NN scattering from the dispersive N/D method including two-pion exchange

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Outline

Introduction

NN interaction is important for nuclear structure, nuclear reactions, nuclear matter, neutron star, nucleosynthesis, etc ...

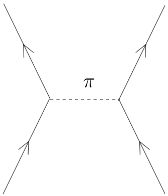
Application of Chiral Perturbation Theory (ChPT) to *NN*
S. Weinberg, PLB **251** (1990) 288; NPB **363** (1991) 3; PLB **295** (1992) 114.

Weinberg's counting: Calculate the two-nucleon irreducible graphs in ChPT (the *NN* potential V_{NN}) and then solve the Lippmann-Schwinger (LS) equation

$$T_{NN}(\mathbf{p}',\mathbf{p}) = V_{NN}(\mathbf{p}',\mathbf{p}) + \int d\mathbf{p}'' V_{NN}(\mathbf{p}',\mathbf{p}'') \frac{m}{\mathbf{p}^2 - \mathbf{p}''^2 + i\epsilon} T_{NN}(\mathbf{p}'',\mathbf{p})$$

C. Ordóñez, L. Ray and U. van Kolck, PRL **72** (1994) 1982; PRC **53** (1996) 2086.

In 1935 H. Yukawa introduced the pion as the carrier of the strong nuclear force



The pion mass was inferred from the range of strong nuclear forces

This was estimated from the radius of the atomic nucleus
Relativistic-Quantum-Mechanical argument

Thanks to ChPT we can calculate TPE and its role in *NN* scattering is also well established N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A **625** (1997) 758.

Heisenberg uncertainty principle: $\Delta t \Delta E \geq \hbar$

Relativity: Velocity of light is the Maximum velocity c

$$\Delta t \Delta E = \frac{\Delta \ell}{c} \Delta E \geq \hbar$$
$$\Delta E = \frac{\hbar c}{\Delta \ell}$$

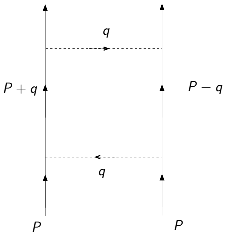
$\Delta \ell \sim 2$ fm $\quad (1 \text{ fm} = 10^{-15} \text{ m})$

$$m_\pi \sim \frac{\hbar c}{2 \text{ fm}} \sim 100 \text{ MeV}$$

$m_\pi = 138$ MeV

- A typical three-momentum cut-off $\Lambda \sim 600$ MeV (fine tuned to data) is used in order to regularize the Lippmann-Schwinger equation because chiral potentials are singular.
E.g. The tensor part of One-Pion Exchange (OPE) diverges as $1/r^3$ for $r \to 0$

- *NN* scattering is nonperturbative: Presence of bound states (deuteron) in $^3S_1$ and anti-bound state in $^1S_0$.
Spectroscopic notation $^{2S+1}L_J$



$$\int d^4q\,(q^0 + i\epsilon)^{-1}(q^0 - i\epsilon)^{-1}(q^2 + m_\pi^2)^{-2}\,P(q)$$

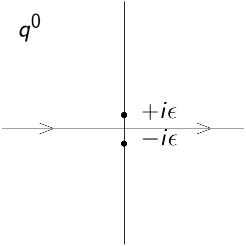
Infrared enhancement
$1/|\mathbf{q}| \to 1/|\mathbf{q}| \times m/|\mathbf{q}|$.
$1/q^0 \to 1/[q^0 - \mathbf{q}^2/(2m)]$,
non-relativistic nucleon propagator

Extreme non-relativistic propagator (or Heavy-Baryon propagator)

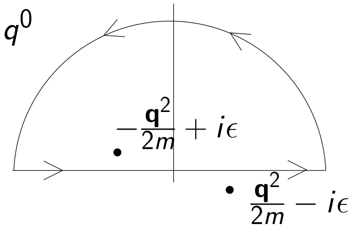
$$\frac{1}{q^0 + i\epsilon}$$

Non-relativistic propagator

$$\frac{1}{q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon}$$



"Pinch" singularity
The integration contour cannot be deformed



$$\int dq^0 \, (q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon)^{-1}(q^0 + \frac{\mathbf{q}^2}{2m} - i\epsilon)^{-1} = -2\pi i \frac{m}{\mathbf{q}^2}$$

- V_{NN} is calculated up to next-to-next-to-next-to-leading order (N^3LO) and applied with great phenomenological success

Entem and Machleidt, PLB **254** (2002) 93; PRC **66** (2002) 014002; PRC **68** (2003) 041001
Epelbaum, Glöckle, Meißner, NPA **637** (1998) 107; **671** (2000) 195; **747** (2005) 362

- **On the cut-off dependence**

Chiral counterterms introduced in V_{NN} following naive chiral power counting are not enough to reabsorb the dependence on cut-off when solving the LS equation

Nogga, Timmermans and van Kolck, PRC **72** (2005) 054006
Pavón Valderrama and Arriola, PRC **72** (2005) 054002; **74** (2006) 054001; **74** (2006) 064004
Kaplan, Savage, Wise NPB **478** (1996) 629
Birse, PRC **74** (2006) 014003 ; C.-J. Yang, Elster and Phillips, PRC **80** (2009) 034002; *idem* 044002.

▷ In Nogga *et al.* one counterterm is promoted from higher to lower orders in $^3P_0$, $^3P_2$ and $^3D_2$ and then stable results for $\Lambda < 4$ GeV are obtained.

▷ *Higher order contributions could be treated perturbatively*

Pavón Valderrama, PRC **83** (2011) 024003; **84** (2011) 064002
B. Long, C.-J. Yang, PRC **84** (2011) 057001; **85** (2011) 034002; **86** (2012) 024001

N/D Method

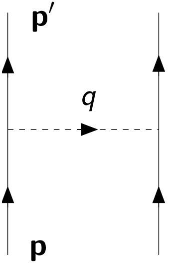
Chew and Mandelstam, Phys. Rev. **119** (1960) 467

A *NN* partial wave amplitude has two type of cuts:

Unitarity or Right Hand Cut (RHC)

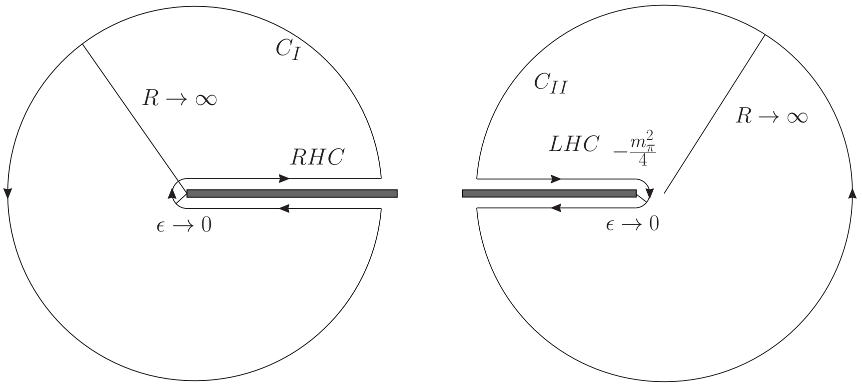
$$\Im T = \frac{m|\mathbf{p}|}{4\pi} T T^\dagger \quad , \ \mathbf{p}^2 > 0$$

Left Hand Cut (LHC)



$$\frac{1}{(\mathbf{p}-\mathbf{p}')^2 + m_\pi^2}$$

$$\mathbf{p}^2 = -\frac{m_\pi^2/2}{1-\cos\theta} \to \mathbf{p}^2 \in]-\infty, -m_\pi^2/4]$$



$$T_{J\ell S}(A) = \frac{N_{J\ell S}(A)}{D_{J\ell S}(A)}$$

$N_{J\ell S}(A)$ has Only LHC

$D_{J\ell S}(A)$ has Only RHC

• In connection with ChPT this dispersive method was recently applied to NN scattering in

M. Albaladejo and J.A. Oller, PRC **84** (2011) 054009; **86** (2011) 034005

employing OPE

• An alike N/D method was later used in

A. M. Gasparyan, M. F. M. Lutz and E. Epelbaum, arXiv:1212.3057

Once-iterated OPE and irreducible TPE are considered

They make reasonable assumptions but

- LHC integrals of infinity extension are truncated
  This is why there are insensitive to the divergent behavior of $\Delta(A)$ from TPE when $A \to -\infty$
- Analytical properties are lost
- Particular way to parameterize local terms

Uncoupled Partial Waves

$$T_{J\ell S}(A) = N_{J\ell S}(A)/D_{J\ell S}(A)$$
$$\Im D_{J\ell S}(A) = -N_{J\ell S}(A)\frac{m\sqrt{A}}{4\pi}\ ,\ \ A > 0$$
$$\Im N_{J\ell S}(A) = D_{J\ell S}(A)\,\Im T_{J\ell S}(A)\ ,\ \ A < -m_\pi^2/4$$
$$A \equiv |\mathbf{p}|^2$$

E.g. taking one subtraction in $D(A)$ and $N(A)$

$$\oint_{C_I} dz \frac{D_{J\ell S}(z)}{(z-A)(z-D)} = 2\pi i \frac{D_{J\ell S}(A) - D_{J\ell S}(D)}{A-D}$$
$$= \int_0^\infty dq^2 \frac{\left[D_{J\ell S}(q^2+i\epsilon) - D_{J\ell S}(q^2-i\epsilon)\right]}{(q^2-A+i\epsilon)(q^2-D+i\epsilon)}$$

Schwartz's reflection principle:

If $f(z)$ is real along an interval of the real axis and is analytic then: $f(z^*) = f(z)^*$

$$D_{J\ell S}(q^2+i\epsilon)-D_{J\ell S}(q^2-i\epsilon)=2i\Im D(q^2+i\epsilon)$$

COUPLED SYSTEM OF LINEAR INTEGRAL EQUATIONS

$$D_{J\ell S}(A)=1-\frac{A-D}{\pi}\int_0^\infty dq^2\frac{\rho(q^2)N_{J\ell S}(q^2)}{(q^2-A)(q^2-D)}$$

$$N_{J\ell S}(A)=N_{J\ell S}(D)+\frac{A-D}{\pi}\int_{-\infty}^{L}dk^2\frac{\Delta_{J\ell S}(k^2)D_{J\ell S}(k^2)}{(k^2-A)(k^2-D)}$$

$$L\equiv-\frac{m_\pi^2}{4}$$

$$\rho(A)=m\sqrt{A}/4\pi\ ,\ \ A>0$$

$$\Delta(A)=\Im T_{J\ell S}(A)\ ,\ \ A<L$$

$$D_{J\ell S}(A) = 1 - A N_{J\ell S}(0)\mathbf{g}(\mathbf{A},\mathbf{0}) + \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{J\ell S}(k^2) D_{J\ell S}(k^2)}{k^2}\mathbf{g}(\mathbf{A},\mathbf{k^2})$$

$$\mathbf{g}(\mathbf{A},\mathbf{k^2}) = \frac{1}{\pi}\int_0^{+\infty} dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$

$$\text{Convergent, } \quad \rho(A) \propto \sqrt{A}$$

Change of variable:

$$A = \frac{L}{x}\ , \ \ x \in [1,0]$$

$$D_{J\ell S}(x) = 1 - \frac{L}{x} N_{J\ell S}(0)\mathbf{g}(\mathbf{x},\mathbf{0}) + \frac{L}{\pi x}\int_0^1 dy \frac{\Delta(y)\mathbf{g}(\mathbf{x},\mathbf{y})}{y} D(y)$$

Fredholm Integral Equation of the Second Kind

$$D_{J\ell S}(x) = f_{J\ell S}(x) + \int_0^1 dy\, K(x,y) D(y)$$
$$K(x,y) = \frac{L}{\pi} \frac{\mathbf{g}(\mathbf{x},\mathbf{y})}{x\, y} \Delta(y)$$

- Not L_2
- Not symmetric

We discretize the equation:

$$\left.\begin{aligned} K(x,y) &= k_{rs} \ \left(\tfrac{r-1}{n} < x \le \tfrac{r}{n}, \tfrac{s-1}{n} < y \le \tfrac{s}{n}\right) \\ f(x) &= f_r \ \left(\tfrac{r-1}{n} < x \le \tfrac{r}{n}\right) \\ \phi(x) &= \phi_r \ \left(\tfrac{r-1}{n} < x \le \tfrac{r}{n}\right) \end{aligned}\right\} \ (r,\ s = 1, 2, \ldots, n)$$

$$\sum_{s=1}^{n} \left(\delta_{rs} - \frac{1}{n} k_{rs}\right) \phi_s = f_r$$

We indeed make use of more efficient numerical methods to calculate integrals !

High-Energy behavior

- Let $|D(A)| \leq A^n$ for $A \to \infty$

$$N(A) = T(A)D(A)$$
$$T(A) = \frac{S(A)-1}{2\rho(A)}$$
$$N(A) \leq A^{n-1/2}$$

We divide $N(A)$ and $D(A)$ by $(A-C)^m$ with $m > n$

$$\frac{D(A)}{A^m} \to 0 \ , \ \text{when } A \to \infty$$
$$L < C < 0$$

$$d(A) = \frac{D(A)}{(A-C)^m}$$
$$n(A) = \frac{N(A)}{(A-C)^m}$$

Unsubtracted dispersion relation (DR)

$$d(A) = \sum_{i=1}^{m} \frac{\delta_i}{(A-C)^i} - \frac{1}{\pi}\int_0^\infty dq^2 \frac{\rho(q^2)n(q^2)}{q^2-A}$$
$$n(A) = \sum_{i=1}^{m} \frac{\nu_i}{(A-C)^i} + \frac{1}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)d(k^2)}{k^2-A}$$

In terms of the original functions $D(A)$ and $N(A)$

$$D(A) = \sum_{i=1}^{m} \delta_i (A-C)^{m-i} - \frac{(A-C)^m}{\pi} \int_0^\infty dq^2 \frac{\rho(q^2) N(q^2)}{(q^2-A)(q^2-C)^m}$$

$$N(A) = \sum_{i=1}^{m} \nu_i (A-C)^{m-i} + \frac{(A-C)^m}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2-A)(k^2-C)^m}$$

$m = 1$ IS THE MINIMUM

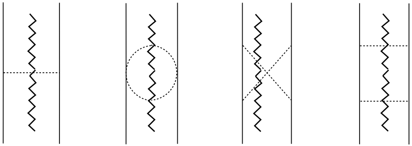
Once-subtracted DR for $N(A)$ and $D(A)$

- C could be taken different in $D(A)$ and $N(A)$
  - $N(A)$: $C = 0$
  - $D(A)$: One subtraction at $C = 0$ and the rest at $C \simeq -m_\pi^2$.

In our study $\Delta(A)$ is given by $\frac{-i}{2}$ the discontinuity across the LHC:

- OPE
- Leading TPE (irreducible)
- Once-iterated OPE

Kaiser, Brockmann and Weise, NPA625(1997)758



$\Delta(A)$ is finite

$$\lim_{A\to\infty} \Delta(A) \to A$$

For once-subtracted DR:

- $D(A)$ should decrease $1/A^{\alpha}$, $\alpha > 0$, for $A \to \infty$
- $N(A)$ should decrease as $1/A^{\alpha+\frac{1}{2}}$ for $A \to \infty$

Threshold Behavior: $\ell \geq 2$

A partial wave amplitude must vanish at least like A^ℓ for $A \to 0$

This implies ℓ constraints on the subtractions

$$\nu_i = 0 \quad (i = m,\ m-1,\ \ldots, m+1-\ell)$$

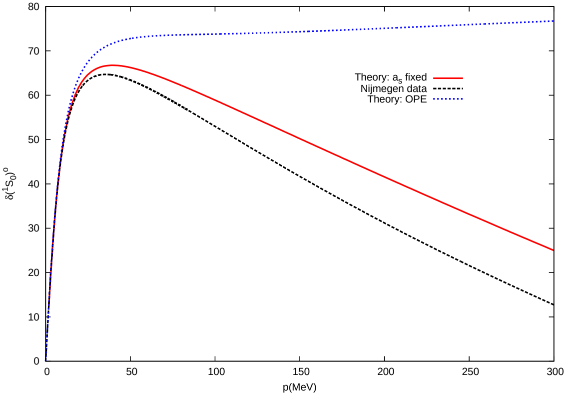
$$N(A) = \sum_{i=1}^{m-\ell} \nu_i A^{m-i} + \frac{A^m}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2 - A)(k^2)^m}$$

$$D(0) = 1$$

Once-subtracted DR

$$D(A) = 1 - A\nu_1 g(A,0) + \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2} g(A,k^2)$$

Fixed in terms of scattering length: $\nu_1 = -4\pi a_s/m$



$$r_s = \frac{m}{2\pi^2 a_s}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2}\left\{\sqrt{-k^2} - \frac{1}{a_s}\right\}$$

Correlation between a_s and r_s

$$r_s = 2.64 \text{ fm}$$

Exp: 2.75 ± 0.05 fm

NijmII: 2.670 fm Arriola, Pavón, nucl-th/0407113

$$-\frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2} g(A,k^2) + D(A) = 1 + A\frac{4\pi a_s}{m} g(A,0)$$

$D(A) = D_0(A) + a_s D_1(A)$ with $D_{0,1}(A)$ independent of a_s

Low-energy theorem:

$$r_s = \alpha_0 + \frac{\alpha_{-1}}{a_s} + \frac{\alpha_{-2}}{a_s^2} \ ,$$

$$\alpha_0 = 2.44 \ \mathrm{fm} \ ,$$

$$\alpha_{-1} = -4.61 \ \mathrm{fm}^2 \ ,$$

$$\alpha_{-2} = 5.26 \ \mathrm{fm}^3 \ .$$

$$\alpha_0 = \frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_1(k^2)}{(k^2)^2}\sqrt{-k^2}$$

$$\alpha_{-1} = \frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\left[D_0(k^2)\sqrt{-k^2} - D_1(k^2)\right]$$

$$\alpha_{-2} = -\frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_0(k^2)}{(k^2)^2}$$

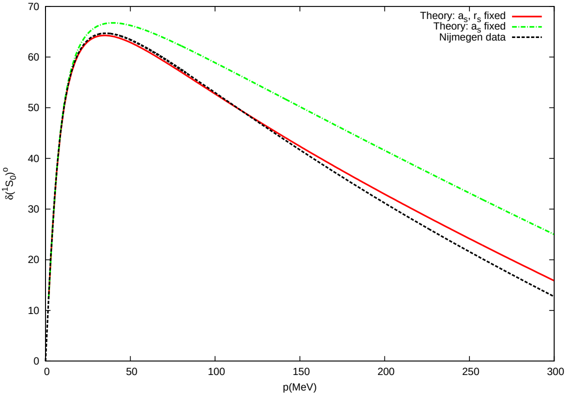
Pavón Valderrama, Ruiz Arriola PRC74(2006)054001: solving a Lippmann-Schwinger equation with V_{NN} that includes OPE+TPE + boundary conditions + orthogonality of wave functions

Twice-subtracted DR: a_s and r_s fixed— ν_2 is fitted

$$D(A) = 1 + A\left\{\frac{a_s}{m_\pi}(1 - \frac{1}{2} r_s m_\pi) + \frac{\nu_2}{\nu_1}\left[1 + \nu_1 m_\pi^2 g(0, -m_\pi^2)\right]\right\}$$

$$- A(A + m_\pi^2)\left[\nu_2 g(A, -m_\pi^2) - \nu_1 \frac{g(A, -m_\pi^2) - g(A, 0)}{m_\pi^2}\right]$$

$$+ \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^2}\left\{\frac{A + m_\pi^2}{k^2 + m_\pi^2}\left[k^2 g(A, k^2) + m_\pi^2 g(A, -m_\pi^2)\right] - m_\pi^2 g(k^2, -m_\pi^2)\right\}$$



Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta(k^2)D(k^2)}{(k^2)^2}\int_0^\infty dq^2\,\frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$

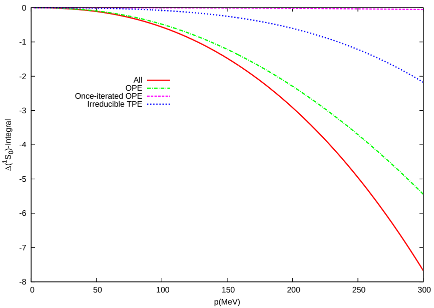
$$D(k^2)\to 1$$

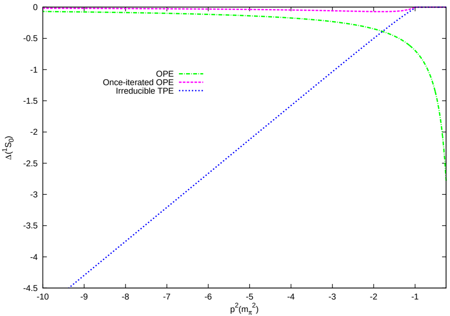
Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$

The integral displays the dominant role played by the nearest region in the LHC

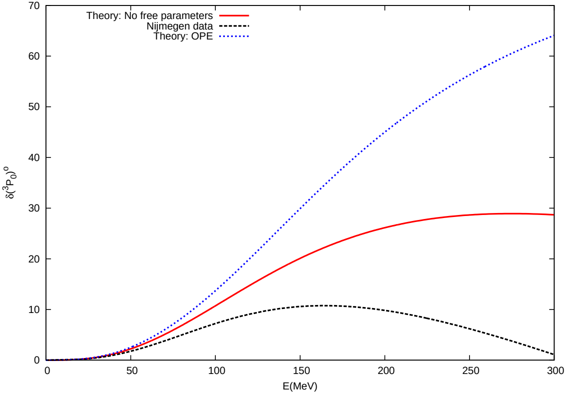




$^3P_0$: **Once-subtracted DR.** No Free Parameters

$\nu_1 = 0$ because for a P-wave $T(0) = 0 = N(0)$, $D(0) = 1$

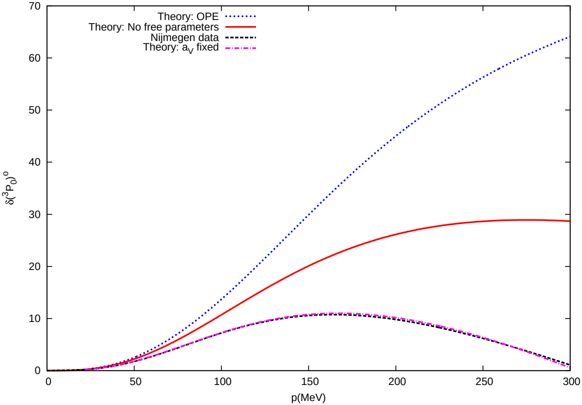
$$D(A) = 1 - \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2} g(A,k^2) \qquad N(A) = \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2(k^2-A)}$$



Twice-subtracted DR

$$N(A) = A\nu_2 + \frac{A^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2(k^2-A)} \qquad \nu_2 = \frac{4\pi a_V}{m}\ ,\ a_V = 0.89\ m_\pi^{-3}$$

$$D(A) = 1 + A\delta_1 - A^2\nu_2 g(A,0) + \frac{A^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} g(A,k^2)$$



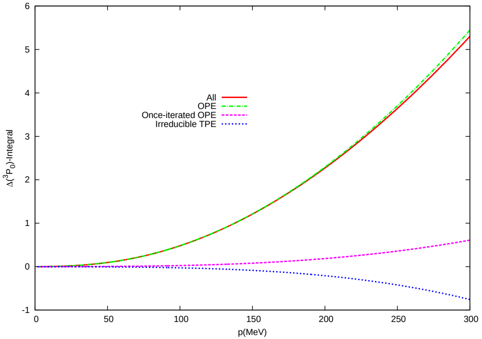
δ_1 is fitted

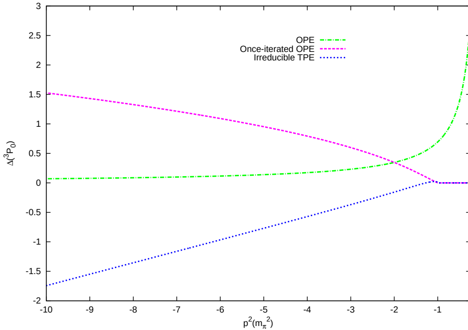
$\delta_1 \simeq -0.30\ M_\pi^{-2}$

Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$



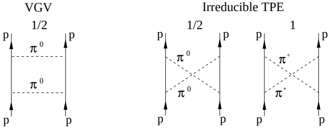


Discussion on LHC and chiral counting

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$

1) Low-energy enhancement in the integrand $1/(k^2)^2$
2) From $-m_\pi^2/4$ to $-m_\pi^2$ large OPE $\Delta(A)$
   OPE dominates the integral.
   Typical value of derivative $1/A^2 \to 16/m_\pi^4$ in an interval of length $3/4m_\pi^2$ relative change ~ 3 (quite steep function)
3) Rapid convergence pattern at low energies:
   $1\pi \gg 2\pi \gg 3\pi > \ldots > n\pi$
   $(e^{-m_\pi r} \gg e^{-2m_\pi r} \gg e^{-3m_\pi r} \gg \ldots)$ $\; r \gg m_\pi^{-1}$
4) $A < -m_\pi^2$ Numeric enhancement of irreducible TPE

- E.g. $pp \to pp$: Irre-TPE has a factor 3 relative to VGV



5) This numerical enhancement makes VGV and Irre-TPE to have similar size.

6) For a given $n\pi$-exchange: Higher order corrections would share this numerical enhancement. Subleading in the chiral counting $\to$ perturbative treatment.

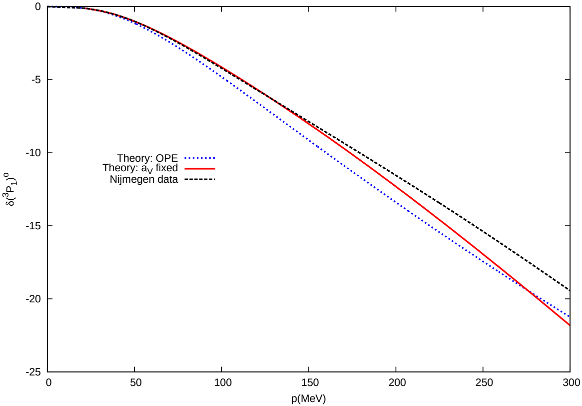
7) Increasing n in multi-π ladder: $VGV \cdots VGV$
Three-(n-)times iterated OPE gives rise to $3\pi(n\pi)$ cut for $A < -9m_\pi^2/4$ ($A < -n^2 m_\pi^2/4$)$\to$ Further suppressed $\sim$ size as irreducible contributions because of numerical enhancement of the latter.

8) We advocate for counting in $\Delta(A)$: each iteration GV as $\mathcal{O}(p^2) \sim$ extra loop in Irre-TPE

$^3P_1$: **Once-subtracted DR.** No solution. It depends on the integration numerical limit

Twice-Subtracted DR

$$\nu_2 = \frac{4\pi a_V}{m} \ , \ a_V = -0.54 \ m_\pi^{-3}$$



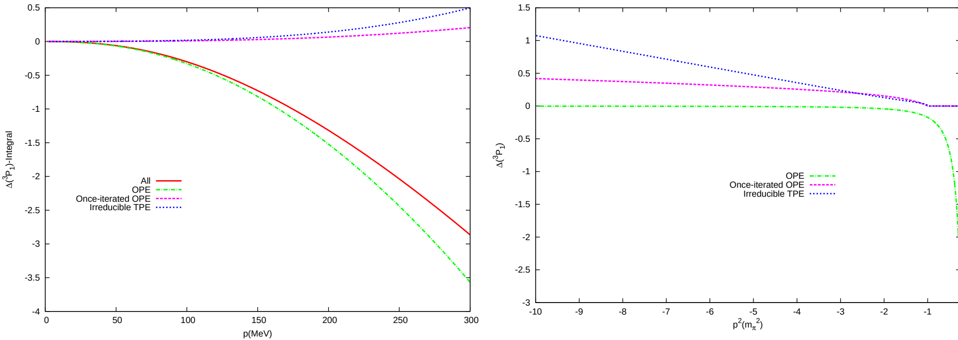
δ_1 is fitted

$\delta_1 \simeq 3.6 \ M_\pi^{-2}$

Quantifying contributions to $\Delta(A)$

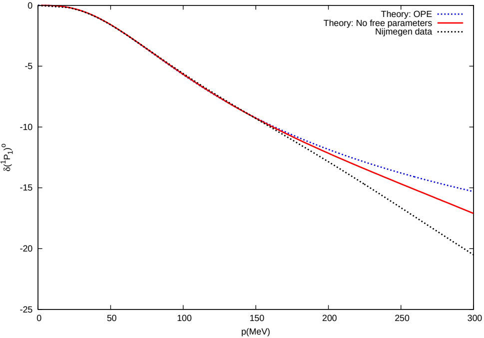
A typical integral from twice-subtracted DR:

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta(k^2)}{(k^2)^2}\int_0^{\infty} dq^2\,\frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$



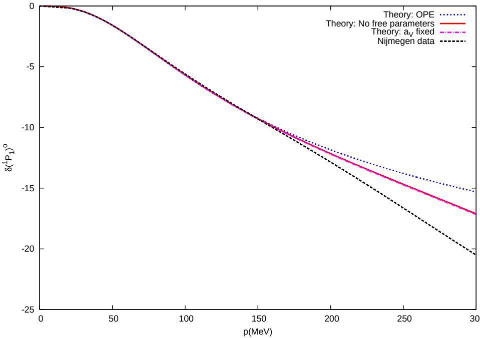
$^1P_1$: **Once-subtracted DR**:

No free parameters



Twice-Subtracted DR:

$a_V = -0.94\ m_\pi^{-3}$

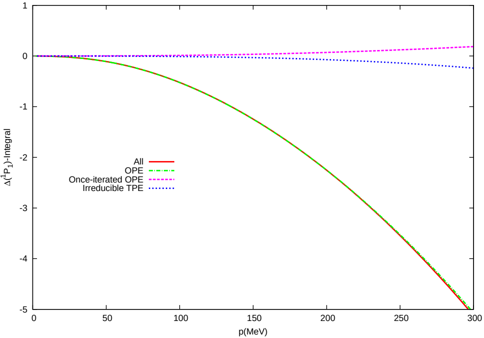


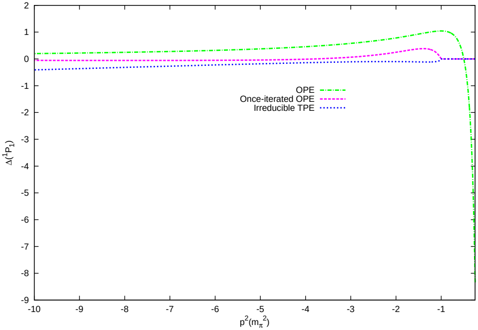
They are the same!

Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L}dk^2\,\frac{\Delta(k^2)}{(k^2)^2}\int_0^\infty dq^2\,\frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$





A partial wave should vanish as A^ℓ in the limit $A \to 0^+$ (threshold)

Method: ℓ-TIMES SUBTRACTED DR

$$N(A) = \frac{A^\ell}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D_{J\ell S}(k^2)}{(k^2)^\ell (k^2 - A)}$$

$$\lim_{A\to 0} N(A) \longrightarrow A^\ell$$

$$D(A) = 1 + \sum_{i=1}^{\ell-1} \delta_i A^i + \frac{A^\ell}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^\ell} g(A, k^2)$$

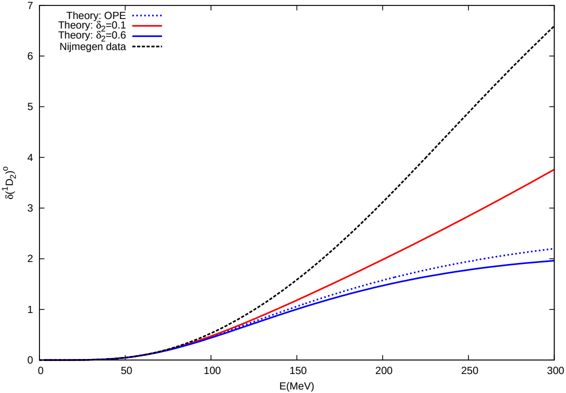
$$\lim_{A\to 0} D(A) \longrightarrow 1 + \mathcal{O}(A)$$

$$\lim_{A\to 0} \frac{N(A)}{D(A)} \longrightarrow A^\ell$$

Price to pay: $\ell - 1$ free parameters:
$\delta_i \quad (i = 1, \ldots, \ell - 1)$
Tend to become irrelevant as ℓ increases

$^1D_2$: Twice-subtracted DR

$$D(A) = 1 + A\delta_1 + \frac{A^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} g(A,k^2)$$



Results for $\delta_1 < 0$ are rejected.

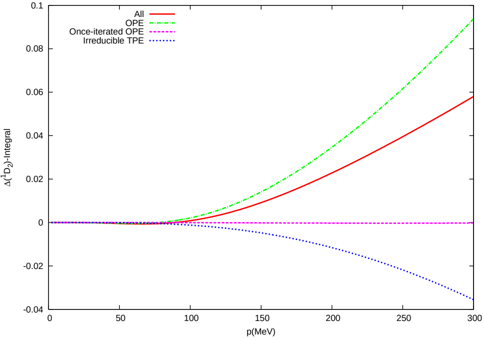
Low-energy resonance.

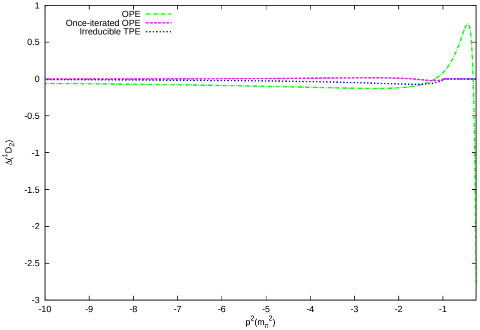
Allowed Range: $0 < \delta_1 < 0.6$

Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A^2}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^{\infty} dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$

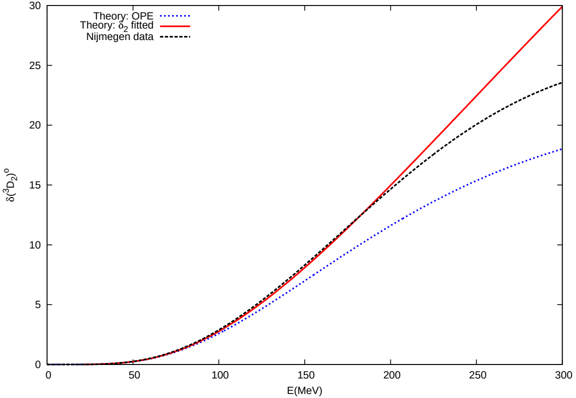




$^3D_2$: Twice-subtracted DR

$$D(A) = 1 + A\delta_1 + \frac{A^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} g(A, k^2)$$

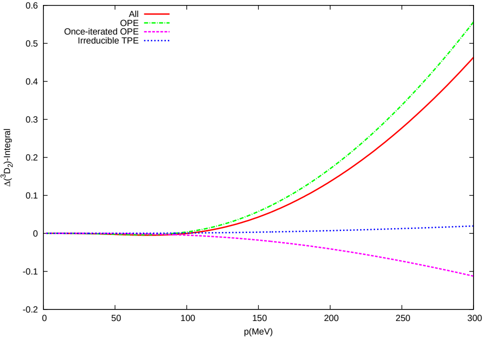
- $\delta_1 = -0.17$ is fitted to data $\sqrt{A} \leq 200$ MeV

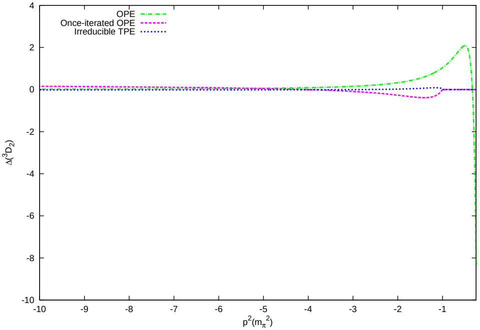


Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A^2}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^{\infty} dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$

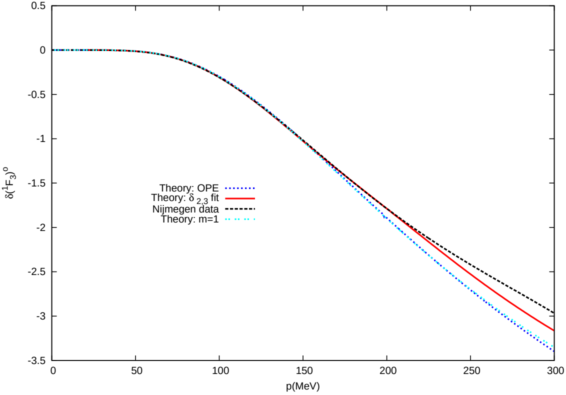




$^1F_3$: Three-times- subtracted DR

$$D(A) = 1 + A\delta_1 + A^2\delta_2 + \frac{A^3}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^3} g(A,k^2)$$

- $\delta_1 = -0.95,\ \delta_2 = 0.38$ is fitted to data $\sqrt{A} \leq 200$ MeV

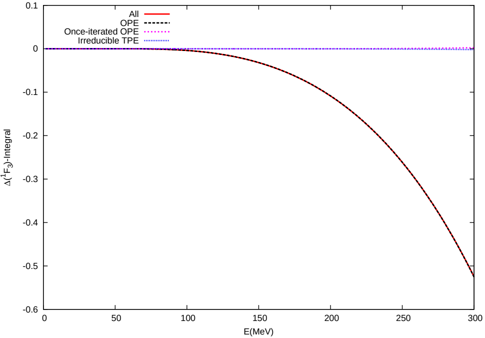


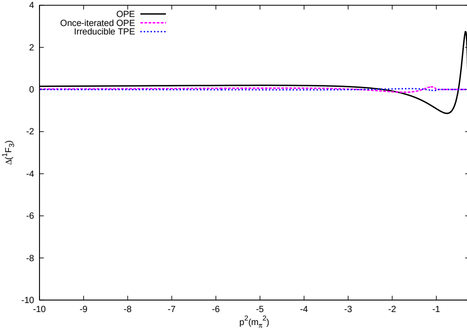
Cuve with Once-Subtracted DR , $m = 1$
Similar Results
F waves are perturbative

Quantifying contributions to $\Delta(A)$

A typical integral from three-times-subtracted DR:

$$\frac{A^3}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^3}\int_0^\infty dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$

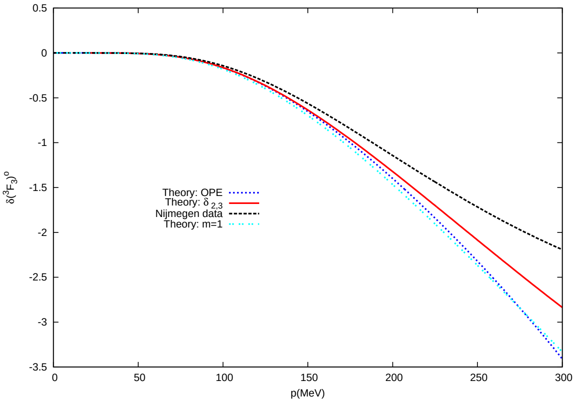




$^3F_3$: Three-times- subtracted DR

$$D(A) = 1 + A\delta_1 + A^2\delta_2 + \frac{A^3}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^3} g(A, k^2)$$

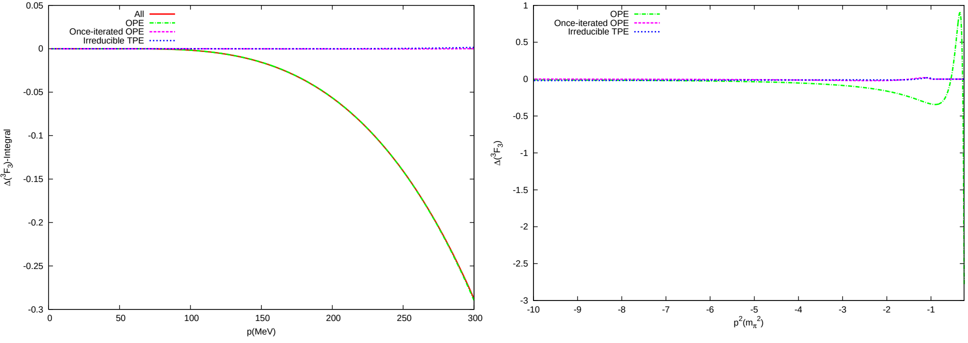
- $\delta_1 = 1^*$, $\delta_2 = 0.02^*$ FIXED
  Smaller δ_2 gives rise to resonances. δ_1 is left undetermined



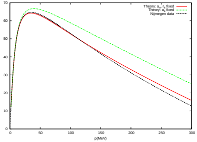
Quantifying contributions to $\Delta(A)$

A typical integral from three-times–subtracted DR:

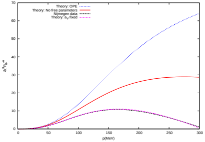
$$\frac{A^3}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^3}\int_0^{\infty} dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$



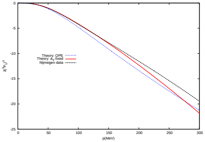
$^1S_0$:



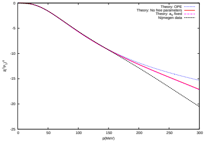
$^3P_0$:



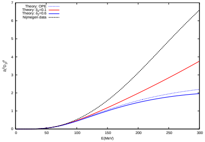
$^3P_1$:



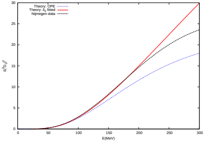
$^1P_1$:



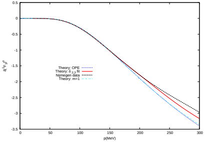
$^1D_2$:



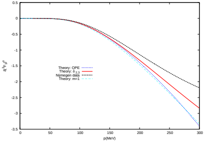
$^3D_2$:



$^1F_3$:



$^3F_3$:



Coupled Waves

$$S_{JIS} = I + i\frac{|\mathbf{p}|m}{4\pi}T$$

Along the RHC $A \geq 0$

$$S_{JIS} \cdot S^{\dagger}_{JIS} = S^{\dagger}_{JIS} \cdot S_{JIS} = I$$

$$S_{JIS} = \begin{pmatrix} \cos 2\epsilon\, e^{i2\delta_1} & i \sin 2\epsilon\, e^{i(\delta_1+\delta_2)} \\ i \sin 2\epsilon\, e^{i(\delta_1+\delta_2)} & \cos 2\epsilon\, e^{i2\delta_2} \end{pmatrix} \quad , \quad |\mathbf{p}|^2 \geq 0$$

ϵ is the mixing angle: $i = 1$ $(\ell = J - 1)$, $i = 2$ $(\ell = J + 1)$

$$\text{Im}\frac{1}{T_{ii}(A)} = -\rho(A)\left[1 + \frac{\frac{1}{2}\sin^2 2\epsilon}{1 - \cos 2\epsilon\, \cos 2\delta_i}\right]^{-1} \equiv -\nu_{ii}(A)$$

$$\text{Im}\frac{1}{T_{12}(A)} = -2\rho(A)\frac{\sin(\delta_1 + \delta_2)}{\sin 2\epsilon} \equiv -\nu_{12}(A)$$

$$T_{ij}(A) = \frac{N_{ij}(A)}{D_{ij}(A)} \quad , \quad (ij = 11,\ 12,\ 22)$$

$$N_{ij}(A) = N(0)\delta_{\ell_{ij}0} + \frac{A^{\ell_{ij}}}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2) D_{ij}(k^2)}{(k^2)^{\ell_{ij}}(k^2 - A)}$$

$$D_{ij}(A) = 1 + \sum_{i=1}^{\ell_{ij}-1} A^i \delta_i - \frac{A(A-C)^{\ell_{ij}-1}}{\pi}\int_0^{\infty} dq^2 \frac{\nu_{ij}(q^2) N_{ij}(q^2)}{q^2(q^2-C)^{\ell_{ij}-1}(q^2-A)}$$

$$= 1 + \sum_{i=1}^{\ell_{ij}-1} A^i \delta_i + \frac{A(A-C)^{\ell_{ij}-1}}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2) D_{ij}(k^2)}{(k^2)^{\ell}_{ij}} \times$$

$$\times \int_0^{\infty} dq^2 \frac{\nu_{ij}(q^2)(q^2)^{\ell_{ij}-1}}{(q^2-A)(q^2-k^2)(q^2-C)^{\ell_{ij}-1}}$$

$$\lim_{A\to 0^+} \nu_{22}(A) \propto A^{-3/2}$$

$C \neq 0$ to avoid infrared divergences for $ij = 22$

One proceeds in a coupled-iterative way:

1. We take an input.
2. Solve the integral equations and get new $\nu_{ij}(A)$.
3. Repeat the process until convergence is obtained.

$$\boxed{\text{Typically, } C = -m_\pi^2}$$

${}^3P_2 - {}^3F_2$

- ${}^3P_2$: $\ell_{11} = 1$ Two types of DR are included:
  1. Minimal: Once-subtracted DR
  2. Twice-subtracted DR

$$\nu_2 = \frac{4\pi a_V}{m}\ ,\ a_V = 0.0964\ m_\pi^{-3}$$

- ${}^3P_2 - {}^3F_2$: $\ell_{12} = 2 \rightarrow$Twice-subtracted DR

$$\delta_1^{(12)} = \frac{D_{12}(C) - 1}{C}$$

- ${}^3F_2$: $\ell_{22} = 3 \rightarrow$ Three-times subtracted DR

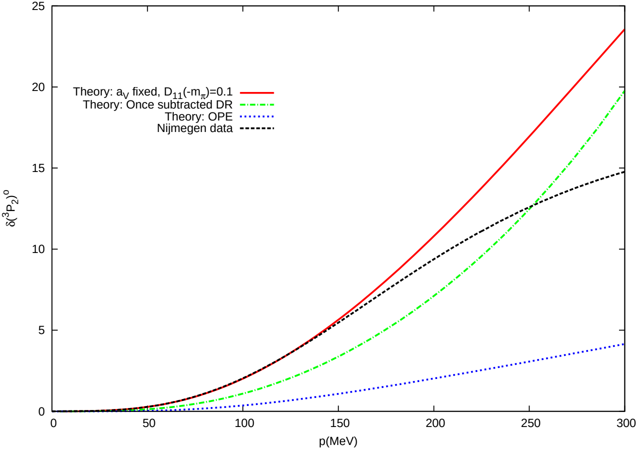
$$\delta_1^{(22)} = -\frac{2 - 2D_{22}(C) + CD'_{22}(C)}{C}$$

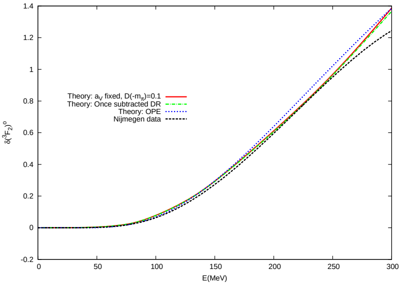
$$\delta_2^{(22)} = \frac{1 - D_{22}(C) + CD'_{22}(C)}{C^2}$$

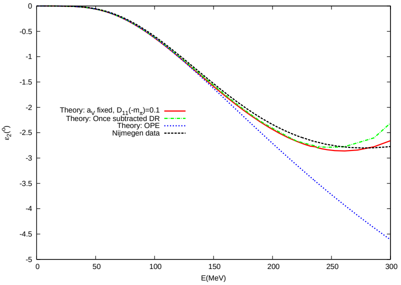
Results are very similar for $D'_{22}(C) \lesssim -1$ and insensitive to $D_{22}(C)$ fixed to 1^*

Phase shifts

From fit to data: $D_{11}(-m_\pi^2) \simeq 0.1$, $\delta_1^{(12)} \simeq -0.1\ m_\pi^{-2}$

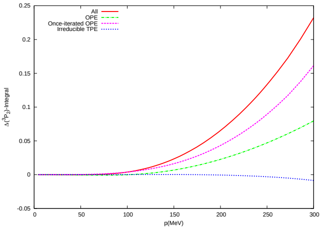


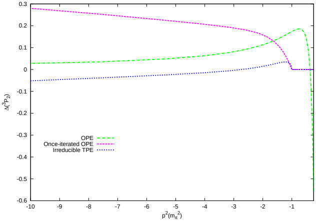

Theory: a_V fixed, $D(-m_\pi)$=0.1
Theory: Once subtracted DR
Theory: OPE
Nijmegen data
$\delta({}^3F_2)^o$
E(MeV)

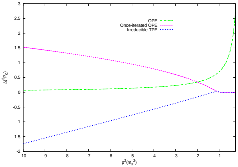

Theory: a_V fixed, $D_{11}(-m_\pi)$=0.1
Theory: Once subtracted DR
Theory: OPE
Nijmegen data
$\epsilon_2(^o)$
E(MeV)

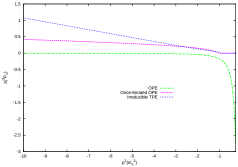
Quantifying contributions to $\Delta(A)$: $^3P_2$

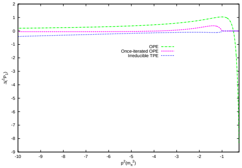
$$\frac{A^2}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2)}{(q^2-A)(q^2-k^2)}$$





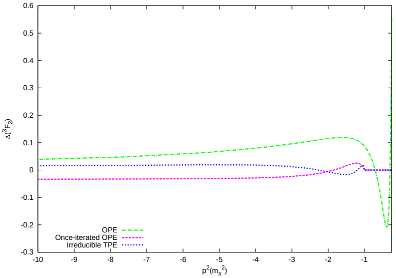


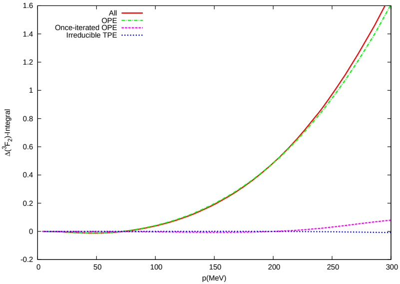




- The OPE contribution to $\Delta(A)$ for ${}^3P_2$ has an anomalously small size compared to the other P-waves
- ${}^3F_2$

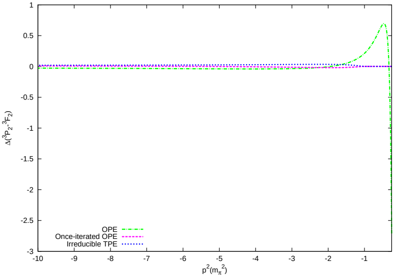
$$\frac{A(A+m_\pi^2)^2}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta(k^2)}{(k^2)^3}\int_0^\infty dq^2\,\frac{\nu_{22}(q^2)(q^2)^2}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)^2}$$

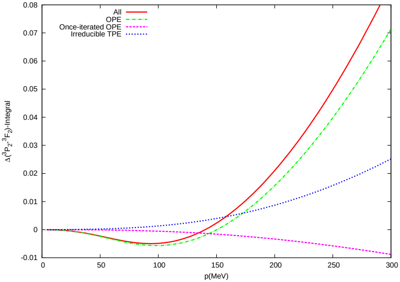




- $^3P_2 - {}^3F_2$

$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{\rho(q^2)q^2}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$





$^3D_3 - {}^3G_3$

- $^3D_2$: $\ell_{11} = 2 \to$ Twice-subtracted DR

$$\delta_1^{(11)} = \frac{D_{11}(C) - 1}{C}$$

- $^3D_3 - {}^3G_3$: $\ell_{12} = 3 \to$ Three-times subtracted DR

$$\delta_1^{(12)} = -\frac{2 - 2D_{12}(C) + CD'_{12}(C)}{C}$$

$$\delta_2^{(12)} = \frac{1 - D_{12}(C) + CD'_{12}(C)}{C^2}$$

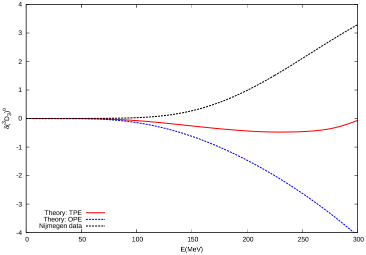
- $^3G_3$: $\ell_{22} = 4 \to$ Four-times subtracted DR

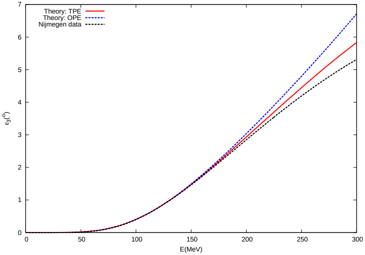
$$\delta_1^{(22)} = \frac{-6 + 6D_{22}(C) - 4CD'_{22}(C) + C^2 D''_{22}(C)}{2C}$$

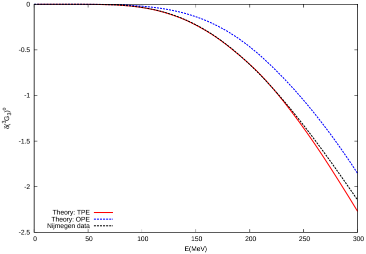
$$\delta_2^{(22)} = \frac{3 - 3D_{22}(C) + 3CD'_{22}(C) - C^2 D''_{22}(C)}{C^2}$$

$$\delta_3^{(22)} = \frac{-2 + 2D_{22}(C) - 2CD'_{22}(C) + C^2 D''_{22}(C)}{2C^3}$$

Phase shifts





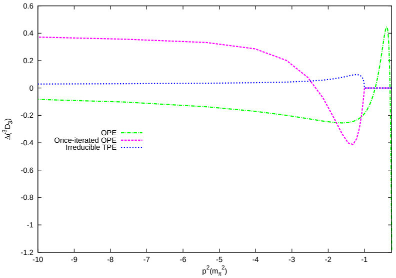


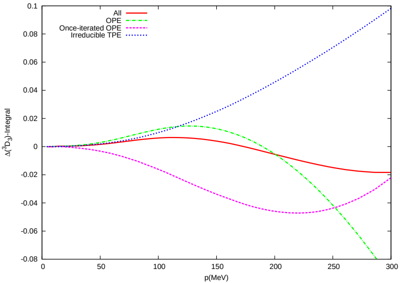
$$D_{11}(C) = 1^*$$
$$D_{12}(C) = 1^*,\ \ D'_{12}(C) = 0$$
$$D_{22}(C) = 0.91 \pm 0.15\ ,\ \ D'_{22}(C) = 0^*$$
$$D''_{22}(C) = 0^*$$

Quantifying contributions to $\Delta(A)$

- $^3D_3$

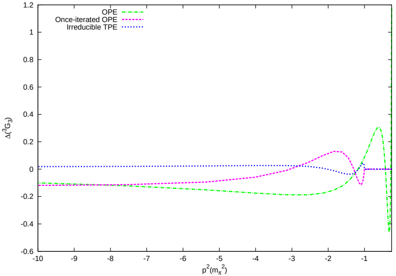
$$\frac{A(A+m_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta_{11}(k^2)}{(k^2)^2}\int_0^\infty dq^2\,\frac{\nu_{11}(q^2)q^2}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)}$$

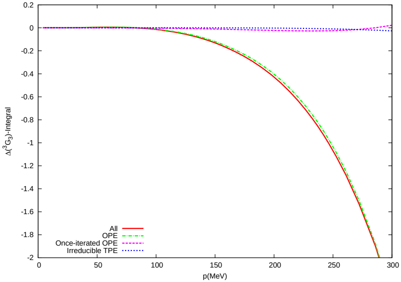




- ${}^3G_3$

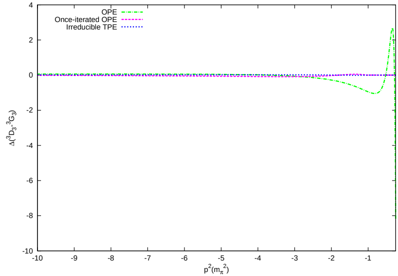
$$\frac{A(A+m_\pi^3)^3}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta_{22}(k^2)}{(k^2)^4}\int_0^\infty dq^2\,\frac{\nu_{22}(q^2)(q^2)^3}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)^3}$$

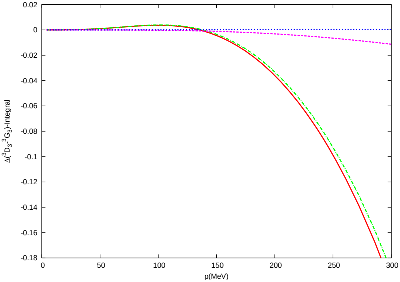




- $^3D_3 - {}^3G_3$

$$\frac{A(A+m_\pi^2)^2}{\pi^2}\int_{-\infty}^{L} dk^2\,\frac{\Delta_{12}(k^2)}{(k^2)^3}\int_0^\infty dq^2\,\frac{\rho(q^2)(q^2)^2}{(q^2-A)(q^2-k^2)(q^2+m_\pi^2)^2}$$





${}^3S_1 - {}^3D_1$

- ${}^3S_1$: The ${}^3S_1$ scattering length $a_t = 5.424$ fm is fixed
- ${}^3D_1$ and **mixing wave**: The deuteron is located at the same position as it is obtained in ${}^3S_1$.

$$D_{11}(A) = 1 + A\frac{4\pi a_t}{m} g_{11}(A,0) + \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2) D_{11}(k^2)}{k^2} g_{11}(A,k^2)$$

$$g_{11}(A,k^2) = \frac{1}{\pi}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2)}{(q^2-A)(q^2-k^2)}$$

For $(i,j=1 \text{ or } 2)$: $k_D^2 = -E_D({}^3S_1)/m$

$$D_{ij}(A) = 1 - \frac{A}{k_D^2} + \frac{A(A-k_D^2)}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2) D_{ij}(k^2)}{k^{2\ell_{ij}}} g_{ij}^d(A,k^2)$$

$$g_{ij}^d(A,k^2) = \frac{1}{\pi}\int_0^{\infty} dq^2 \frac{\nu_{ij}(q^2) q^{2(\ell_{ij}-1)}}{(q^2-A)(q^2-k^2)(q^2-k_D^2)}$$

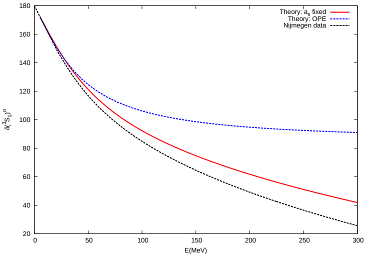
- There is dependence on the input used to solve the integral equations
- We require the maximum stability under changes in the input.

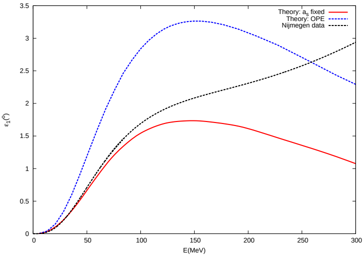
E.g.

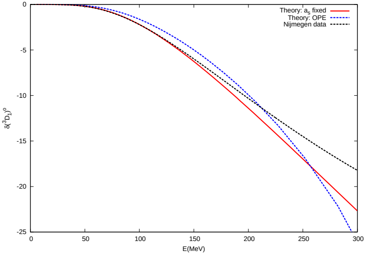
$$a_\epsilon \equiv \lim_{p\to 0} \frac{\sin \epsilon_1}{p^3} = 1.128\ m_\pi^{-3}|_{\text{experiment}}$$

has its minimum value for our best results $a_\epsilon = 1.1 - 1.14\ m_\pi^{-3}$

Phase Shifts:







- Great improvement of the OPE results

▷ **Deuteron binding energy:** $E_D = 2.37$ MeV,
experimentally $E_D = 2.22$ MeV,
with OPE we obtained $E_D = 1.7$ MeV.

▷ **Effective range:** $r_t = 1.36 - 1.39$ fm,
experimentally $r_t = 1.75$ fm,
with OPE we obtained $r_t = 0.46$ fm

$$r_t = -\frac{m}{2\pi^2 a_t}\int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2)D_{11}(k^2)}{(k^2)^2}\left\{\frac{1}{a_t} + \frac{4\pi k^2}{m} g_{11}(0,k^2)\right\}$$
$$- \frac{8}{m}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2) - \rho(q^2)}{(q^2)^2}$$
$$g_{11}(0,k^2) = \frac{1}{\pi}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2)}{q^2(q^2-k^2)}$$

More complicated correlation betwen r_t–a_t than in $^1S_0$: $\nu_{11}(A)$ depends nonlinearly on $D_{11}(A)$

Diagonalizing S-matrix

$$S = \mathcal{O} \begin{pmatrix} S_0 & 0 \\ 0 & S_2 \end{pmatrix} \mathcal{O}^T \qquad \mathcal{O} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$

Asymptotic D/S ratio of the deuteron

$$\eta = -\tan\varepsilon$$

Residue of S_0 at the deuteron pole

$$S_0 = \frac{N_p^2}{\sqrt{-k_D^2 + i\sqrt{A}}} + \mathrm{reg.terms}$$

Ours results: $\eta = 0.029 \quad , \quad N_p^2 = 0.73$

Other determinations:

Ericson, Rosa-Clot, 1983: $\eta = 0.02741(4)$

Conzett *et al.*, 1979: $\eta = 0.0263(13)$

Nijmegen PWA: $\eta = 0.02543(7) \quad , \quad N_p^2 = 0.7830(7)\ \mathrm{fm}^{-1}$

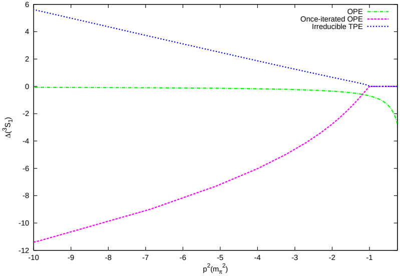
- We also tried other possibilities for the integral equations by including more subtractions
- **They did not work:**
  - Either the coupled-channel iterative process did not converge
  - Or it converged to the uncoupled-wave case

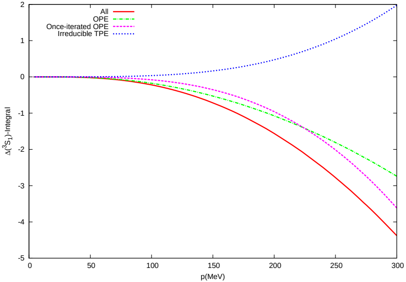
Case 1 Fixing from data: a_t and a_ϵ
Case 2 Fixing from data: a_t, r and E_d
Case 3 Fixing from data: a_t, r, E_d and a_ϵ

Quantifying contributions to $\Delta(A)$

- $^3S_1$

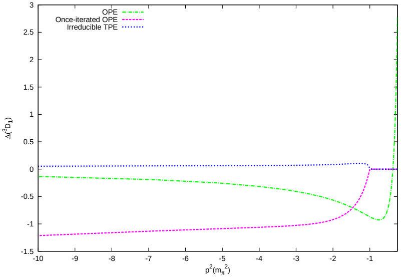
$$\frac{A^2}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2)}{(k^2)^2}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2)}{(q^2-A)(q^2-k^2)}$$

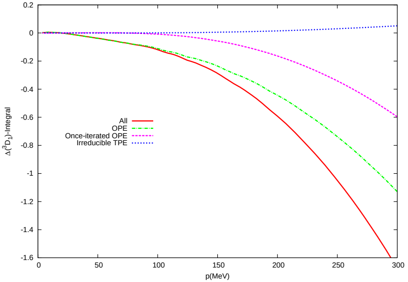




- $^3D_1$

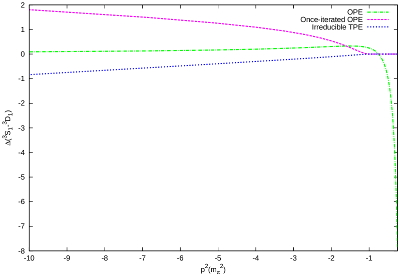
$$\frac{A(A-k_D^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{\nu_{22}(q^2)q^2}{(q^2-A)(q^2-k^2)(q^2-k_D^2)}$$

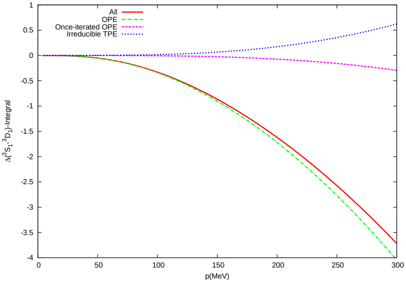




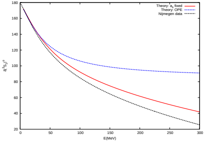
- ${}^3S_1 - {}^3D_1$

$$\frac{A(A-k_D^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta_{12}(k^2)}{(k^2)^2}\int_0^{\infty} dq^2 \frac{\rho(q^2)q^2}{(q^2-A)(q^2-k^2)(q^2-k_D^2)}$$

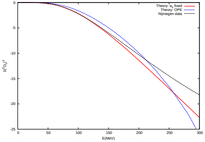




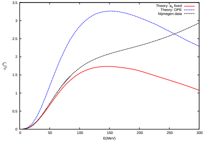
$^3S_1$:



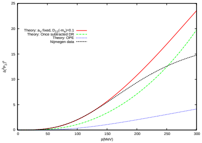
$^3D_1$:



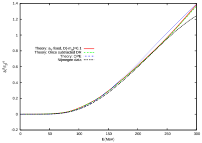
ϵ_1:



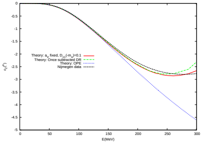
$^3P_2$:



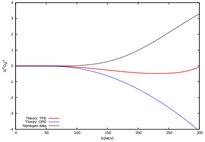
$^3F_2$:



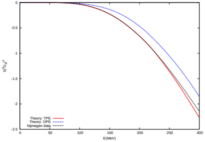
ϵ_2:



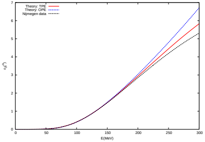
$^3D_3$:



$^3G_3$:



ϵ_3:



Conclusions:

1. Great improvement of the results from OPE to TPE. Our results typically reproduce data better than pure NLO Weinberg scheme.
2. Contributions to $D(A)$, $A>0$, from LHC integrals of $\Delta(A)$ are suitable for a chiral expansion:
   - OPE is $\mathcal{O}(p^0)$: Dominant.
   - Once-iterated OPE and irreducible TPE can be booked of the same size: Subleading.
3. We count iterated and irreducible two-pion loops on the same footing, $\mathcal{O}(p^2)$. Numerical enhancement of the latter.
4. Perturbative treatment of higher order contributions with a fixed number of exchanged pions.
5. Adding one more lowest-order pion ladder in reducible NN diagrams is suppressed by $\mathcal{O}(p^2)$

1. This should be further confronted with calculations of $\Delta(A)$ at $\mathcal{O}(p^3)$ and $\mathcal{O}(p^4)$.
2. At $\mathcal{O}(p^3)$ one only needs extra irreducible diagrams Kaiser, Brockmann and Weise, NPA625(1997)758.
3. At $\mathcal{O}(p^4)$ one also needs, among others, twice-iterated OPE (2 pion ladders)