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Renormalising nuclear forces: pions and their perturbations

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Review and further references:

M. C. Birse, Phil Trans Roy Soc A **369** (2011) 2662 [arXiv:1012.4914]

Effective field theories for nuclear forces

Weinberg (1990): extended chiral perturbation theory to two- and three-nucleon systems

- effective field theory expanded in powers of Q/Λ_0
  low-energy scales, Q: momenta, m_π ($\lesssim$ 200 MeV)
  scales of underlying physics, Λ_0: $4\pi F_\pi$, M_N, m_ρ ($\gtrsim$ 800 MeV)
- convergent expansion of potential and observables
  provided Q/Λ_0 is small enough (good separation of scales)
- terms organised by naive dimensional analysis
  aka “Weinberg power counting”
  (simply counts powers of low-energy scales)

Interactions with ranges $\sim 1/\Lambda_0$ not resolved at scales Q

- replaced by contact interactions
- infinite number of terms, constrained only by symmetries of QCD
- iterations (loop diagrams) usually infinite

$\longrightarrow$ need to renormalise

- we can, provided we have a consistent expansion

But...

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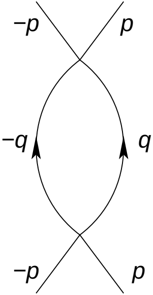
But. . .

- simply counting powers of low-energy scales: perturbative
- may work for weakly interacting systems: ≤ 1 nucleon
- but nucleons interact strongly at low-energies
- bound states exist (nuclei!)

$\longrightarrow$ need to treat some interactions nonperturbatively

Problem: basic nonrelativistic loop diagram of order Q

$$\frac{M}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{p^2-q^2+\mathrm{i}\varepsilon}=-\mathrm{i}\,\frac{Mp}{4\pi}+\text{analytic}$$



- lowest-order potential, Q^0: contact term and one-pion exchange
- each iteration suppressed by power of Q/Λ_0
- perturbative provided $Q<\Lambda_0$
- integral linearly divergent
- cut off (or subtract) at $q=\Lambda$
- still perturbative provided we keep $\Lambda<\Lambda_0$

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- expand potential to some order in Q
- then iterate to all orders in favourite dynamical equation (Schrödinger, Lippmann-Schwinger, ...)
- widely applied and even more widely invoked

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- but we need them to generate bound states (and we don’t want just to play “Steven says...”)

How can we iterate interactions consistently?

Identify new low-energy scales

- promote leading-order terms to order Q^{-1}

$\longrightarrow$ cancels Q from loop and so iterations not suppressed

- can, and must, then be iterated to all orders
  (all other terms: perturbations)

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Then use the renormalisation group to determine the power counting

- general tool for analysing scale dependence

The renormalisation group

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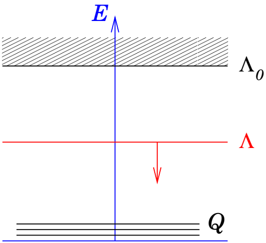

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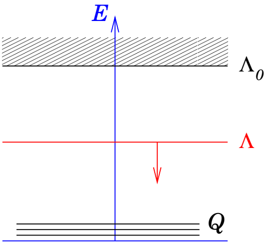

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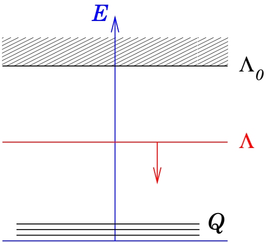

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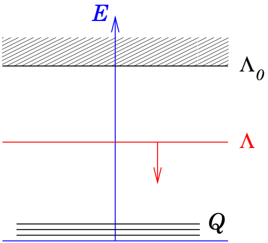

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- rescale express all dimensioned quantities in units of Λ
  (potential and all low-energy scales)

Λ is highest acceptable low-energy scale

- do not take it above breakdown scale Λ_0
  (unless you know the rules and follow them to the letter!)
- really of order Q
- rescaling $\longrightarrow$ power of Λ counts low-energy scales

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Endpoints of flow of effective potential as $\Lambda \to 0$

- fixed points: rescaled theories independent of Λ
  (except for some three-body systems $\longrightarrow$ limit cycles)
- correspond to scale-free systems
- expand around one using perturbations that scale like Λ^ν

$\longrightarrow$ EFT with power counting: Q^d where $d = \nu - 1$

RG for short-range potentials

- potential analytic in all scales Q

$$V(k',k,p;\Lambda) = b_{00}(\Lambda) + b_{20}(\Lambda)(k^2 + k'^2) + b_{02}(\Lambda)\,p^2 + \cdots$$

$k,\ k'$: initial, final relative momenta, $p = \sqrt{ME}$: on-shell momentum

- reactance matrix K satisfies Lippmann-Schwinger equation

$$K(k',k,p) = V(k',k,p;\Lambda) + \frac{M}{2\pi^2}\,\mathcal{P}\int_0^\Lambda q^2\mathrm{d}q\,\frac{V(k',q,p;\Lambda)K(q,k,p)}{p^2 - q^2}$$

$\mathcal{P}$: principal value (standing wave bc's)

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- demand full off-shell K matrix be independent of cutoff
$\partial K/\partial\Lambda = 0$
(need off-shell to disantangle scaling of redundant operators)

- leads to

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda)$$

  - RHS: contribution to scattering of states at cutoff $q = \Lambda$
  - removed from loop integrals as we lower Λ

$\longrightarrow$ effects added into the potential to compensate

- leads to

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2}\, V(k', \Lambda, p, \Lambda)\, \frac{\Lambda^2}{\Lambda^2 - p^2}\, V(\Lambda, k, p, \Lambda)$$

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  $\longrightarrow$ effects added into the potential to compensate

- rescale: express all low-energy scales in units of Λ: $\hat{k} = k/\Lambda$ etc and define $\widehat{V}(\hat{k}', \hat{k}, \hat{p}; \Lambda) = \frac{M\Lambda}{2\pi^2} V(\Lambda\hat{k}', \Lambda\hat{k}, \Lambda\hat{p}; \Lambda)$

$\longrightarrow$ RG equation [Birse, McGovern, Richardson (1998)]

$$\Lambda \frac{\partial \widehat{V}}{\partial \Lambda} = \hat{p}\, \frac{\partial \widehat{V}}{\partial \hat{p}} + \hat{k}'\, \frac{\partial \widehat{V}}{\partial \hat{k}'} + \hat{k}\, \frac{\partial \widehat{V}}{\partial \hat{k}} + \widehat{V} + \widehat{V}(\hat{k}', 1, \hat{p}; \Lambda)\, \frac{1}{1 - \hat{p}^2}\, \widehat{V}(1, \hat{k}, \hat{p}; \Lambda)$$

- two interesting fixed-point solutions $\partial \widehat{V}_0 / \partial \Lambda = 0$

Trivial fixed point $V_0 = 0$

Describes free particles (scale-free system)

Expansion around $V_0 = 0$ in powers of momenta

- p^{2n} is an eigenfunction of the RG equation: scales as Λ^{2n+1}
- order in EFT given by naive dimensional analysis: Q^{2n}
- perturbative $\longrightarrow$ appropriate EFT for weakly interacting systems

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Nontrivial fixed point

$$V_0(p,\Lambda) = -\frac{2\pi^2}{M\Lambda}\left[1 - \frac{p}{2\Lambda}\ln\frac{\Lambda+p}{\Lambda-p}\right]^{-1} \quad \text{(sharp cutoff)}$$

- order Q^{-1} (so must be iterated)

$\longrightarrow$ scatteringmatrix $T(p) = \mathrm{i}4\pi/Mp$

- describes “unitary limit”: scattering length $a \to \infty$ or bound state exactly at threshold (also scale-free)

Expanding around this point

$$V(p,\Lambda) = V_0(p,\Lambda) + V_0(p,\Lambda)^2 \frac{M}{4\pi} \left(-\frac{1}{a} + \frac{1}{2} r_e p^2 + \cdots \right)$$

- factor $V_0^2 \propto \Lambda^{-2}$ promotes terms by two orders compared to naive expectation: Q^{-2}, Q^0, ...
- coefficients of perturbations related to effective-range expansion [Bethe (1949)]

First example of new scales

NN scattering lengths $1/a \lesssim 40$ MeV

[van Kolck; Kaplan, Savage and Wise (1998)]

- for $p \ll m_\pi$ only contact interactions: "pionless EFT"
- (effective-range) expansion around unitary limit: $1/a \to 0$

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Promotion of potential follows from form of wave functions as $r \to 0$

- Schrödinger equation at zero energy for $r > 1/\Lambda_0$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]\psi_0(r) = 0 \qquad \text{(S wave)}$$

- unitary limit $\to$ irregular solutions: $\psi(r) \propto r^{-1}$
- cutoff smears contact interaction over range $R \sim \Lambda^{-1}$

$\to$ need extra factor Λ^{-2} to cancel cutoff dependence from $|\psi(R)|^2 \propto \Lambda^2$ in matrix elements of potential

One-pion exchange

- important for nuclear physics at energies $\sim$ 100 MeV
- order Q^0 in chiral counting

$\longrightarrow$ treat as a perturbation [Kaplan, Savage and Wise (1998)]

- S waves: series coverges slowly, if at all
  [Fleming, Mehen and Stewart (1999)]
- OPE "unnaturally" strong

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- strength of OPE set by scale

$$\lambda_{NN} = \frac{16\pi F_\pi^2}{g_A^2 M_N} \simeq 290 \text{ MeV}$$

built out of high-energy scales ($4\pi F_\pi$, M_N) but $\sim 2m_\pi$

$\longrightarrow$ another low-energy scale?

- promotes OPE to order Q^{-1}

Central potential

- Yukawa form

$$V_{\pi C}(r) = \frac{1}{3}\,\frac{m_\pi^2}{M_N \lambda_\pi}\,\frac{e^{-m_\pi r}}{r}\,(\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)$$

- behaves like $1/r$ for small r
- not singular enough to alter powers of r in wave functions

$\longrightarrow$ same power countings as for short-range potential alone
(except for a few additional logarithms)

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Tensor potential

- much more singular

$$V_{\pi T}(r) = \frac{1}{3}\,\frac{1}{M_N\lambda_\pi}\left(3+3m_\pi r+m_\pi^2 r^2\right)\frac{e^{-m_\pi r}}{r^3}\,S_{12}(\tau_1\cdot\tau_2)$$

- dominated by $1/r^3$ for small r

$\longrightarrow$ very different forms for wave functions

Schrödinger equation for spin-triplet channels at short distances
(uncoupled waves, keep only most singular term in potential)

- tends to energy-independent form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+1)}{r^2} + \frac{\beta_{LJ}}{r^3}\right]\psi_0(r) = 0 \qquad \beta_{LJ} \propto \frac{1}{\lambda_{NN}}$$

- can be converted to Bessel's equation by defining $x = 2\sqrt{|\beta_{LJ}|/r}$

$$\psi_0(r) \propto r^{-1/2}\left[\sin\alpha\, J_{2L+1}\left(2\sqrt{\frac{\beta_{LJ}}{r}}\right) + \cos\alpha\, Y_{2L+1}\left(2\sqrt{\frac{\beta_{LJ}}{r}}\right)\right]$$

- $\beta_{LJ} > 0$: solutions undetermined as $r \to 0$

$\longrightarrow$ α: fixes phase of short-distance oscillations
(self-adjoint extension of Hamiltonian)

- equivalent to fixing leading contact interaction

Attractive potential $\beta > 0$

- $r \ll 1/\beta$: oscillatory behaviour ($1/r^3$)

$$r^{-1/2} Y_{2L+1}\left(2\sqrt{\frac{\beta}{r}}\right) \sim r^{-1/4} \sin\left(2\sqrt{\frac{\beta}{r}} - \left(L + \frac{3}{4}\right)\pi\right)$$

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$$r^{-1/2} Y_{2L+1}\left(2\sqrt{\frac{\beta}{r}}\right) \sim r^{L}$$

- fine tune $\alpha = \pi/2$

$$r^{-1/2} J_{2L+1}\left(2\sqrt{\frac{\beta}{r}}\right) \sim r^{-(L+1)}$$

irregular solution for $r \gg 1/\beta \rightarrow$ unitary limit

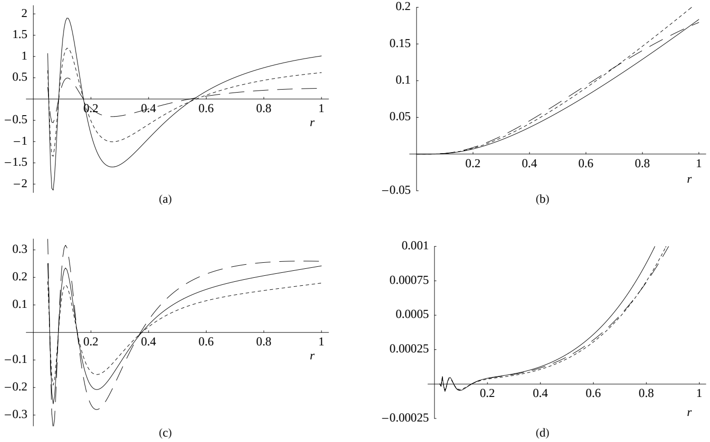
Repulsive case $\beta < 0$: modified Bessel functions

- $r \ll 1/\beta$: exponential behaviour (regular as $r \to 0$)

$$r^{-1/2} K_{2L+1}\left(2\sqrt{\frac{\beta}{r}}\right) \sim r^{-1/4} \exp\left(-2\sqrt{\frac{\beta}{r}}\right)$$

- $r \gg 1/\beta$: power law

$$r^{-1/2} K_{2L+1}\left(2\sqrt{\frac{\beta}{r}}\right) \sim r^{L}$$



Wave functions $\psi(r)/p^L$ for (a) $^3P_0$, (b) $^3P_1$, (c) $^3D_2$, (d) $^3G_4$.

Solid lines: energy-independent asymptotic form

Short-dashed lines: $T = 5$ MeV; long-dashed lines: $T = 300$ MeV

New power countings

Power-law behaviour of wave functions in presence of tensor OPE:

$$\psi(r) \sim r^{-1/4} \times \text{sine or exponential}$$

Renormalised contact interactions

- extra factor of $\Lambda^{-1/2}$ to cancel $|\psi(R)|^2 \propto \Lambda^{1/2}$

$\longrightarrow$ promoted by half order compared to naive dimensional analysis for S waves, but in all partial waves

- leading term: order $Q^{-1/2}$ (not quite relevant)
- matches results of full RG analysis [Birse (2006)]

RG analysis $\longrightarrow$ second fixed point

- unstable, like effective-range point
- contact interactions promoted by further power of Q^{-1}
- leading term: order $Q^{-3/2}$ $\longrightarrow$ iterate
- presumably related to bound state with momentum scale $p \lesssim \beta$
  equivalent to taking α close to $\pi/2$

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Leading term is relevant only in small regions of α

- explains "new leading order"
  [Nogga, Timmermans and van Kolck (2005)]
- also "plateaux" in Λ-dependence seen there
- need to fix α to get well-defined wave functions but away
  low-energy scattering depends weakly on α
  except around $\alpha = \pi/2$

$1/r^3$ “short-ranged” for a long-range potential

- nonperturbative region not resolved by long-wavelength S waves
- higher partial waves shielded by centrifugal barrier below critical momentum

$$p_c \sim [L(L+1)]^{3/2}/|\beta|$$

$\longrightarrow$ perturbative treatment of tensor potential for $p \ll p_c$

Critical momenta in chiral limit

Channel	p_c
$^3S_1$–$^3D_1$	66 MeV
$^3P_0$	182 MeV
other P, D waves	$\sim$ 400 MeV
F waves and above	$\gtrsim$ 2000 MeV

Summary

Identifying λ_{NN} as another low-energy scale justifies (requires) iteration of OPE

- scale $\propto \lambda_{NN}$ defining nonperturbative region depends on L
- “natural” systems: scattering depends weakly on short-distance parameter α or leading contact interactions
- “unnatural” systems: fine-tuned to give low-energy bound state (cf effective-range expansion around unitary limit)

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RG analysis

- gives the possible power countings
- can explain features seen by Nogga, Timmermans and van Kolck, and Pavòn Valderrama and Ruiz Arriola
- but does not say whether separation of scales is good enough

$\longrightarrow$ need to examine specific system and its scales

Suggested power countings for triplet waves:

- naive dimensional analysis ("Weinberg")
  - F waves and above
  - P and D waves for $p \ll \lambda_{NN}$
- "natural" counting with iterated tensor potential: leading contact term promoted to order $Q^{-1/2}$
  - P and D waves for $p \gtrsim \lambda_{NN}$
- "unnatural" counting: leading contact term of order $Q^{-3/2}$
  - ${}^3S_1$–${}^3D_1$

Open questions

- Is the new power counting needed for all P and D waves?
- Is the unnatural counting required in the $^3S_1$–$^3D_1$ waves?
(And what does it mean in terms of wave functions?)
- Do the same power countings also apply to waves where tensor OPE is repulsive?
- Shold we identify factors of $1/\Lambda_{NN}$ in two-pion exchange potentials?
- What is the counting for three-body forces in presence of tensor OPE?