Time-splitting for the random nonlinear Schrödinger equation

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The equation

We aim to study the following NLS equation :

NLS with random dispersion

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = i\, \overset{\circ}{W}(t) \dfrac{\partial^2 u}{\partial x^2}(t,x) + g(u(t,x)) \quad \forall (t,x) \in (0,1] \times \mathbb{R} \\ u(t=0,x) = u_0(x) \end{cases}$$

where

- u is unknown and u_0 is the initial condition,
- g is a nonlinear function,
- $\overset{\circ}{W}$ is a white noise (i.e. the "derivative" of a Brownian motion W) :

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Roughly speaking, $\overset{\circ}{W}$ can be seen as a stationary Gaussian process satisfying for every $t, s \in [0,1]$,

$$\mathbb{E}(\overset{\circ}{W}(t)) = 0 \quad \& \quad \mathbb{E}(\overset{\circ}{W}(s)\,\overset{\circ}{W}(t)) = \delta(t-s).$$

It corresponds to the normalized independent infinitesimal increments of a **Brownian motion** W, which is a Gaussian process satisfying for every $t, s \in [0,1]$,

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Sample path of the white noise $\overset{\circ}{W}$

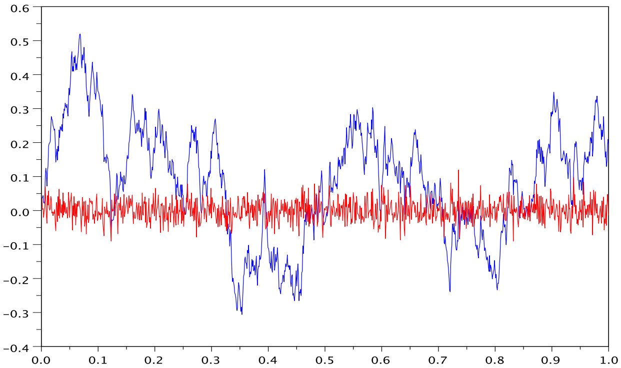

FIGURE: Blue : W ; Red : $\overset{\circ}{W}$.

Origin : nonlinear optic

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- z is the position on the fiber, τ is the time,
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is solved by following the procedure :

- $v_0(x) \xrightarrow{FT} \widehat{v_0}(\xi)$
- we solve the Fourier linear equation $\dfrac{\partial \widehat{v}}{\partial t}(t,\xi) = -i\,\overset{\circ}{W}(t)\xi^2\widehat{v}(t,\xi)$, which gives $\widehat{v}(t,\xi) = \widehat{v_0}(\xi)\exp(-i\xi^2(W(t) - W(t_0)))$ (Itô-Stratonovich calculus)
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Existence and uniqueness

Theorem

Let $u_0 \in L^2_x$. We assume that g is Lipschitz. Then, there exists a unique solution $u \in \mathcal{C}([0,1], L^2_x)$ a. s. Moreover, if $u_0 \in H^2_x$, g two times differentiable and its derivatives up the order 2 are bounded, then $u \in \mathcal{C}([0,1], H^2_x)$ a. e.

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The scheme : definition and convergence

We subdivise $[0,1]$ into N intervals of lenght $h = 1/N$ and we aim to construct $\{u_n^h\}_{n\in\{0,1,\cdots,N\}}$ such that $\{u_n^h\}_{n\in\{0,1,\cdots,N\}} \approx_{h\to 0} \{u(nh,\cdot)\}_{n\in\{0,1,\cdots,N\}}$. We let :

- initial condition :

$$u_0^h := u_0,$$

- for every $n \in \{1,\cdots,N\}$: $u_n^h(t)$ is defined by

$$u_n^h = Y(h)X((n-1)h, nh)u_{n-1}^h.$$

Theorem (M. (2006))

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We subdivise $[0,1]$ into N intervals of lenght $h = 1/N$ and we aim to construct $\{u_n^h\}_{n\in\{0,1,\cdots,N\}}$ such that $\{u_n^h\}_{n\in\{0,1,\cdots,N\}} \approx_{h\to 0} \{u(nh,\cdot)\}_{n\in\{0,1,\cdots,N\}}$. We let :

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The order of convergence is bounded below by 1/2.
We aim to illustrate the previous result by simulation.
We let $\varepsilon(h) = \|u_N^h - u(1,\cdot)\|_{L^2(\Omega, L^2(\mathbb{R},\mathbb{C}))}$ for every $h = 2^{-m}$ where m is integer.
The order of convergence will be given by

$$\alpha(m) = \log_2\left(\frac{\varepsilon(2^{-m})}{\varepsilon(2^{-m-1})}\right).$$

Indeed : assume that $\varepsilon(h) \approx h^{\alpha^*}$ for $h \to 0$. Then

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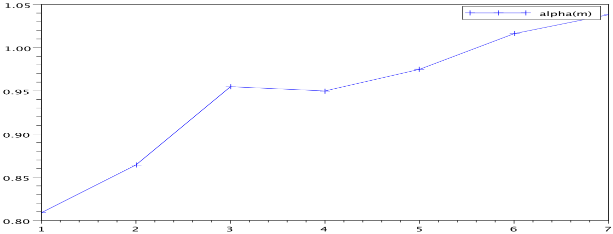

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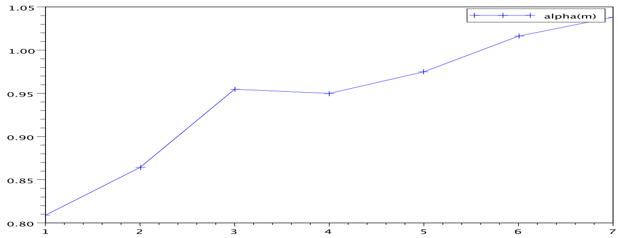

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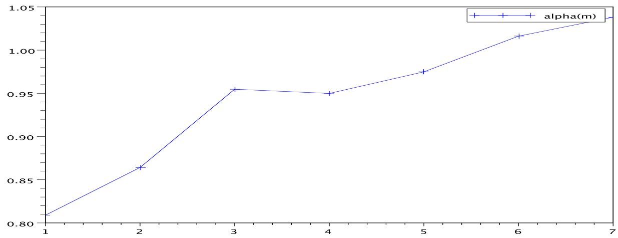

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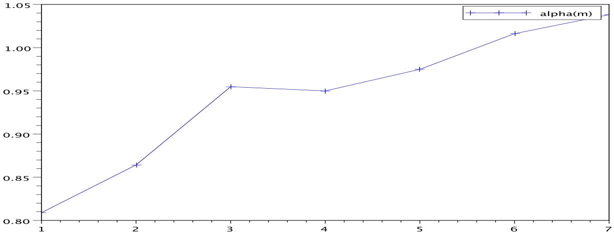

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Previously we established an estimate on the local error :

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In order to study more precisely the scheme we establish :

Theorem (M. (preprint 2011))

We assume that $v \in H^4$ and g sufficiently regular. Then for every $t_0 \leq t \in [0,1]$,

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with $\mathcal{G}(v) = i\frac{\partial^2 g(v)}{\partial x^2} - \left\langle (\nabla g)(v), i\frac{\partial^2 v}{\partial x^2}\right\rangle$ and $\mathcal{I}(t_0,t) = \int_{t_0}^t (W_t - W_\theta)d\theta$.

$$\left(\left\langle (\nabla g)(v), i\frac{\partial^2 v}{\partial x^2}\right\rangle = (\partial_1 g)(v)\Re\left(i\frac{\partial^2 v}{\partial x^2}\right) + (\partial_2 g)(v)\Im\left(i\frac{\partial^2 v}{\partial x^2}\right)\right).$$

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Previously we established an estimate on the local error :

$$\mathcal{L}[t_0, t, v] = ||S(t_0, t)v - Y(t - t_0)X(t_0, t)v||_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))} \quad \leq \quad C(t - t_0)^{3/2}.$$

In order to study more precisely the scheme we establish :

Theorem (M. (preprint 2011))

We assume that $v \in H^4$ *and* g *sufficiently regular. Then for every* $t_0 \leq t \in [0, 1]$,

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Applying the norm $\|\cdot\|_{L^2}$, we get

$$||u_n^h - u(nh,\cdot)||_{L^2}^2 = \mathcal{E}_1(n,h) + 2\mathcal{R}\mathcal{E}_2(n,h)$$

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We first have (M. (2006), thanks to $\mathcal{L}[t_0,t,v_0]^2 \leq C(\|v_0\|_{H^2})(t-t_0)^3$)

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- $Z_l^h v \approx v + iW_l^h \partial_x^2 v, \qquad (W_l^h = (W_{lh} - W_{(l-1)h}))$
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- $\mathcal{L}[(l-1)h, lh, v] \approx \mathcal{G}(v)\mathcal{I}_l^h, \qquad (\mathcal{I}_l^h = \mathcal{I}((l-1)h, lh))$
- $S_l^h v \approx v + iW_l^h \partial_x^2 v,$

in $\langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2}$ with $E_j^{(n,h)} = Z_n^h \cdots Z_j^h S_{j-1}^h \cdots S_1^h u_0 - Z_n^h \cdots Z_{j+1}^h S_j^h \cdots S_1^h u_0$
Very roughly, something "like this" will appear :

$$\begin{aligned}\langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2} \quad &\approx \quad |Id + iW_n^h|^2 \cdots |Id + iW_{k+1}^h|^2 (Id + iW_k^h)\mathcal{I}_k^h |Id + iW_{k-1}^h|^2 \cdots \\ &\quad \cdots |Id + iW_{j+1}^h|^2 (Id - iW_j^h)\mathcal{I}_j^h |Id + iW_{j-1}^h|^2 \cdots |Id + iW_1^h|^2\end{aligned}$$

Taking the expectation (and by independence of the increments of BM) :

$$\mathbb{E}[\langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2}] \quad \approx \quad (1+h)^{n-k} h^2 (1+h)^{k-1-j} h^2 (1+h)^{j-1} \leq Ch^4$$

Exact order of convergence

Finally, we get

$$\begin{aligned}\mathbb{E}(\mathcal{E}_2(n,h)) &= \mathbb{E}\left(\sum_{j,k=1,j<k}^{n} \langle E_j^{(n,h)}, E_k^{(n,h)}\rangle_{L^2}\right) \\ &\leq C\sum_{j,k=1,j<k}^{n} h^4 \leq Ch^2,\end{aligned}$$

then

$$\left\|u_n^h - u(nh,\cdot)\right\|^2_{L^2(\Omega,L^2(\mathbb{R},\mathbb{C}))} = \mathbb{E}[||u_n^h - u(nh,\cdot)||^2_{L^2}] \leq Ch^2.$$

This concludes the "proof".

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