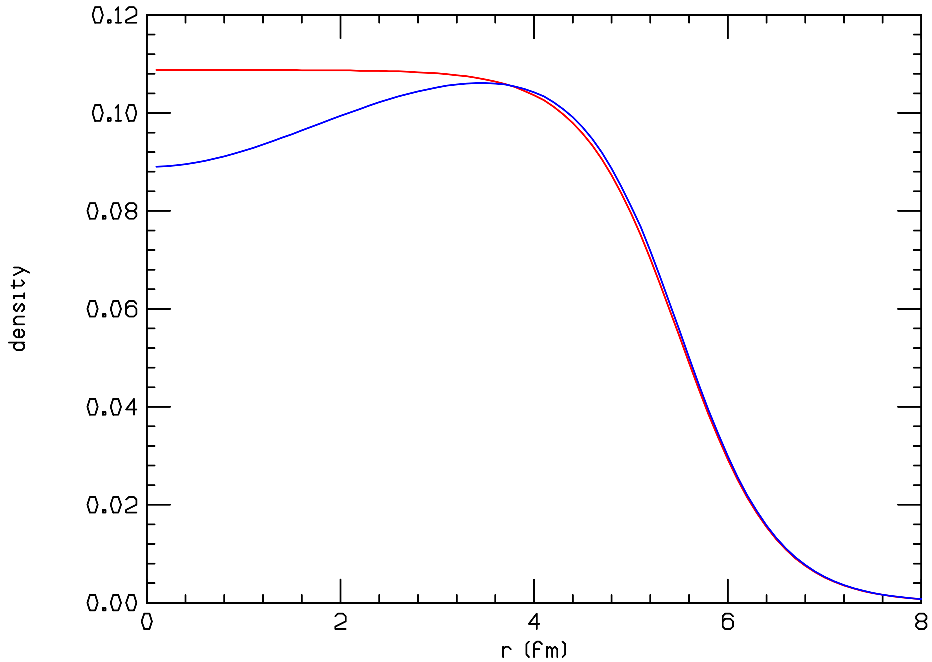

$w$ with precise data to $q = 1.5 fm^{-1}$ measurable
but: represents depression, or shape $\rho(r)$ at large $r$?

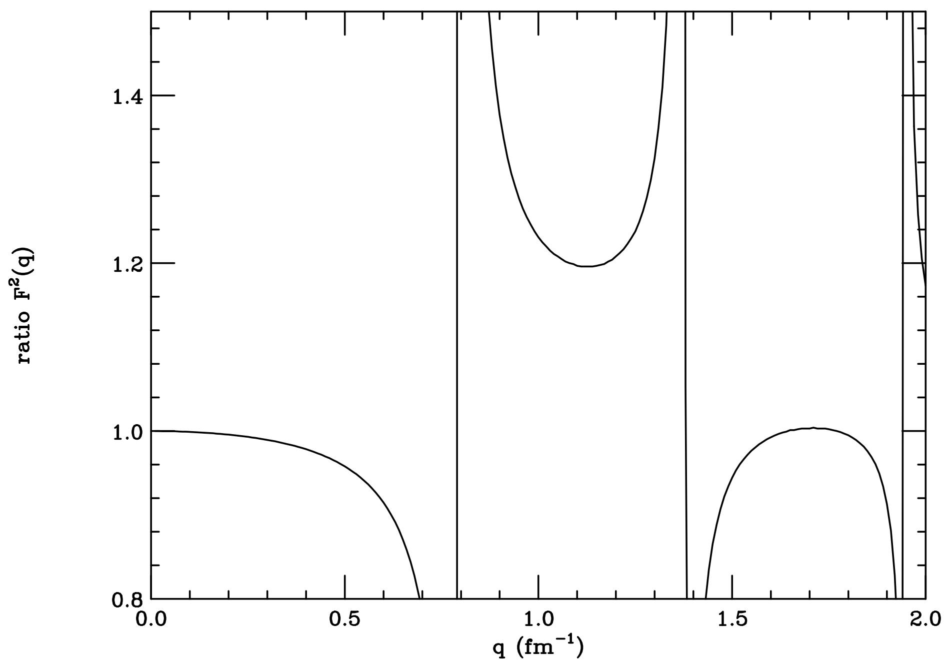
My conclusion: $w$ mainly used to affect large-$r$ tail, *not* central depression
$\rightarrow$ keep away from $w \sim$ central depression
$w$ affects *both* small and large $r$

## Better parameterization of eventual central dip:

$$\begin{aligned}\rho(r) &= 2PF - ae^{-r^2/b^2} \quad \text{for } r \leq c \\ &= 2PF \quad\quad\quad\quad\quad\; \text{for } r \geq c\end{aligned}$$

## Result for $\rho(r)$ and $F^2$-ratio in PWBA




## Fairly characteristic change in $F^2$

increase of $F^2$ in 2. maximum, little change in third maxi
comparatively little change below first mini, above second mini

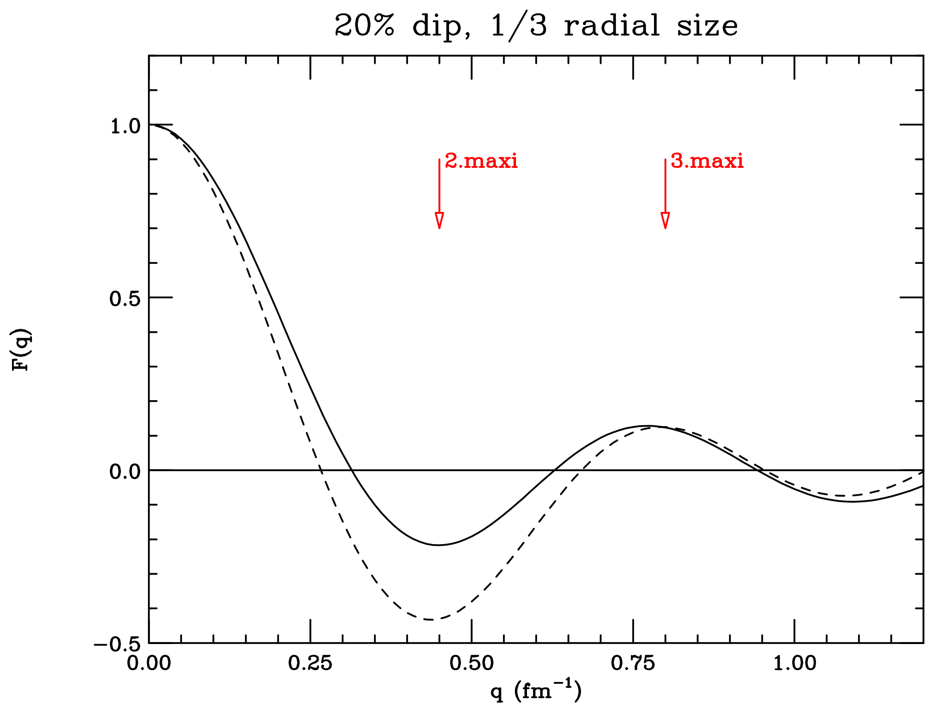
## Measurable if can reach third maxi of $F^2$

## Qualitative understanding from $q$-dependent term in integrand for F(q)

**compare for $r_1 = 0.3 r_0$**

$$sin(qr_0)/qr_0$$

$$0.8\ sin(qr_0)/qr_0 + 0.2\ sin(qr_1)/qr_1$$




**Confirms: main signal of central dip = enhancement in second maxi of F(q)**

**should be within reach of experiment**

## Light vs. heavy nuclei

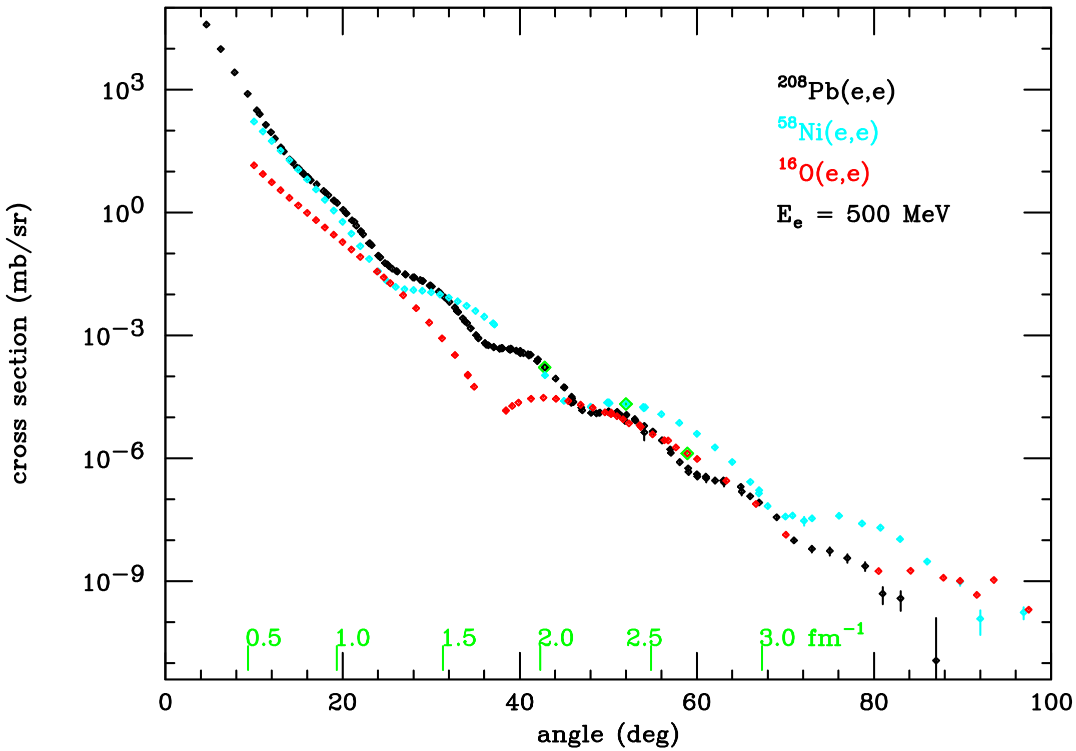
above studies mainly performed for $A$=208

how about lighter nuclei?

## Fall-off of $d\sigma/d\Omega$

quite similar for different $A$

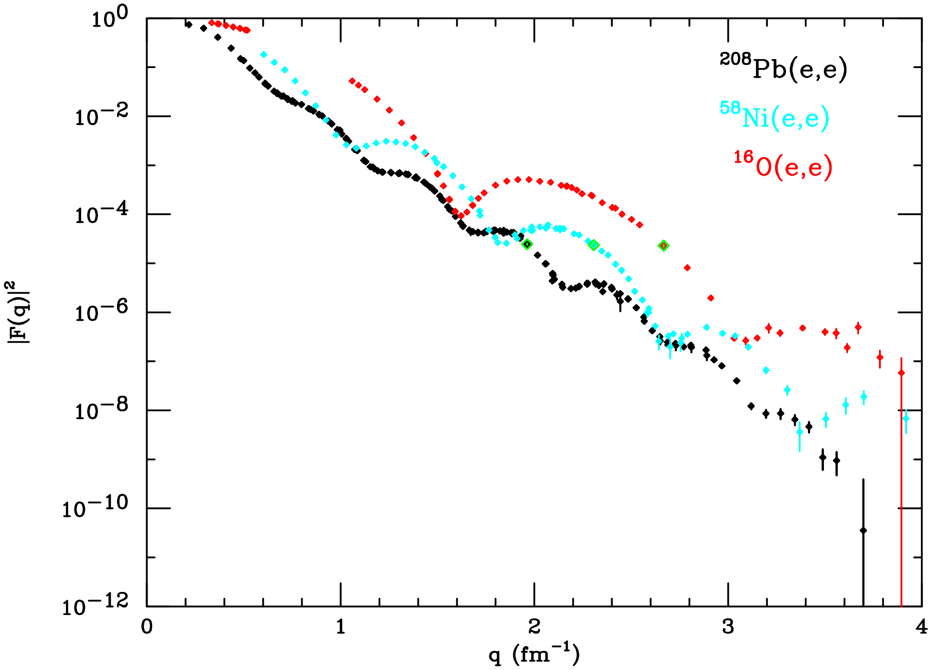
somewhat slower for small $A$



Relevant for accuracy of $\rho(r)$:

smallest $F(q)$ measured, not $q$ or number of diffraction maxima

$$\rho(r, q_{max}) = \int_0^{q_{max}} F(q)\, \frac{sin(qr)}{qr}\, q^2 dq$$




For small $A$

need larger $q$ to get same, small $F(q)$

need larger $q$ for same completeness error $\rightarrow$ somewhat lower $\sigma$

determination of $c$, $z$, '$w$' somewhat more difficult

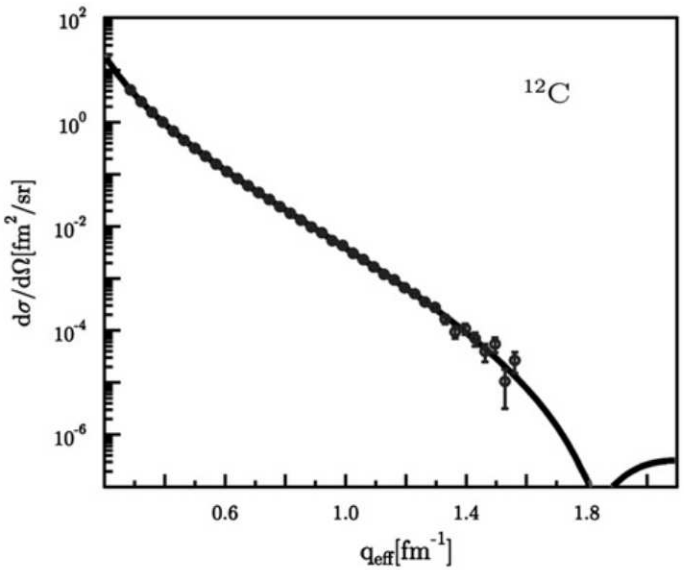
Need/can determine further integral quantities?

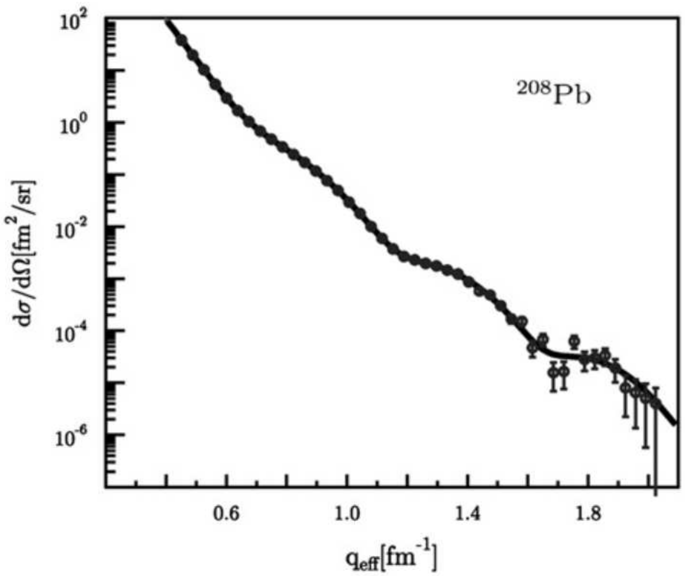
can determine *more* than $c$, $z$, dip?

depends strongly on achievable luminosity L
depends on quality of e-spectrometer
evaluation beyond scope of present study

Use result of study for ELISe (Antonov *et al.* [12] )

assume $L = 10^{28}$, 100msr, 4 weeks



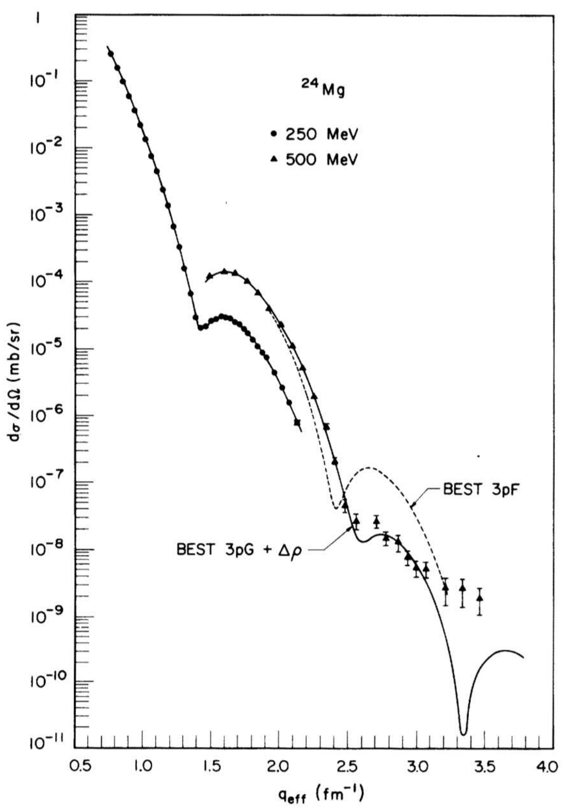



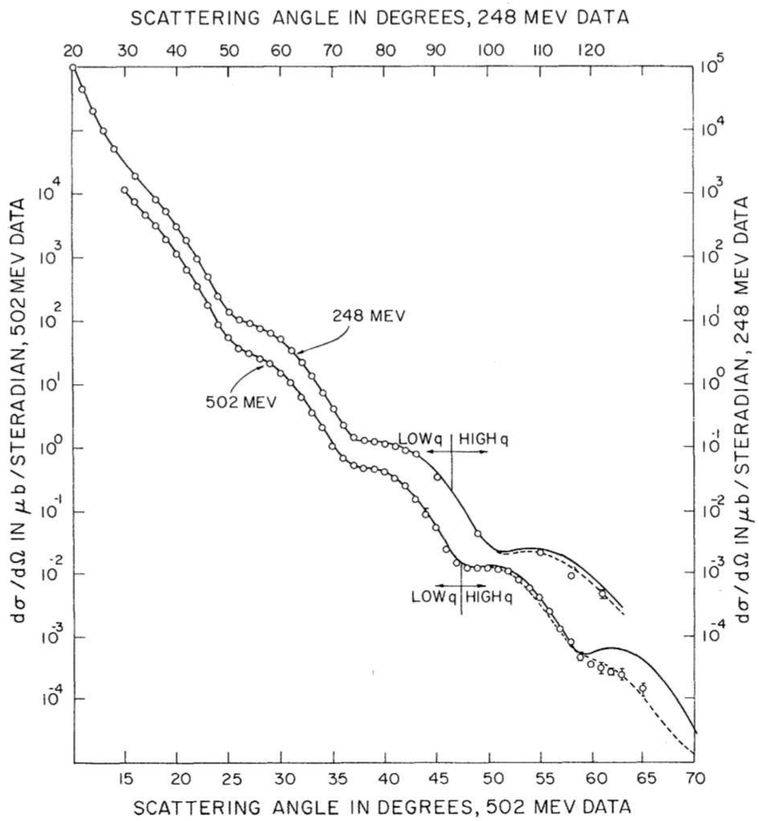

$q_{max} \sim 1.5(2.0) fm^{-1}$ seems to be limit for A=12(208)

Implications for quantities that can be determined

experience from past fits to stable isotope data:
3PF, 3PG typically flexible enough for $q_{max} \sim 2fm^{-1}$

Examples: Mg, Pb [13, 14]




Data from ETIC-type collider can be described using 3pF, 3pG, ...

For exotic nuclei can determine $R$, $z$, central dip, large-$r$ tail

## (e,e) and trapped ions: conclusions

great interest + potential

## Limitations due to achievable luminosity

interpretation of data with models for $\rho(r)$
extraction of integral properties: $R$, $z$, dip, halo
look feasible given present estimates for L

## Precise results for $rms$-radius $R$

need precise (absolute) cross sections
only achievable via *ratio* measurements to stable isotopes
requires precise monitors for (relative) luminosity

## For precise extraction of $R$

use realistic large-$r$ tail of $\rho$ (not properly done in past)
reanalyze data for (stable) reference nucleus

## For nuclei far from valley

must expect halo's, *i.e.* long large$r$ tail
parameterize as halo (R(r) from WS potential) + model-density
emphasizes interest in measuring SE via precise masses, (e,e'p)

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