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Basics of Group Theory for Quantum Systems

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Lecture series of the *Espace de Structure Nucléaire Théorique*

- Introduction
- Finite Groups
  - *Representations of Groups*
  - *Reducible/irreducible representations*
- Continuous groups
  - *Lie groups & Lie algebras*
  - *Adjoint representation*
  - *Roots & weights of the Lie algebra*
  - *Rotation matrices*
  - *Group measure*

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Introduction: further reading

Literature

- H. F. Jones - Groups, Representations and Physics
  IOP publishing,1998 (ISBN 978-0750305044)
- H. Georgi - Lie Algebras in Particle Physics: From Isospin to Unified Theories
  Westview Press, 1999 (ISBN 978-0738202334)
- Fl. Stancu - Group Theory in Subnuclear Physics
  Oxford Univ. Press, 1996 (ISBN 978-0198517429)
- M. Creutz - Quarks, Gluons and Lattices
  Cambridge Univ. Press, 1983 (ISBN 0 521 24405 6)
- W. K. Tang, Group theory in physics,
  World Scientific, 1985 (ISBN 9971-966-57-3)

Introduction: Symmetries & conservation laws

Noether's theorem: symmetries imply conservation laws

For quantum mechanical systems most easily seen in the Heisenberg picture:

expectation value for observable corresponding to Hermitian operator A_H

$$\langle A_H \rangle = \langle \Psi_H | A_H | \Psi_H \rangle$$

time-dependence of the expectation value due to the time-dependence of the operator

$$\frac{dA_H}{dt} = \frac{\partial A_H}{\partial t} + \frac{1}{i\hbar} [A_H, H]$$ (Ehrenfest theorem)

usually no explicit time-dependence (no time-dependence of Schrödinger operator) $\frac{\partial A_H}{\partial t} = 0$

→ A_H **corresponds to a conserved quantity, if** $[A_H, H] = 0$

simplest case: invariance under time-translations

$\frac{\partial H}{\partial t} = 0$ and $[H, H] = 0$ → the energy $E = \langle H \rangle$ is conserved

Usually, symmetry transformations form „groups"

→ **Group theory helps to explore the consequences of symmetries**

Finite groups

Groups: a definition

A group G is a set of group elements $\{g\}$ accompanied by a „product" with the properties

1. $g_1, g_2 \in G \quad \Longrightarrow \quad g_1 \cdot g_2 \in G$ (closure)
2. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ (associativity)
3. there is one identity element e with $e \cdot g = g \cdot e = g$ for all $g \in G$
4. for all $g \in G$ the inverse g^{-1} exists with $g^{-1} \cdot g = g \cdot g^{-1} = e$

In many cases, the number of elements N_G (order of the group) in the group is finite

Simple physical examples of finite groups:

- space reflection $P: \quad \vec{x} \longrightarrow -\vec{x}$

  $\{e = P^2, P\} \qquad N_G = 2$

- permutations, e.g. of 3 particles $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$: 1 2 3 $\longrightarrow$ $i\ j\ k$

$$\left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\} \qquad N_G = 6$$

Representations of groups I

1. first idea to define a group: group multiplication table

$$
\begin{array}{c|ccc}
\cdot & g_1 & g_2 & \cdots \\
\hline
g_1 & g_1 \cdot g_1 & g_1 \cdot g_2 & \cdots \\
g_2 & g_2 \cdot g_1 & g_2 \cdot g_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

- *not well suited for applications and calculations!*
- *not well suited for larger or infinite groups*

2. most natural: (linear) representations on a vector space

- *all tools of linear algebra available*
- *direct connection to the action of group elements (symmetry transformations) on the quantum mechanical states in the Hilbert space*

Representations of groups II

Ingredients of linear representations

1. vector space $\mathcal{L}$: $\vec{v}, \vec{w} \in \mathcal{L}$ (e.g. vectors of $\mathbb{R}^n$)

addition $\vec{v} + \vec{w} \in \mathcal{L}$ and multiplication with a scalar $\lambda\vec{v} \in \mathcal{L}$ where $\lambda \in \mathbb{R}, \mathbb{C}$

inner product (scalar product) $(\vec{v}, \vec{w}) : \mathcal{L} \times \mathcal{L} \to \mathbb{R}, \mathbb{C}$

is bilinear: for example $(\vec{u}, \lambda_1\vec{v} + \lambda_2\vec{w}) = \lambda_1(\vec{u}, \vec{v}) + \lambda_2(\vec{u}, \vec{w})$ and

$(\vec{v}, \vec{v}) > 0$ for $\vec{v} \neq 0$

2. linear operators $D : \mathcal{L} \to \mathcal{L}$ where $D(\lambda_1\vec{v} + \lambda_2\vec{w}) = \lambda_1 D(\vec{v}) + \lambda_2 D(\vec{w})$

for example $n \times n$ matrices

3. mapping $g \in G \longrightarrow D(g) : \mathcal{L} \to \mathcal{L}$

which associates a linear operator with each group element such that the product is preserved

$$D(g_1 \cdot g_2) = D(g_1) \cdot D(g_2) \quad \text{and} \quad D(e) = \mathbb{1}$$

Note that this implies that $D(g)$ has an inverse (Why?)

It can be shown (for finite groups) that $D(g)$ can always be chosen as a unitary matrix

$$D(g)^\dagger = D(g)^{-1}$$

Example of a representation of S_n

vector space: Hilbert space of two spins spanned by

$$|\alpha\rangle = |m_1 m_2\rangle = |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \quad \alpha = 1, \ldots, 4$$

group S_2: $\left\{e = P^2, P\right\}$

define the representation as naturally expected: $D\left(\begin{pmatrix} 1 & 2 \\ i & j \end{pmatrix}\right) |m_1 m_2\rangle \equiv |m_i m_j\rangle$

explicit matrix representations of the group elements are:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad D(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- the unity is mapped on the unit matrix ✔
- the group product is mapped on the product of matrices, e.g. $D(P) \cdot D(P) = D(e)$ ✔

mapping $g \longrightarrow D(g)$ is indeed a representation!

Note the representation is defined on the quantum mechanical Hilbert space.

Reducible/Irreducible representation

A representation may have a block diagonal form

$$D(g) = \begin{pmatrix} \square & & 0 \\ & \square & \\ 0 & & \square \end{pmatrix} \quad \text{for all } g$$

with the same blocks for all g → The vector space decomposes into **invariant subspaces**.

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \cdots$$

And this can be done until there are no more small invariant subspaces anymore.
The representations within the invariant subspaces are then called **irreducible**.

Reducible representations can be transformed so that subspace separate into even smaller subspaces.

What is special about the irreducible representations?

Group operation on the Hamiltonian

If the group is a symmetry group of the Hamiltonian, we find

$D^{-1}(g)\ H\ D(g) = H$ or $[H, D(g)] = H$ for all members of the group g

Let's assume that $|\psi\rangle$ is an eigenstate of the Hamiltonian: $H|\psi\rangle = E|\psi\rangle$

Then also $D(g)|\psi\rangle$ is an eigenstate: $H\ D(g)|\psi\rangle = E\ D(g)|\psi\rangle$

On the other hand, all states in an irreducible representation of the symmetry group can be obtained by $D(g)$.

This implies that all states of the irreducible representation are eigenstates of H with the same eigenvalue ($\Rightarrow$ multiplet of states).

Note that this is only true for irreducible representations.

If the representation is still reducible, then an eigenstate of H could be in a subspace for which no $D(g)$ connects to the other states.

Diagonalization of a spin-spin Hamiltonian

Back to our simple example of two spins.

Toy Hamiltonian: $$H = \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

H is symmetric under the exchange of particle 1 & 2: symmetry group S_2 $\{e = P^2, P\}$

Start with our standard basis:

$$|\alpha\rangle = |m_1 m_2\rangle = |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \quad \alpha = 1, \ldots, 4$$

Hamiltonian is non-diagonal in this basis
and our representation is reducible, e.g.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find four 1-dimensional irreducible representations of $\{e = P^2, P\}$ spanned by

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \qquad |\uparrow\uparrow\rangle \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad |\downarrow\downarrow\rangle$$

They turn out to be the eigenstates of H

$$H\,\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = -3\,\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \qquad H|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \qquad H|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

$$H\,\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

although the degeneracy is larger ($\Rightarrow$ additional symmetry)

Further subjects

Central theorems are **Schur's Lemmas** (first & second)

Criteria whether irreducible representations are equivalent

There are only a finite number of non-equivalent, irreducible representations

E.g. orthogonality theorem for two representations D^{μ} and D^{ν} of the group G of order N (d_{μ} is the dimensions of D^{μ})

$$\Rightarrow \qquad \frac{1}{N}\sum_{g\in G} D^{\mu}_{ij}(g) D^{\mu}_{kl}(g^{-1}) = \frac{\delta_{\mu\nu}}{d_{\mu}} M_{il}\,(M^{-1})_{kj}$$

where $\delta_{\mu\nu} = \begin{cases} 0 \text{ for not equivalent representations} \\ 1 \text{ for equivalent representations} \end{cases}$

and for $\delta_{\mu\nu} = 1$ $\qquad D^{\mu} = M D^{\nu} M^{-1}$

This enables systematic decomposition of representations in irreducible representations using so called **characters** (traces of matrices)

Continuous groups

Continuous groups

Most physical symmetries are related to **continuous groups**

Typically the group elements depend on (real) parameters in this case

The group is defined using a **defining representation**

E.g. group of rotations around the z-axis is defined by $R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The group product is given by the matrix product.

$\{R(\theta) \mid \theta = 0 \ldots 2\pi\}$ comprises all group elements

This is a representation on $\mathbb{R}^3$ and, at the same time, defines the group.

Usual convention: $R(0) = \mathbb{1}$

It is useful to specifically look to the vicinity of $R(0) = \mathbb{1}$ by Taylor expanding the expression

$$R(\theta) = \mathbb{1} + \theta \left. \frac{\partial R(\theta)}{\partial \theta} \right|_{\theta=0} + \cdots = \mathbb{1} + i\,\theta \underbrace{\frac{1}{i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\equiv J_3} + \cdots$$

The hermitian operator J_3 is called a generator (for rotations, it is an angular momentum operator)

Lie groups

If (as in the example) the group „smoothly“ depends on a set of real parameters, it is called a **Lie group**

In this case $g = g(\alpha_1, \ldots, \alpha_r)$ and $g(0) = \mathbb{1}$

and the Taylor expansion $$g(\epsilon) = \mathbb{1} + i\epsilon_a \underbrace{\left(\frac{1}{i} \left. \frac{\partial g(\alpha)}{\partial \alpha_a} \right|_{\alpha=0} \right)}_{\equiv X_a} + \cdots$$

defines the hermitian **generators** X_a (hermitian since we assume a unitary representation)

It can be shown that already the generators completely define the group!

In most cases, one formally defines the groups elements as

$$g(\alpha) = \exp\left(i\alpha_a X_a\right)$$

- generators completely define the group
- generators are hermitian and will commute with the Hamiltonian ⇒ conserved observables

Structure constants

We will be interested in other representations of the group than the defining ones

e.g. action of rotations on quantum mechanical wave functions

(vector space = function space)

What is unique to the group? ⟹ **Group multiplication law!**

For continuous groups, the multiplication $g(\alpha) \cdot g(\beta) \equiv g(\gamma)$

defines a function $\gamma = \gamma(\alpha, \beta)$ that is unique, **not representation dependent**

Convention implies $\gamma(0,0) = 0 \qquad \gamma(\alpha, 0) = \alpha \qquad \gamma(0, \beta) = \beta$

Taylor expanding up to second order of $g(\alpha) \cdot g(\beta) \equiv g(\gamma)$ and of $\gamma = \gamma(\alpha, \beta)$ one finds

$$[X_a, X_b] = i \underbrace{\left[-2 \left. \frac{\partial^2 \gamma(\alpha, \beta)}{\partial \alpha_a \partial \beta_b} \right|_{\alpha, \beta = 0} \right]}_{\equiv f_{abc}} X_c$$

which defines the **structure constants** f_{abc}

and shows that the commutators are a group property

Lie algebras

The generators are the „basis“ of a vector space $\mathcal{L}$ with elements $X = \sum_a \alpha_a \; X_a$

The commutators are defined by the structure constants and are closed within this vector space

$$X, Y \in \mathcal{L} \quad \Rightarrow \quad [X, Y] \in \mathcal{L}$$

The define a „product“ with the properties

- anticommutativity $\quad [X, Y] = -[Y, X]$
- Jacobi identity $\quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
- distributivity $\quad [X, Y + Z] = [X, Y] + [X, Z]$
- linearity $\quad [\alpha X, Y] = \alpha \; [X, Y]$

vector space with a product (with the listed properties) $\Rightarrow$ **„Lie algebra“**

- representations of the Lie algebra are directly related to representations of the group
- Lie algebra is finite dimensional

Adjoint representation

The generators are the basis states of the Lie algebra and can be the basis of its own representation $\Rightarrow$ **adjoint representation**

We need a (matrix) representation that fulfills the commutator relations $[X_a, X_b] = i\ f_{abc}\ X_c$

Then the generators will generate group elements that obey the desired group product

Define the action of the matrix $T_a = T(X_a)$ by

$$T_a\ X_b = \{T_a\}_{cb} X_c = i f_{abc} X_c = [X_a, X_b]$$

which means $\{T_a\}_{cb} \equiv i f_{abc}$ if each X_a is represented by the standard basis in $\mathbb{C}^r$

Using the Jacobi identity and the antisymmetry of the structure constants f_{abc} in all indices

$$\Rightarrow \quad [T_a, T_b] = i\ f_{abc}\ T_c$$

The structure constants define a representation of the Lie algebra **(adjoint representation)**

Example of SU(2) - generators

special (with determinant = 1) unitary matrices in 2 dimensions (defining representation)

parameter count: 4 complex matrix elements: 8 parameters
4 relations because of unitarity: -4 parameters
1 relation since det=+1: -1 parameter $\Rightarrow$ 3 parameters!

elements of the group can be represented using generators $M = \exp(i\ \vec{\alpha} \cdot \vec{X})$

generators are hermitian matrices, from det=+1 follows that they are traceless

One possible set of generators is given by the Pauli matrices

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that this choice is orthogonal with respect to the scalar product defined by the trace

$$\mathrm{Tr}\left(X_i^\dagger \cdot X_j\right) = \frac{1}{2}\delta_{ij}$$

(the overall constant depends on the representation)

Generators obey the commutator relations of angular momentum operators

$$[X_i, X_j] = i\ \epsilon_{ijk}\ X_k$$

Example of SU(2) - Cartan generators

Defining representation is a 2-dimensional irreducible representation

Find set with maximal number of commuting generators, e.g. X_3
and use eigenstates of X_3 as a basis

$$X_3 \; |\mu = \pm\frac{1}{2}\rangle = \pm\frac{1}{2} \;\; |\mu = \pm\frac{1}{2}\rangle$$

Note that such a basis can always be found in any irreducible representation.

The states of any irreducible representation can be labeled by the eigenvalues of a set of commuting generators **(Cartan generators)**

In the defining representation of SU(2) the two states have the eigenvalues

$$\mu = \pm\frac{1}{2}$$

This is called the **weight** of the states,
possible weights depend on the irreducible representation

There is one Cartan generator for SU(2)

Example of SU(2) - Adjoint representation

The adjoint representation in matrix form $\{T_i\}_{jk} = i\ \epsilon_{ikj}$ reads

$$T_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad T_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad T_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

again, the eigenstates E_μ of the Cartan generator T_3 define a basis

$$E_0 = (0,0,1) \qquad E_1 = \frac{1}{\sqrt{2}}(1,i,0) \qquad E_{-1} = \frac{1}{\sqrt{2}}(1,-i,0)$$

Now, in the adjoint representation, the vector space is the algebra, therefore the eigenvalue equation can also be written in form of commutators

$$[X_3, E_\mu] = \mu\ E_\mu$$

Since the commutators are independent of the representation, this is a general result

The weights of the adjoint representations are called **„roots“**

All „eigenstates“ with $\mu = 0$ are Cartan operators, since they commute

The other E_μ are called **ladder operators**, since the commutator implies

$$E_\alpha\ |\mu\rangle \propto |\mu + \alpha\rangle \qquad \text{(Why?)}$$

Using the three eigenvectors one find for the ladder operators in the defining representation

$$E_{-1} \propto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad E_1 \propto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad ✔$$

Example of SU(2) - irreducible representation

SU(2) commutator relations imply many constraints on irreducible representations

$$[E_0, E_{\pm 1}] = \pm E_{\pm 1} \qquad\qquad [E_{+1}, E_{-1}] = E_0$$

$\Rightarrow$

- *states within an irreducible representation are identified by the eigenvalues* μ *of* $E_0 = T_3$

(third component of angular momentum)

$$E_0 \; |j \mu\rangle = \mu \; |j \mu\rangle$$

- *the different irreducible representations are distinguished by the maximal weight* $j = \mu_{max}$

- j *is half-integer and* $\mu = j, j-1, \ldots, -j$

The irreducible representations of the algebra are the ones of the group. ✔

The generators define the matrix (Killing form) $g_{ij} = 2 \; \mathrm{Tr}(T_i T_j) = \delta_{ij}$
which allows to define the operator

$$T^2 \equiv g_{ij} \; T_i T_j = T_1^2 + T_2^2 + T_3^2$$

for which $[T^2, T_i] = 0$

Often it makes physically sense to identify the irreps using such a **Casimir operator**

Classification of Lie groups I

More general now: semi-simple, compact & connected Lie groups

(e.g. SU(N), but there are more examples)

Classification according to Cartan & Weyl

- *find the maximal number of commuting generators: SU(2) $\longrightarrow$ 1 ,SU(3) $\longrightarrow$ 2, ...*

  $\Rightarrow$ $\{H_i\}$ ***Cartan generators***
- *perform simultaneous diagonalization in the adjoint representation*

$$[H_i, H_j] = 0 \qquad [H_i, E_{\vec{\alpha}}] = \alpha_i \; E_{\vec{\alpha}} \qquad E^\dagger_{\vec{\alpha}} = E_{-\vec{\alpha}}$$

- *orthogonalize the new set of generators with respect to scalar product*

$$(A, B) = k \cdot \mathrm{Tr}(A^\dagger B)$$

- *for each pair* $\pm\alpha$ *define a set of SU(2) like generators*

$$E^\pm = \frac{E_{\pm\alpha}}{|\alpha|} \qquad E_0 = \frac{\vec{\alpha}\cdot\vec{H}}{\alpha^2} \qquad \Rightarrow \qquad [E_0, E^\pm] = \pm E^\pm \qquad [E^+, E^-] = E_0$$

each non-zero pair of **roots** defines an independent **SU(2) subalgebra**

(the new set of generators requires the extention of the parameters space to $\mathbb{C}^r$)

Classification of Lie groups II

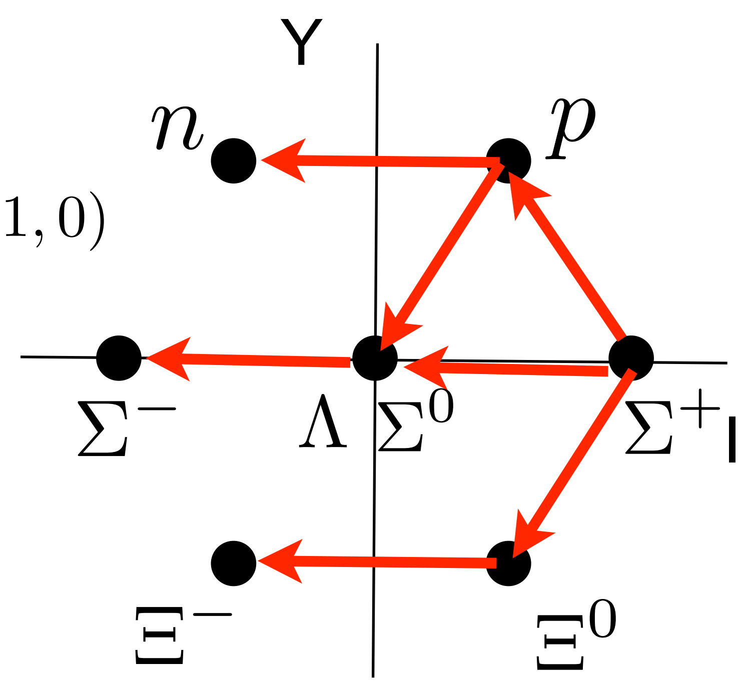
All constraints for the weights & roots of SU(2) translate into geometrical constraints of arbitrary semi-simple,compact,... Lie groups/algebras!

Geometrical constraints on roots (and weights), e.g. $\frac{\vec{\alpha}_i \cdot \vec{\alpha}_j}{\vec{\alpha}_i^2} = \text{half-integer}$

Label states of an irrep by eigenvalues of the Cartan generators $\Rightarrow$ weight diagram

E.g. construction of the baryon octet (flavor SU(3)):

- *Cartan generators: isospin I_3, hypercharge Y*
- *roots:* $\vec{\alpha}_1 = \pm(1/2, \sqrt{3}/2) \quad \vec{\alpha}_2 = \pm(1/2, -\sqrt{3}/2) \quad \vec{\alpha}_3 = \pm(1, 0)$
- *start from hights weight* Σ^+
- *possible steps are given by root vectors*
- *number of possible can be deduced*
- *here: multiplet of 8 states which are expected to be degenerate for flavor-symmetric QCD*



Label irreps by highest weight or by as many Casimir operators as there are Cartan generators

Simplest Casimir operator is again given by $g_{ij} T_i T_j$

Representation of group elements

Once an irreducible representation is defined, the matrix elements of all generators can be obtained

Group elements are given by, e.g. $M = \exp(i\ \vec{\alpha} \cdot \vec{X})$

Therefore, the irreducible representation also defines the action of the group on all states within this representation!

Note:

- *the matrix elements of the group elements do not depend on the actual states, but just on the position within the multiplet*
- *the group cannot rotate an element of an irreducible representation to other irreducible representations (by definition)*

The resulting functions have been calculated in some cases and be used to apply a group element to an arbitrary state.

Example: rotation group SO(3)

The rotation group is generate by angular momentum operators $\{J_i\} \quad i = 1, \ldots, 3$

The algebra fulfills SU(2) commutator relations $[J_i, J_j] = i\;\epsilon_{ijk}\;J_k$

Find a convenient parameterization,e.g. using Euler angles

$$M = \exp(-i\,\alpha\,J_3)\;\;\exp(-i\,\beta\,J_2)\;\;\exp(-i\,\gamma\,J_3)$$

$$\alpha, \gamma = 0, \ldots, 2\pi$$

$$\beta = 0, \ldots, \pi$$

this choice enables to evaluate the outer rotations trivially

$$\Rightarrow \quad D^J_{M'M}(\alpha,\beta,\gamma) \equiv \langle JM'|\exp(-i\,\alpha\,J_3)\;\;\exp(-i\,\beta\,J_2)\;\;\exp(-i\,\gamma\,J_3)|JM\rangle$$

$$D^J_{M'M}(\alpha,\beta,\gamma) = \exp(-i\,\alpha\,M' - i\,\gamma\,M)\langle JM'|\exp(-i\,\beta\,J_2)\;|JM\rangle \equiv \exp(-i\,\alpha\,M' - i\,\gamma\,M)d^J_{M'M}(\beta)$$

This defines the Wigner D-and d-functions (analytically known)

As expected, the functions only depend on quantum numbers of irreducible representations

(and parameters of the Lie group)

Group measure I

Finite groups:

$$\frac{1}{N}\sum_{g\in G} f(g) \qquad \text{well defined}$$

Easy to see that sum is invariant under „group translation“ by g'

$$\frac{1}{N}\sum_{g\in G} f(g'g) = \frac{1}{N}\sum_{g\in G} f(g)$$

by construction the sum is normalized

$$\frac{1}{N}\sum_{g\in G} 1 = 1$$

How to generalize this to continous groups?

→ group integration/group measure

Group measure II

For the continuous group, we define integrals with a similar „translational invariance“ and normalization

$$\int dg\ f(g) = \int dg\ f(g'g) \qquad \int dg\ 1 = 1$$

Start with the ansatz
$$\int dg\ f(g) \equiv \int d^r\alpha\ J(\alpha)\ f(g(\alpha))$$

What is $J(\alpha)$?

$$g(\alpha) = g(\gamma) \cdot g(\beta)$$

Look at
$$\int dg\ f(g) = \int d^r\beta\ J(\beta)\ f(g(\beta))$$

$$= \int d^r\beta\ J(\beta)\ f(g(\alpha(\gamma,\beta))) \ = \int d^r\alpha\ \underbrace{J(\beta)\ \left|\frac{\partial\alpha(\gamma,\beta)}{\partial\beta}\right|^{-1}}_{\equiv J(\alpha)} f(g(\alpha))$$

since γ is arbitrary, we choose it such that

$$J(\alpha) = J(\beta'(\gamma,\alpha))\ \left|\frac{\partial\alpha(\gamma,\beta)}{\partial\beta}\right|^{-1} \longrightarrow J(\alpha) = J(0)\ \left|\frac{\partial\alpha(\gamma,\beta)}{\partial\beta}\right|^{-1}_{\beta=0}$$

The invariant group measure is given by the Jacobian of the group multiplication law

$J(0)$ is fixed using the normalization condition

Examples - group measure

simple example: U(1)

1 parameter group, defining representation is one-dimensional and the only irreducible one

(expect equivalent ones)

$$D = \exp(i\alpha) \qquad \alpha = 0, \ldots, 2\pi$$

we read off the group multiplication

$$\alpha(\gamma, \beta) = \beta + \gamma \quad \Rightarrow \quad \frac{\partial \alpha}{\partial \beta} = 1$$

$$\Rightarrow \quad \int dg\ f(g) = \int_0^{2\pi} d\alpha\ J(0)\ f(g(\alpha)) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\ f(g(\alpha))$$

Less trivial example: measure for SO(3)

$$\int dg\ f(g) = \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin(\beta) d\beta \int_0^{2\pi} d\gamma\ \ f(g(\alpha, \beta, \gamma))$$

where

$$g(\alpha, \beta, \gamma) = \exp(-i\,\alpha\, J_3)\ \exp(-i\,\beta\, J_2)\ \exp(-i\,\gamma\, J_3)$$

Example - usage group measure

Generalization of orthogonality theorem to SO(3) irreps

The orthogonality theorem for finite groups can be generalized to Lie groups

We assume two identical irreps of SO(3) given by *j* $\Rightarrow M = M^{-1} = \mathbb{1}$

The theorem then directly translates from discrete group elements to

$$\frac{1}{N}\sum_{g\in G} D^{\mu}_{ij}(g)D^{\mu}_{kl}(g^{-1}) = \frac{1}{N}\sum_{g\in G} D^{j}_{ij}(g)D^{j*}_{lk}(g) = \frac{\delta_{il}\delta_{kj}}{2j+1}$$

to continuous for form using the measure

$$\int dg\ D^{\mu}_{ij}(g)D^{\mu}_{kl}(g^{-1}) = \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin(\beta)d\beta \int_0^{2\pi} d\gamma\ \ D^{j}_{ij}(\alpha,\beta,\gamma)D^{j*}_{lk}(\alpha,\beta,\gamma) = \frac{\delta_{il}\delta_{kj}}{2j+1}$$