How to calculate the nuclear wave function of ^{4}He ?

The Faddeev-Yakubowsky approach

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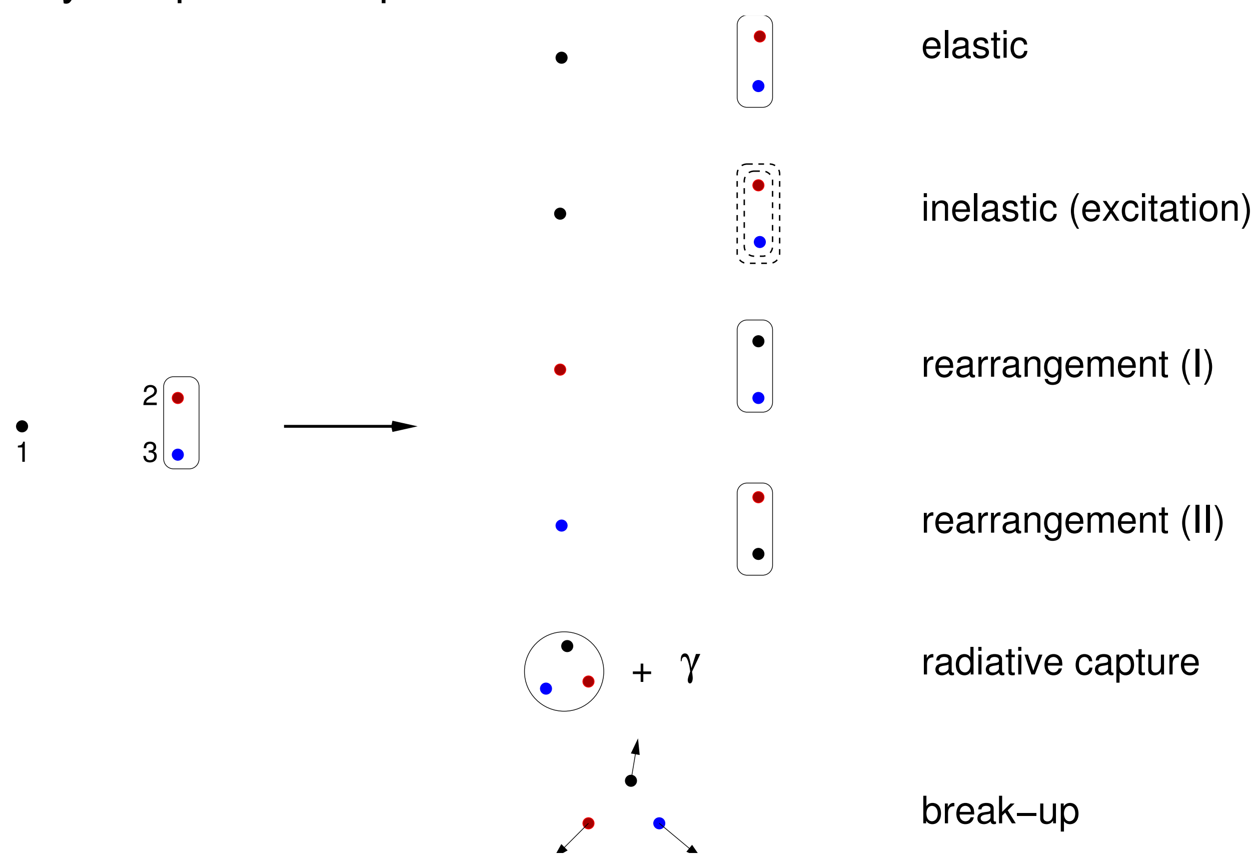
INTRODUCTION (I)

In view of solving the A=3 Schrodinger equation

$$(E - H_0)\Psi = V\Psi \qquad\qquad V = \sum_{i<j} V_{ij} \qquad\qquad (1)$$

Faddeev wrote in 1960 a set of equations, equivalent to (1), which provides a proper mathematical scheme for the variety of physical situations involved.
Apart from the 3-body bound state, the scattering of one particle on a 2-body bound state gives rise to a very complex description

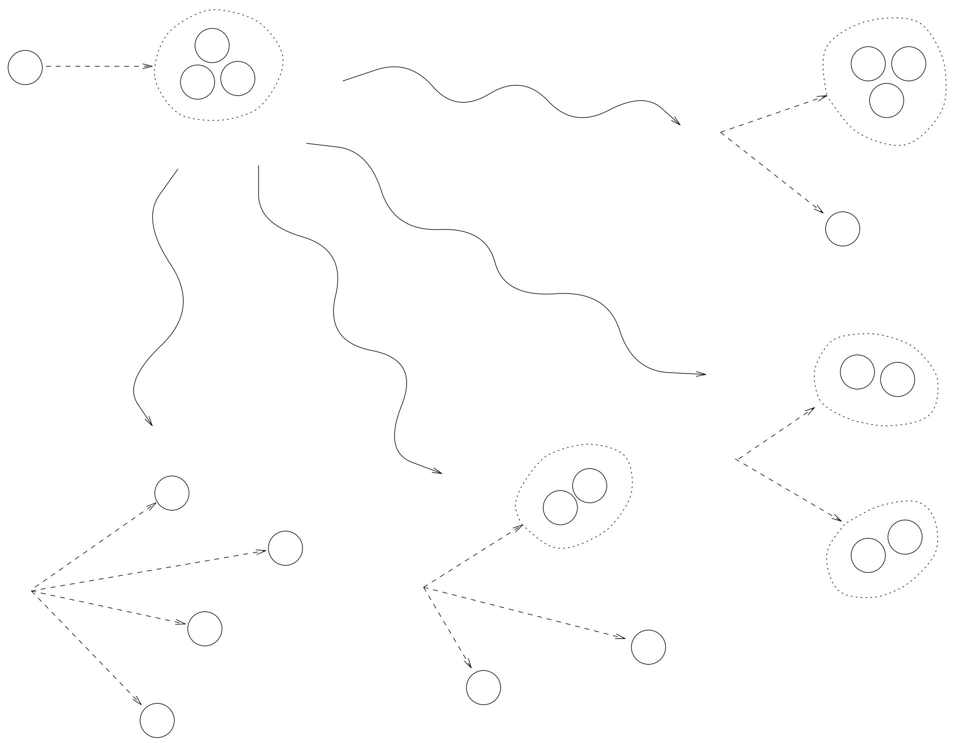


INTRODUCTION (II)

Soon latter (19676), Yakubovsky generalized the equations to A>4 thus providing a complete mathematical tool for the “exact” solution of the A-body problem

$$(E - H_0)\Psi = V\Psi \qquad V = \sum_{i<j} V_{ij} \qquad (1)$$

Till now, only the A=3 and (partially) the A=4 problem have been solved in their “full complexity”



The 3- and 4-body break-up is still “on the way” (R. Lazauskas, using Complex Scaling)

INTRODUCTION (III)

A “Fadeev-like” - but independent - approach exists based on AGS(*) equations
They have been developped recently by A. Fonseca and A. Deltuva (Lisbon) with great succes

All this machinery is superfluous when dealing only with bound states

For solving this problem, other independent methods methods have been developped leading in the last 10 years to a **spectacular** progress in the field:

No Core Shell Model (cf B. Barret talk) A=14 (?)
combined with RGM solve “simple” scattering problems (A=4,5 Navratil, Sofia)
Green Function Monte Carlo A=12
also “simple” scattering problems (A=4,5)
CCM (cf. M Dufour, last FUTIPEN workshop)
Hyperspherical Harmonics (Pisa A=3-4, + M. Gattobigio A=6)
Based on a Faddeev decomposition of the wf and Khon variational principle
Applied to full scattering results A=3,4

(*) Alt, Grasberger, Sandhas

THE MACHINERY A=3 (I)

The first step is to isolated the intrinsc dynamics of the 3-body Hamiltonian

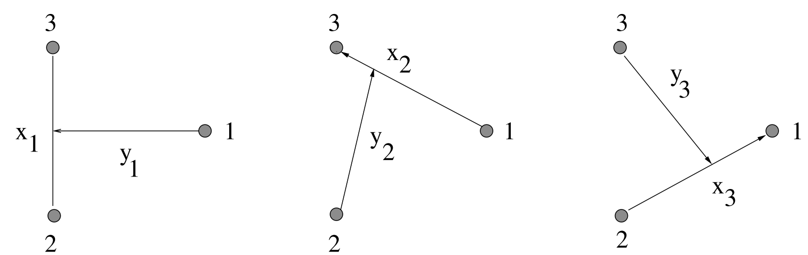
$$\mathcal{H} = \mathcal{H}_0 + V \qquad \mathcal{H}_0 = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\Delta_{\vec{r}_1} + \frac{1}{m_2}\Delta_{\vec{r}_2} + \frac{1}{m_3}\Delta_{\vec{r}_3}\right)$$

$$V = V_1(\vec{r}_2 - \vec{r}_3) + V_2(\vec{r}_3 - \vec{r}_1) + V_3(\vec{r}_1 - \vec{r}_2)$$

The first step is to isolated the intrinsic dynamics of the 3-body Hamiltonian.
This is done by introducing the Jacobi coordinates (3 sets!)

$$\vec{x}_i = \sqrt{\frac{2m_j m_k}{m_0(m_j + m_k)}}(\vec{r}_j - \vec{r}_k), \quad i = 1,2,3$$

$$\vec{y}_i = \sqrt{\frac{2m_i(m_j + m_k)}{m_0 M}}\left[\vec{r}_i - \frac{m_j\vec{r}_j + m_k\vec{r}_k}{m_j + m_k}\right]$$



And the center of mass coordinate R. In terms of them

$$\mathcal{H}_0 = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\Delta_{\vec{r}_1} + \frac{1}{m_2}\Delta_{\vec{r}_2} + \frac{1}{m_3}\Delta_{\vec{r}_3}\right) = -\frac{\hbar^2}{m_0}\left[\Delta_{\vec{x}_i} + \Delta_{\vec{y}_i} + \frac{m_0}{2M}\Delta_{\vec{R}}\right]$$

$$V = V_1(x_1) + V_2(x_2) + V_3(x_3)$$

The total 3-body wf factorizes into an intrinsic part Φ and a c.o.m. plane wave

$$\Psi(\vec{x}_i, \vec{y}_i, \vec{R}) = \Phi(\vec{x}_i, \vec{y}_i)\, e^{i\vec{P}\cdot\vec{R}}$$

THE MACHINERY A=3 (II)

Φ is a solution of the 3-body « intrinsic » Schrodinger equation

$$[E - H_0 - V_1(x_1) - V_2(x_2) - V_3(x_3)]\,\Phi = 0 \qquad (1) \qquad\qquad H_0 = -\frac{\hbar^2}{m_0}\left[\Delta_{\vec{x_i}} + \Delta_{\vec{y_i}}\right]$$

None of the Jacobi sets is privileged: all are necessary to properly describe the interaction region and the asymptotic behaviours of the differents channels.
And Φ ?

The seminal idea of Faddeev was to split the total 3-body wavefunction in a sum(*) of as many components (Faddeev Amplitudes) as asymptotic channels

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3$$

Φ_i fulfill a set of coupled equations – the Faddeev Equations- strictly equivalent to (1)

$$\left[E - H_0 - V_1(x_1)\right]\Phi_1(\vec{x}_1,\vec{y}_1) = V_1(x_1)\left[\Phi_2(\vec{x}_2,\vec{y}_2) + \Phi_3(\vec{x}_3,\vec{y}_3)\right]$$
$$\left[E - H_0 - V_2(x_2)\right]\Phi_2(\vec{x}_2,\vec{y}_2) = V_2(x_2)\left[\Phi_3(\vec{x}_3,\vec{y}_3) + \Phi_1(\vec{x}_1,\vec{y}_1)\right]$$
$$\left[E - H_0 - V_3(x_3)\right]\Phi_3(\vec{x}_3,\vec{y}_3) = V_3(x_3)\left[\Phi_1(\vec{x}_1,\vec{y}_1) + \Phi_2(\vec{x}_2,\vec{y}_2)\right]$$

- Each FC is « naturally » expressed in its own Jacobi set
- Coupling is ensured by the rhs. Is strongly non local, given by the linear reletions between different J sets
- In the «non interacting region » V_i=0, the FE decouple and corresponding FC has simple asymtpotics

(*) not a product !!! as one could expect from the N-body approximate solutions

THE MACHINERY A=3 (III)

In case of 3 identical particles
- The 3 potentials are the same V_i=V
- The 3 Faddeev equations are the same
- The functional form of the FA - in its own Jacobi set - is the same

$$[E - H_0 - V(x_1)]\,\Psi(\vec{x}_1,\vec{y}_1) = V(x_1)\,[\Psi(\vec{x}_2,\vec{y}_2) + \Psi(\vec{x}_3,\vec{y}_3)]$$

$$\Phi = \Psi(\vec{x}_1,\vec{y}_1) + \Psi(\vec{x}_2,\vec{y}_2) + \Psi(\vec{x}_3,\vec{y}_3)$$

Introducing the Permutation operators $P^{\pm}\Psi(\vec{x}_i,\vec{y}_i) = \Psi(\vec{x}_{i\pm1},\vec{y}_{i\pm1})$

$$[E - H_0 - V(x)]\,\Psi(\vec{x},\vec{y}) = V(x)\left[P^+ + P^-\right]\Psi(\vec{x},\vec{y})$$

$$\Phi = (1 + P^+ + P^-)\Psi$$

If we impose $P_{23}\Psi(\vec{x},\vec{y}) = \epsilon\Psi(\vec{x},\vec{y})$ with $\varepsilon = \pm 1$

One has $P_{ij}(1 + P^+ + P^-)\Psi = \epsilon\Psi$

and the total 3-body wavefunction has the desired symetrie

Wow do we do it in practice ?

THE MACHINERY A=3 (IV)

To solve in practice equation

$$[E - H_0 - V(x)]\,\Psi(\vec{x},\vec{y}) = V(x)\,[P^+ + P^-]\,\Psi(\vec{x},\vec{y})$$

One expands the FC in terms of Bipolar Harmonics

$$\Psi^{LM}(\vec{x},\vec{y}) = \sum_\alpha \frac{1}{xy}\varphi_\alpha^{LM}(x,y)\; B_\alpha^{LM}(\hat{x},\hat{y}) \qquad\qquad \alpha = \{l_x, l_y\}$$

$$B_{l_1 l_2}^{LM}(\hat{x}_1,\hat{x}_2) = \sum_{m_1 m_2} < l_1 m_1; l_2 m_2 | l_1 l_2; LM > Y_{l_1 m_1}(\hat{x}_1) Y_{l_2 m_2}(\hat{x}_2)$$

and obtain, after projecion, a set of integro-differential equations for the radial components

$$[E-H_0-V]\varphi_\alpha(x,y) = V(x)\left[\sum_{\alpha'}\int_{-1}^{1} du H_{\alpha,\alpha'}(x,y,u)\varphi_{\alpha'}(x',y') + \sum_{\alpha''}\int_{-1}^{1} du H_{\alpha,\alpha''}(x,y,u)\varphi_{\alpha''}(x'',y'')\right]$$

The integral comes from the P's

$$\frac{\phi_{\alpha_1}(x_1,y_1)}{x_1 y_1} = \sum_{\alpha_2}\int du_1\; H_{\alpha_1\alpha_2}(x_1,y_1,u_1)\; \frac{\phi_{\alpha_2}[x_2(x_1,y_1,u_1), y_2(x_1,y_1,u_1)]}{x_2 y_2}$$

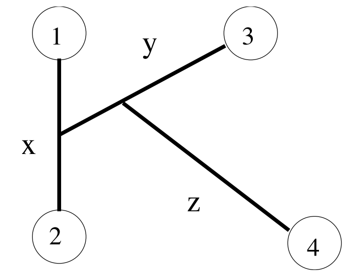
To get the right symetry one must include only components such that:

$$P_{23}\Psi(\vec{x},\vec{y}) = (-)^{l_x+\sigma_x+\tau_x} = \epsilon\Psi(\vec{x},\vec{y})$$

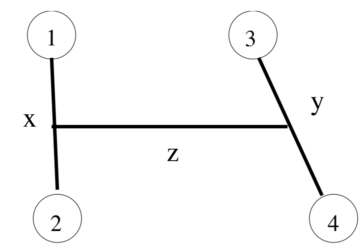
THE FY EQUATIONS FOR A=4

Two diferent types of coordinates « K » and « H »

$$\begin{aligned}
\vec{x}_K(ijkl) \equiv \vec{x}^{\,l}_{ij,k} &= \lambda\sqrt{2\mu_{ij}}(\vec{r}_j - \vec{r}_i) \\
\vec{y}_K(ijkl) \equiv \vec{y}^{\,l}_{ij,k} &= \lambda\sqrt{2\mu_{ij,k}}\left(\vec{r}_k - \frac{m_i\vec{r}_i + m_j\vec{r}_j}{m_{ij}}\right) \\
\vec{z}_K(ijkl) \equiv \vec{z}^{\,l}_{ij,k} &= \lambda\sqrt{2\mu_{ijk,l}}\left(\vec{r}_l - \frac{m_i\vec{r}_i + m_j\vec{r}_j + m_k\vec{r}_k}{m_{ijk}}\right)
\end{aligned}$$



$$\begin{aligned}
\vec{x}_H(ijkl) \equiv \; \vec{x}_{ij,kl} &= \lambda\sqrt{2\mu_{ij}}(\vec{r}_j - \vec{r}_i) \\
\vec{y}_H(ijkl) \equiv \; \vec{y}_{ij,kl} &= \lambda\sqrt{2\mu_{kl}}(\vec{r}_k - \vec{r}_l) \\
\vec{z}_H(ijkl) \equiv \; \vec{z}_{ij,kl} &= \lambda\sqrt{2\mu_{ij,kl}}\left(\frac{m_k\vec{r}_k + m_l\vec{r}_l}{m_{kl}} - \frac{m_i\vec{r}_i + m_j\vec{r}_j}{m_{ij}}\right)
\end{aligned}$$



$$\begin{aligned}
\mathcal{H}_0 &= \frac{\hbar^2}{2}\left(\frac{1}{m_1}\Delta_{\vec{r}_1} + \frac{1}{m_2}\Delta_{\vec{r}_2} + \frac{1}{m_3}\Delta_{\vec{r}_3} + \frac{1}{m_4}\Delta_{\vec{r}_4}\right) \\
&= -\hbar^2\lambda^2(\Delta_{\vec{x}_K} + \Delta_{\vec{y}_K} + \Delta_{\vec{z}_K}) - \frac{\hbar^2}{8m}\Delta_R \\
&= -\hbar^2\lambda^2(\Delta_{\vec{x}_H} + \Delta_{\vec{y}_H} + \Delta_{\vec{z}_H}) - \frac{\hbar^2}{8m}\Delta_R
\end{aligned}$$

$$V = \sum_{i<j=1}^{4} V_{ij} = V_{12} + V_{13} + V_{14} + V_{23} + V_{24} + V_{34}$$

THE FY EQUATIONS FOR A=4

To solve the 4-body (intrinsic) Schrodinger equation

$$(E - H_0)\Psi = V\Psi \qquad\qquad V = \sum_{i<j=1}^{4} V_{ij} \qquad\qquad (3)$$

First step

Split Ψ in the usual Faddeev amplitudes, Ψ_{ij}, associated with each interacting pair (i, j).

$$\Psi = \sum_{i<j} \Psi_{ij} = \Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}$$

Equation (3) is equivalent to the system of 6 coupled equations

$$\begin{aligned}
(E - H_0)\Psi_{12} &= V_{12}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}) \\
(E - H_0)\Psi_{13} &= V_{13}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}) \\
(E - H_0)\Psi_{14} &= V_{14}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}) \\
(E - H_0)\Psi_{23} &= V_{23}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}) \\
(E - H_0)\Psi_{24} &= V_{24}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}) \\
(E - H_0)\Psi_{34} &= V_{34}\,(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34})
\end{aligned}$$

Second step

Each Ψ_{ij} is in its turn splitted in 3, the FY amplitudes, corresponding to the different asymptotics of the remaining two particles

Let us consider e.g:

$$(E - H_0)\Psi_{12} = V_{12}\left(\Psi_{12} + \Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}\right)$$

writen in the form

$$(E - H_0 - V_{12})\Psi_{12} = V_{12}\left(\Psi_{13} + \Psi_{14} + \Psi_{23} + \Psi_{24} + \Psi_{34}\right) \quad (3)$$

We make the following partition

$$\Psi_{12} = \Psi_{12,3}^{4} + \Psi_{12,4}^{3} + \Psi_{12,34}$$

and split equation (3) into a system

$$\begin{aligned}
(E - H_0 - V_{12})\Psi_{12,3}^{4} &= V_{12}\left(\Psi_{13} + \Psi_{23}\right) \\
(E - H_0 - V_{12})\Psi_{12,4}^{3} &= V_{12}\left(\Psi_{14} + \Psi_{24}\right) \\
(E - H_0 - V_{12})\Psi_{12,34} &= V_{12}\left(\Psi_{34}\right)
\end{aligned}$$

If we do the same for the Faddeev amplitudes on the r.h.s.

$$\Psi_{ij} = \Psi^{l}_{ij,k} + \Psi^{k}_{ij,l} + \Psi_{ij,kl} \quad i < j; k < l$$

and for each Faddeev equation, we end with the set of 18 coupled equations equivalents to (1)

$$\begin{aligned}
(E - H_0 - V_{12})\Psi^4_{12,3} &= V_{12}\left(\Psi^2_{13,4} + \Psi^4_{13,2} + \Psi_{13,24} + \Psi^1_{23,4} + \Psi^4_{23,1} + \Psi_{23,14}\right)\\
(E - H_0 - V_{12})\Psi^3_{12,4} &= V_{12}\left(\Psi^3_{14,2} + \Psi^2_{14,3} + \Psi_{14,23} + \Psi^3_{24,1} + \Psi^1_{24,3} + \Psi_{24,13}\right)\\
(E - H_0 - V_{12})\Psi_{12,34} &= V_{12}\left(\Psi^2_{34,1} + \Psi^1_{34,2} + \Psi_{34,12}\right)
\end{aligned}$$

$$\begin{aligned}
(E - H_0 - V_{13})\Psi^2_{13,4} &= V_{13}\left(\Psi^3_{14,2} + \Psi^2_{14,3} + \Psi_{14,23} + \Psi^2_{34,1} + \Psi^1_{34,2} + \Psi_{34,12}\right)\\
(E - H_0 - V_{13})\Psi^4_{13,2} &= V_{13}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34} + \Psi^1_{23,4} + \Psi^4_{23,1} + \Psi_{23,14}\right)\\
(E - H_0 - V_{13})\Psi_{13,24} &= V_{13}\left(\Psi^3_{24,1} + \Psi^1_{24,3} + \Psi_{24,13}\right)
\end{aligned}$$

$$\begin{aligned}
(E - H_0 - V_{14})\Psi^3_{14,2} &= V_{14}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34} + \Psi^3_{24,1} + \Psi^1_{24,3} + \Psi_{24,13}\right)\\
(E - H_0 - V_{14})\Psi^2_{14,3} &= V_{14}\left(\Psi^2_{13,4} + \Psi^4_{13,2} + \Psi_{13,24} + \Psi^2_{34,1} + \Psi^1_{34,2} + \Psi_{34,12}\right)\\
(E - H_0 - V_{14})\Psi_{14,23} &= V_{14}\left(\Psi^1_{23,4} + \Psi^4_{23,1} + \Psi_{23,14}\right)
\end{aligned}$$

$$\begin{aligned}
(E - H_0 - V_{23})\Psi^1_{23,4} &= V_{23}\left(\Psi^3_{24,1} + \Psi^1_{24,3} + \Psi_{24,13} + \Psi^2_{34,1} + \Psi^1_{34,2} + \Psi_{34,12}\right)\\
(E - H_0 - V_{23})\Psi^4_{23,1} &= V_{23}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34} + \Psi^2_{13,4} + \Psi^4_{13,2} + \Psi_{13,24}\right)\\
(E - H_0 - V_{23})\Psi_{23,14} &= V_{23}\left(\Psi^3_{14,2} + \Psi^2_{14,3} + \Psi_{14,23}\right)
\end{aligned}$$

$$\begin{aligned}
(E - H_0 - V_{24})\Psi^3_{24,1} &= V_{24}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34} + \Psi^3_{14,2} + \Psi^2_{14,3} + \Psi_{14,23}\right)\\
(E - H_0 - V_{24})\Psi^1_{24,3} &= V_{24}\left(\Psi^1_{23,4} + \Psi^4_{23,1} + \Psi_{23,14} + \Psi^2_{34,1} + \Psi^1_{34,2} + \Psi_{34,12}\right)\\
(E - H_0 - V_{24})\Psi_{24,13} &= V_{24}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34} + \Psi^2_{13,4} + \Psi^4_{13,2} + \Psi_{13,24}\right)
\end{aligned}$$

$$\begin{aligned}
(E - H_0 - V_{34})\Psi^2_{34,1} &= V_{34}\left(\Psi^2_{13,4} + \Psi^4_{13,2} + \Psi_{13,24} + \Psi^3_{14,2} + \Psi^2_{14,3} + \Psi_{14,23}\right)\\
(E - H_0 - V_{34})\Psi^1_{34,2} &= V_{34}\left(\Psi^1_{23,4} + \Psi^4_{23,1} + \Psi_{23,14} + \Psi^3_{24,1} + \Psi^1_{24,3} + \Psi_{24,13}\right)\\
(E - H_0 - V_{34})\Psi_{34,12} &= V_{34}\left(\Psi^4_{12,3} + \Psi^3_{12,4} + \Psi_{12,34}\right)
\end{aligned}$$

Case of 4 identical particles

In that case the 18 FY amplitudes can be obtained by the action of the permutation operators P_{ij} on two of them, one $\Psi^{l}_{ij,k}$ and one $\Psi_{ij,kl}$
Let us take for instance $K \equiv \Psi^{4}_{12,3}$ and $H \equiv \Psi_{12,34}$.
The K-H amplitudes satisfy the following equations

$$
\begin{aligned}
(E-H_0-V)K &= V\left[(P_{23}+P_{13})\,(\varepsilon+P_{34})\,K+\varepsilon(P_{23}+P_{13})\,H\right] && (4)\\
(E-H_0-V)H &= V\left[(P_{13}P_{24}+P_{14}P_{23})\,K+P_{13}P_{24}\,H\right] && (5)
\end{aligned}
$$

in which $\varepsilon=\pm 1$ depending on bosons or fermions.
Each amplitude $F=K,H$ is considered as function of its own set of Jacobi coordinates $\vec{x},\vec{y},\vec{z}$

$$
\begin{aligned}
\vec{x}_K &= (\vec{r}_2-\vec{r}_1) & \vec{x}_H &= (\vec{r}_2-\vec{r}_1)\\
\vec{y}_K &= \sqrt{\tfrac{4}{3}}\left(\vec{r}_3-\tfrac{\vec{r}_1+\vec{r}_2}{2}\right) & \vec{y}_H &= (\vec{r}_4-\vec{r}_3)\\
\vec{z}_K &= \sqrt{\tfrac{3}{2}}\left(\vec{r}_4-\tfrac{\vec{r}_1+\vec{r}_2+\vec{r}_3}{3}\right) & \vec{z}_H &= \sqrt{2}\left(\tfrac{\vec{r}_3+\vec{r}_4}{2}-\tfrac{\vec{r}_1+\vec{r}_2}{2}\right)
\end{aligned}
$$

and expanded in angular momentum variables for each coordinate

$$
<\vec{x}\vec{y}\vec{z}|F>=\sum_{\alpha}\int d\hat{x}d\hat{y}d\hat{z}\;\frac{F_{\alpha}(xyz)}{xyz}\;Y_{\alpha}(\hat{x},\hat{y},\hat{z}) \qquad (6)
$$

Y_{α} are generalized tripolar harmonics containing spin, isospin and angular momentum variables

The label α holds for the set of intermediate quantum numbers defined in a given coupling scheme and includes the specification for the type of amplitudes K or H.

We have used the following couplings:

$$\text{K amplitudes} \quad \left\{[(t_1 t_2)_{\tau_x} t_3]_{T_3}\, t_4\right\}_T \otimes \left\{\left[(l_x (s_1 s_2)_{\sigma_x})_{j_x}\, (l_y s_3)_{j_y}\right]_{J_3} (l_z s_4)_{j_z}\right\}_J$$

$$\text{H amplitudes} \quad \left[(t_1 t_2)_{\tau_x} (t_3 t_4)_{\tau_y}\right]_T \otimes \left\{\left[(l_x (s_1 s_2)_{\sigma_x})_{j_x}\, \left(l_y (s_3 s_4)_{\sigma_y}\right)_{j_y}\right]_{j_{xy}}\, l_z\right\}_J$$

Each component F_α labelled by a set of 12 quantum numbers

The total 4-body wavefunction is obtained by the action of permutation operators on the two FY amplitudes K and H

$$\begin{aligned} |\Psi\rangle &= [1+\varepsilon(P_{23}+P_{13})][1+\varepsilon(P_{14}+P_{24}+P_{34})] \quad |K\rangle \\ &+ [1+\varepsilon(P_{13}+P_{23}+P_{14}+P_{24})+P_{13}P_{24}] \quad |H\rangle \end{aligned}$$

One can show that, by imposing to
K the « right symetrie » in P12
H the « right symetrie » in P12 and P34
The total wf is also well symetrized.

$$(-1)^{\sigma_x+\tau_x+l_x} = (-1)^{\sigma_y+\tau_y+l_y} = \varepsilon$$

After projection, we end with a sytem of 3d integro-differential equations, similr to the 3-body case

$$
\begin{aligned}
\sum_{\alpha'} \hat{D}_{\alpha\alpha'} \phi_{\alpha'}(x,y,z) &= \sum_{\alpha'} V_{\alpha\alpha'}(x) \sum_{\alpha"} f_{\alpha'\alpha"} \phi_{\alpha"}(x^f_{\alpha'\alpha"}, y^f_{\alpha'\alpha"}, z^f_{\alpha'\alpha"}) \\
&+ \sum_{\alpha'} V_{\alpha\alpha'}(x) \sum_{\alpha"} \int_{-1}^{+1} du \; h_{\alpha'\alpha"}(x,y,z,u) \, \phi_{\alpha"}(x^h_{\alpha'\alpha"}, y^h_{\alpha'\alpha"}, z^h_{\alpha'\alpha"}) \\
&+ \sum_{\alpha'} V_{\alpha\alpha'}(x) \sum_{\alpha"} \int_{-1}^{+1} du \int_{-1}^{+1} dv \; g_{\alpha'\alpha"}(x,y,z,u,v) \, \phi_{\alpha"}(x^g_{\alpha'\alpha"}, y^g_{\alpha'\alpha"}, z^g_{\alpha'\alpha"})
\end{aligned}
$$

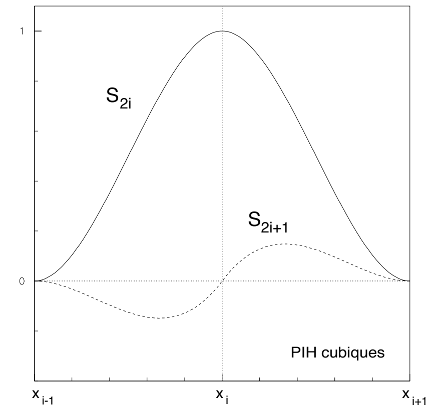
with $\hat{D}_{\alpha\alpha'} = (E + \Delta_\alpha)\delta_{\alpha\alpha'} - V_{\alpha\alpha'}$

and f,h,g are known function to account for the permutation operators

Search solutions in the form

$$\varphi(x,y,z) = \sum_{ijk} c_{ijk} \; S_i(x) \; S_j(y) \; S_k(z)$$

where S are spline functions.
This is the only assumption in the calculations



For bound state the boundary conditions are exponentially decreasing in all directions

We have derived analytic expressions for the integral kernels, e.g. :

$$
\begin{aligned}
g_{\alpha,\alpha'}(x,y,z,u,v) &= \frac{1}{2}\cdot t_{\alpha,\alpha'}\cdot \sum_{l_{xy},\sigma,l'_{xy},\sigma',l'_{yz},L,S,\lambda} A\begin{Bmatrix} l_x & \sigma_x & j_x \\ l_y & s_3 & j_y \\ l_{xy} & \sigma & J_3 \end{Bmatrix}\cdot A\begin{Bmatrix} l'_x & \sigma'_x & j'_x \\ l'_y & s_4 & j'_y \\ l'_{xy} & \sigma' & J'_3 \end{Bmatrix} \\
&\cdot\ A\begin{Bmatrix} l_{xy} & \sigma & J_3 \\ l_z & s_4 & j_z \\ L & S & J \end{Bmatrix}\cdot A\begin{Bmatrix} l'_{xy} & \sigma' & J'_3 \\ l'_z & s_2 & j'_z \\ L & S & J \end{Bmatrix}\cdot A\begin{Bmatrix} l'_x & l'_y & l'_{xy} \\ l'_z & L & l'_{yz} \end{Bmatrix} \\
&\cdot\ A\begin{Bmatrix} l'_x & \lambda & l_{xy} \\ l_z & L & l'_{yz} \end{Bmatrix}\cdot(-)^{s_1+s_2-\sigma_x+s_2+\sigma'-S}\cdot A\begin{Bmatrix} s_2 & s_1 & \sigma_x \\ s_3 & \sigma & \sigma'_x \end{Bmatrix} \\
&\cdot\ A\begin{Bmatrix} s_2 & \sigma'_x & \sigma \\ s_4 & S & \sigma' \end{Bmatrix}\cdot\frac{xyz}{x'_{KK}y'_{KK}z'_{KK}} \\
&\cdot\ \mathcal{H}^{l_{xy}}_{l_x l_y,l'_x\lambda}(x,y,u)\cdot\mathcal{H}^{l'_{yz}}_{\lambda l_z,l'_y l'_z}(y_0,z,v)
\end{aligned}
$$

$$
t_{\alpha,\alpha'} = (-)^{t_1+t_2-\tau_x+t_2+T'_3-T}\cdot A\begin{Bmatrix} t_2 & \tau'_x & T_3 \\ t_4 & T & T'_3 \end{Bmatrix}\cdot A\begin{Bmatrix} t_2 & t_1 & \tau_x \\ t_3 & T_3 & \tau'_x \end{Bmatrix}
$$

SOME RESULTS

The ^{4}He bound state

We obtained in 99 the B with local NN potentials

Phys. Lett. B447 (1999) 199

^{4}He binding energy B (MeV) and r.m.s. radius

	$N_c = 5+5$		$N_c = 15+9$	
	B	$\bar{r}$	B	$\bar{r}$
AV14	23.34	1.56	23.81	1.54
NIJM II	23.39	1.54	23.86	1.53
REID 93	23.65	1.53	24.12	1.52

We obtained in 2004 the B with local+TNI and with non-local NN potentials

Phys. Rev C70 (2004) 044002; nucl-th/04080

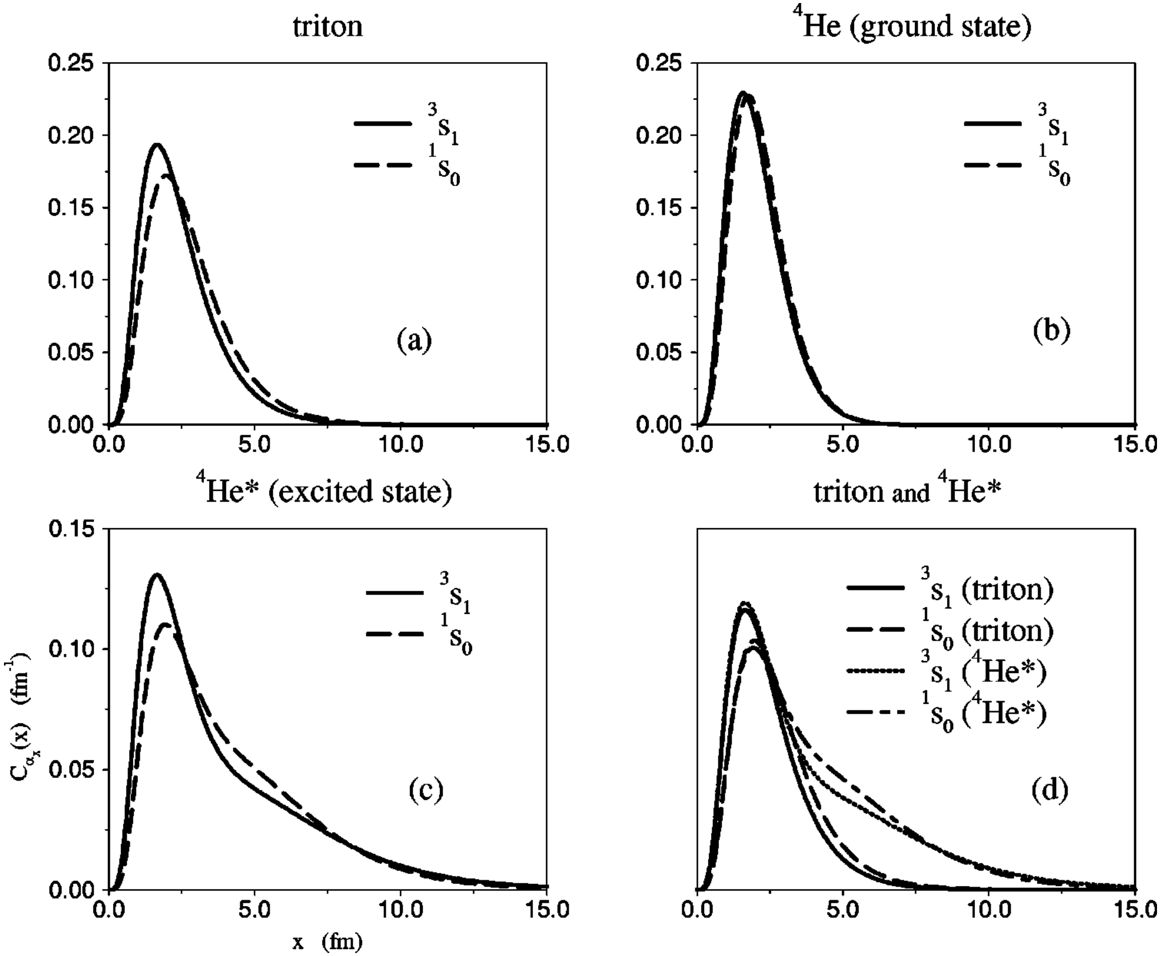
Potential	$\langle T \rangle$	$-\langle V \rangle$	B	R
INOY96	72.80	103.8	31.00	1.353
INOY03	69.89	99.94	30.04	1.369
INOY04	69.49	99.41	29.91	1.372
INOY04′	69.46	99.36	29.88	1.372
AV18	98.69	123.6	24.95	1.511
Potential	$\langle T \rangle$	$-\langle V \rangle$	$-\langle E \rangle$	R
INOY96	72.45	102.7	30.19	1.358
INOY03	69.54	98.79	29.24	1.373
INOY04	69.14	98.62	29.11	1.377
INOY04′	69.11	98.19	29.09	1.376
AV18	97.77	122.1	24.22	1.516
	97.80	122.0	24.23[8,34]	
AV18+UIX	113.2	141.7	28.50[8,34]	1.44 [6]
Expt.			28.30	1.47

The ^{4}He first excitation

Without Coulomb and isospin breaking it appears in most of models as loosely bound

Phys. Rev. C58 (1998) 58-74

$$C_{\alpha_x}(x) = \sum_{\alpha'(\alpha'_x=\alpha_x)} \int\int dy dz |\Psi_{\alpha'}(x,y,z)|^2,$$



Not a « breathing mode »… but a n orbiting aroud a triton !

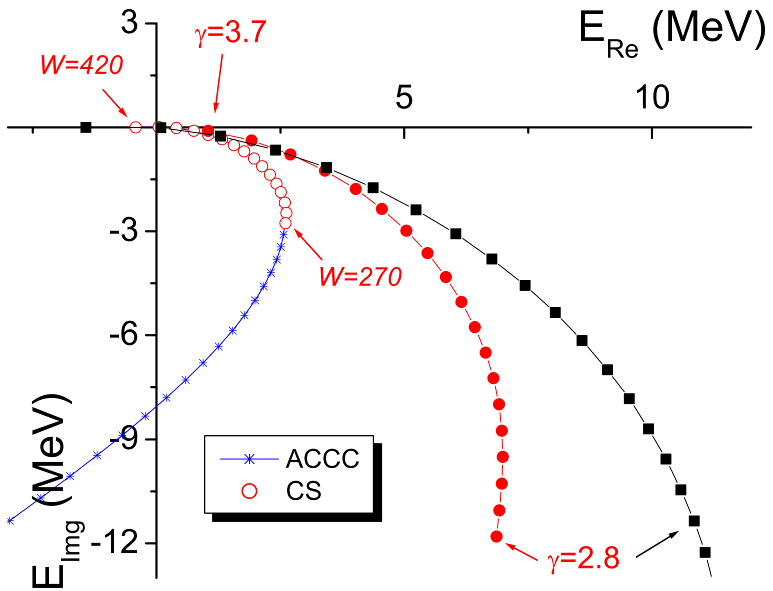
3n and 4n resonances

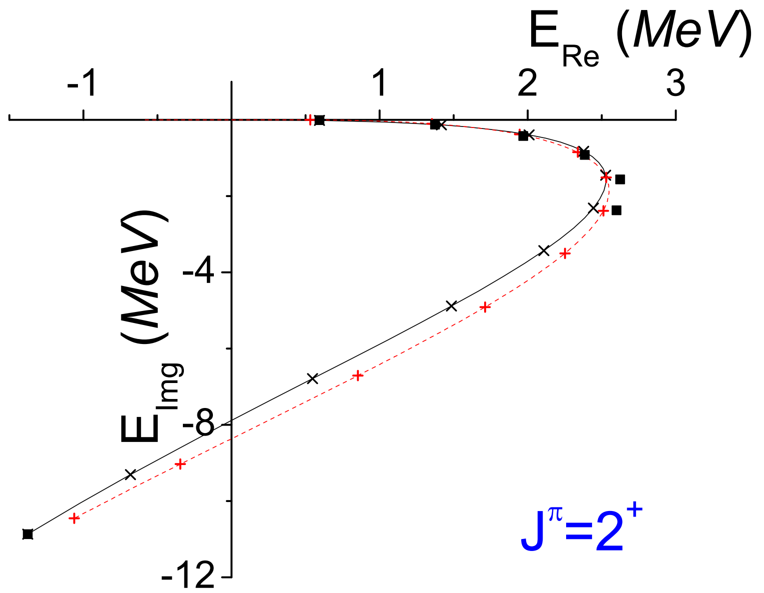
n_3 and n_4 are not bound …. but where are they ?

Computed 3 and 4-n resonances solving full FY in the complex plane (CRM)

Phys. Rev. C71 (2005) 044004; nucl-th/0502037

Phys. Rev. C 72 (2005) 034003; nucl-th/0507022





The A=4 scattering states

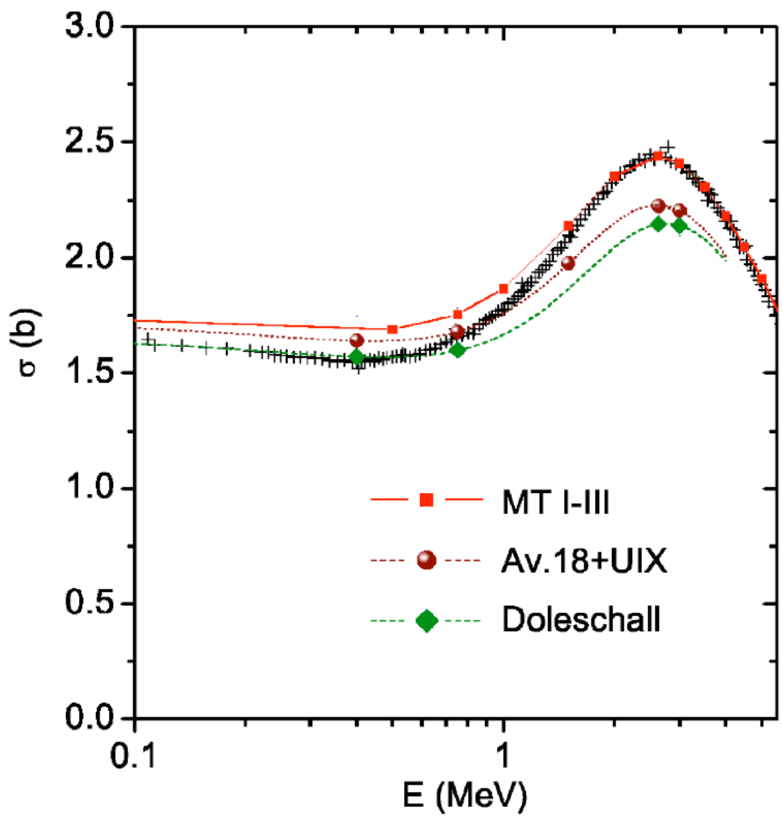
Have described the A=4 scattering prcess below breau-up

$n + {}^3H$

$p + {}^3H \rightarrow \quad n + {}^3He$

$\rightarrow {}^2H + {}^2H$

Maybe the more interesting was to found that the more familar interactions, even reproducing the A=3 and A=4 binding energies, failled to describe the first resonance in nuclear physics !



(*) Phys. Lett. B447 (1999) 199 Phys. Rev C70 (2004) Phys. Rev. C71 (2005) 034004

The He atomic clusters

Our approach allows to deal with strongly repulsive potentials (« hard core »)
Wa could compute(*) the Van-der-Waals 3- and 4-body bound states of 4He atoms

l_{max}	B_3 (mK)	B_3^* (mK)	$a_0^{(1+2)}$ (Å)
0	89.01	2.0093	155.39
2	120.67	2.2298	120.95
4	125.48	2.2622	116.37
6	126.20	2.2669	115.72
8	126.34	2.2677	115.61
10	126.37	2.2679	115.58
12	126.39	2.2680	115.56
14	126.39	2.2680	115.56

max (lx, ly, lz)	B_4 (mK)	$a_0^{(3+1)}$ (Å)
0	348.8	≈−855
2	505.9	190.6
4	548.6	111.6
6	556.0	105.9
8	557.7	103.7

Using the large value of 3+1 sctatering length we predicted a B^*_4 =1.09 mK (below B_3)

(*) Phys. Rev. A73 (2006)

Conclusion

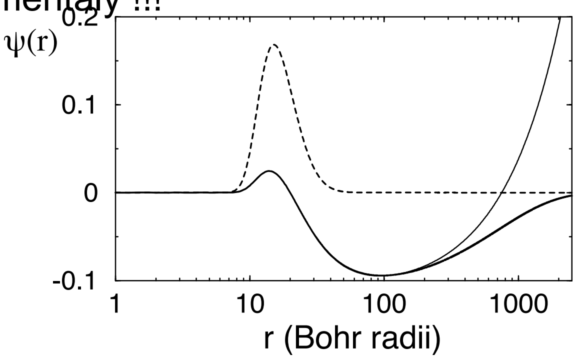
The Faddeev-Yakubowsky methods are the only to provide a solution of the full A-body problem, taking into account the rich variety of channels.

They are numerically quite heavy… and that's why A remains small !
With the present computers they can be extended to A=5 but A=4 brerakup is still "on the way"…

They are not as precise as the ad-hoc variational methods (although 4 digits is relatively easy to get)
However the possibility to acces to bound and scattering states on the same foot, lead in some cases to spectacular predictions.

e.g. the first excited state of a H_2^+ (in the pp S=1 channel)

The binding energies of a H_2^+ were calculated with 12 significant digits
In the pp S=1 channel, only one bound state was found
By computing the p+H scattering we found a scattering length value of A=750 a.u.
We predicted a first excited sate with B= 1.09 10^{-9} a.u.
Latter confirmed by "variational fishing" … and yet not found experimentaly !!!
One of greatest joys …



Lazauskas R. and Carbonell J., *Few-Body Syst.*, **31** (2002) 125.

J. Carbonell, R. Lazauskas, D. Delande, L. Hilico and S. Kılıc,
Europhys. Lett., **64** (3), pp. 316–322 (2003)