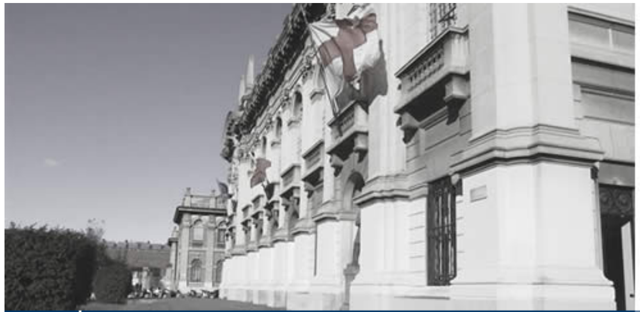


POLITECNICO DI MILANO





Feedback control of a two-level atom and of its fluorescence light

Matteo Gregoratti (joint work with Alberto Barchielli)

Aim

For the prototype case of a two-level atom stimulated by a laser, show how the stochastic Schrödinger equation with **stochastic coefficients** allows to

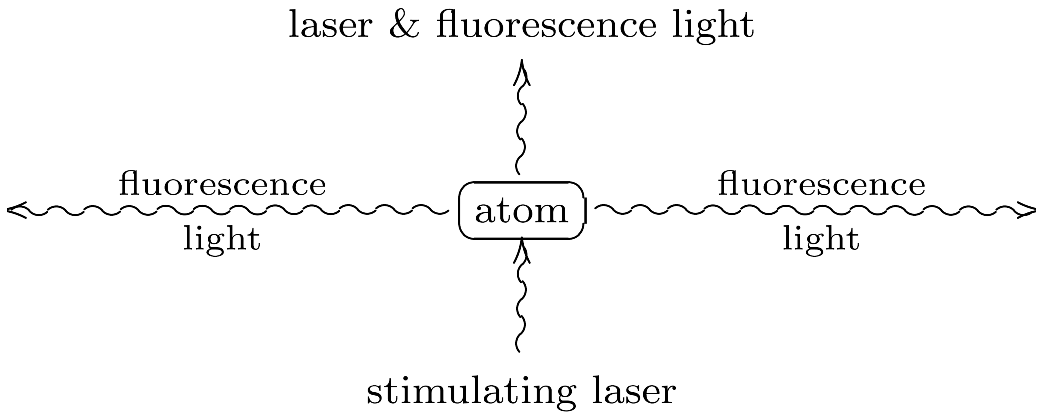
- model **non Markovian** evolutions due to:
  - imperfections in the stimulating laser
  - a feedback loop based on the detection of the fluorescence light

- model measurements in continuous time combined with measurement based **feedback** (including delay)

- compute the **homodyne spectrum** of the fluorescence light in order to control:
  - the atom itself (phase dacay rates)
  - its fluorescence light (squeezing)

Trapped two-level atom stimulated by a coherent monochromatic laser

$\rho \in \mathcal{S}(\mathbb{C}^2)$



resonance frequency

$$H_0 = \frac{\omega_0}{2}\sigma_z$$

$$\mathcal{L}(t)[\rho] = -\mathrm{i}\,[H_0 + H_f(t),\, \rho] + \gamma\Big(\sigma_-\rho\sigma_+ - \frac{1}{2}\{P_+,\rho\}\Big) + \gamma\bar{n}\Big(\sigma_-\rho\sigma_+ + \sigma_+\rho\sigma_- - \rho\Big)$$

$$H_f(t) = \overline{f(t)}\,\sigma_- + f(t)\sigma_+$$

natural line width of the atom

thermal effects intensity

$$f(t) = \frac{\Omega}{2}\exp\{-\mathrm{i}(\vartheta + \omega t)\}$$

coherent monochromatic laser wave

Rabi frequency

laser frequency

Markovian evolution allowing for Markovian unravelling

Linear stochastic Schrödinger equation

$$\mathrm{d}\phi(t) = \left(-\mathrm{i}H_0 - \mathrm{i}H_f(t) - \frac{1}{2}\sum_{i=1}^{2} L_i(t)^* L_i(t) - \frac{1}{2}\sum_{k=1}^{3} R_k^* R_k + \frac{\lambda}{2}\right)\phi(t_-)\,\mathrm{d}t$$
$$+ \sum_{i=1}^{2} L_i(t)\phi(t_-)\,\mathrm{d}B_i(t) + \sum_{k=1}^{3}\left(\frac{R_k}{\sqrt{\lambda_k}} - \mathbb{1}\right)\phi(t_-)\,\mathrm{d}N_k(t)$$

B_1, B_2, N_1, N_2, N_3 independent stochastic processes in some probability space $(\Omega, \mathcal{F}, \mathbb{Q})$

Poisson processes of rates $\lambda_1, \lambda_2, \lambda_3$ $\qquad \lambda_1 + \lambda_2 + \lambda_3 = \lambda$

Wiener processes

$$L_j(t) = \mathrm{e}^{\mathrm{i}\theta_j}\,\frac{\overline{f}(t)}{|f(t)|}\,\alpha_j\,\sigma_-, \qquad \alpha_j > 0$$

$$R_1 = \beta_1\,\sigma_-, \qquad \beta_1 \in \mathbb{C} \qquad |\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 = \gamma$$

$$R_2 = \beta_2\,\sigma_-, \qquad R_3 = \beta_3\,\sigma_+ \qquad |\beta_2|^2 = |\beta_3|^2 = \gamma\bar{n}$$

Linear stochastic Schrödinger equation

$$\mathrm{d}\phi(t) = \left(-\mathrm{i}H_0 - \mathrm{i}H_f(t) - \frac{1}{2}\sum_{i=1}^{2} L_i(t)^* L_i(t) - \frac{1}{2}\sum_{k=1}^{3} R_k^* R_k + \frac{\lambda}{2}\right)\phi(t_-)\,\mathrm{d}t + \sum_{i=1}^{2} L_i(t)\phi(t_-)\,\mathrm{d}B_i(t) + \sum_{k=1}^{3}\left(\frac{R_k}{\sqrt{\lambda_k}} - \mathbb{1}\right)\phi(t_-)\,\mathrm{d}N_k(t)$$

$\exists!$ strong solution $\phi(t)$ for every initial condition $\phi(0)$

$\phi(t)$ Markov process

If we consider the master equation $\dot{\eta}(t) = \mathcal{L}(t)\,\eta(t) \quad$ with $\eta(0) = |\phi(0)\rangle\langle\phi(0)|$

$$\Rightarrow \quad \eta(t) = \int_\Omega |\phi(t)\rangle\langle\phi(t)|\,\mathrm{d}\mathbb{Q} = \mathbb{E}_\mathbb{Q}\Big[|\phi(t)\rangle\langle\phi(t)|\Big]$$

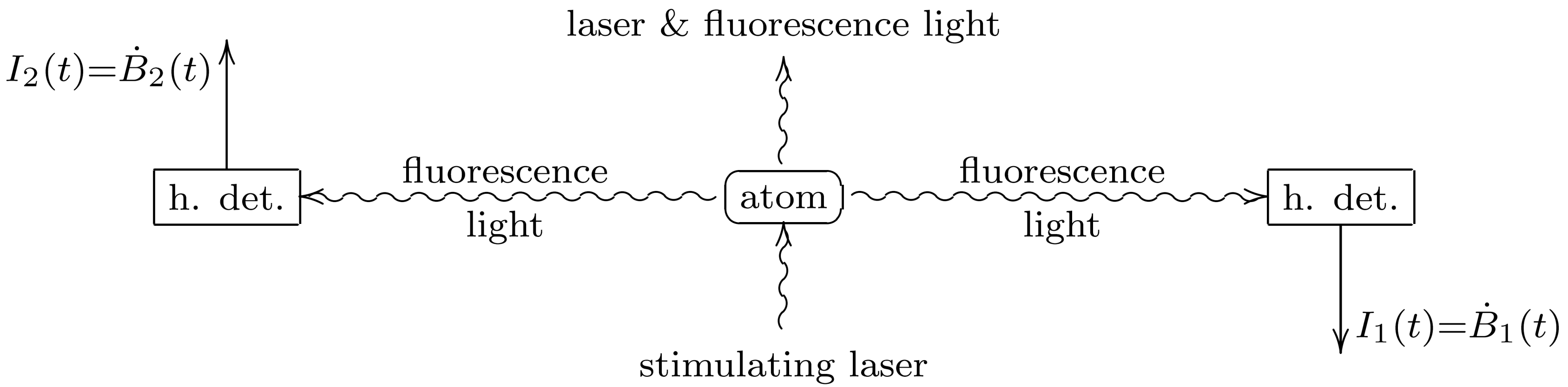
SSE = unravelling of ME

Linear stochastic Schrödinger equation

- Unravelling non unique

  e.g. *B1*, *B2*, *N1* unravel photo emission, *N2*, *N3* unravel thermal effects

- Noise terms can get a physical interpretation by continuous measurements performed on the environment after the interaction (indirect measurements on the atom)



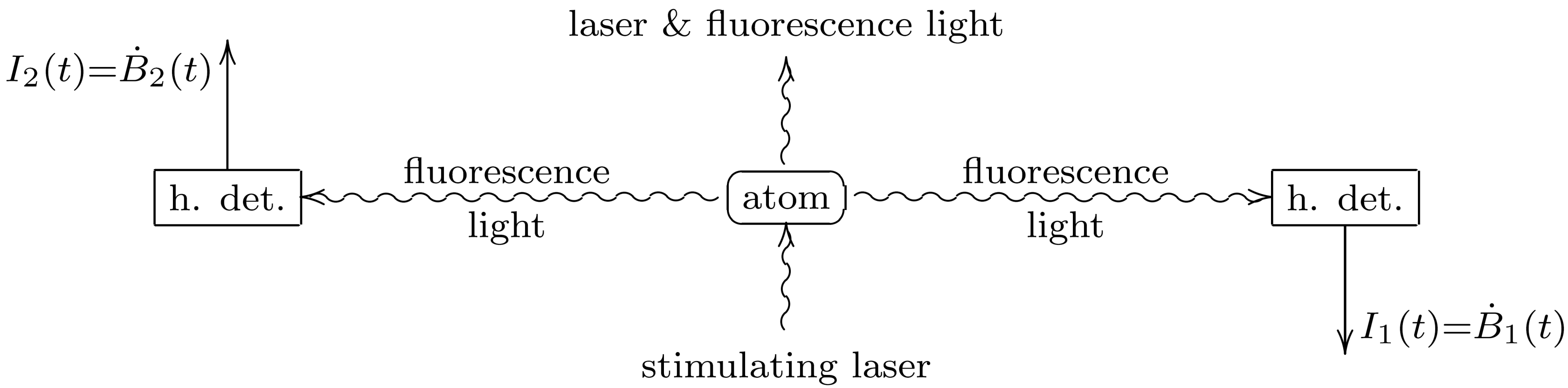
Linear stochastic Schrödinger equation

$$L_j(t) = \mathrm{e}^{\mathrm{i}\theta_j} \frac{\overline{f}(t)}{|f(t)|} \alpha_j \sigma_-$$

select field quadrature

homodyne detection

(square root of) efficiency



Linear stochastic master equation

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\,\mathrm{d}t + \sum_{i=1}^{2}\Big(L_i(t)\,\sigma(t) + \sigma(t)\,L_i(t)^*\Big)\,\mathrm{d}B_i(t)$$

$\sigma(t)$ depends on $B_1(s)$ and $B_2(s)$ for $0 \le s \le t$

$\exists!$ strong solution $\sigma(t)$ for every initial condition $\sigma(0)$

$\sigma(t)$ Markov process

If we consider the master equation $\dot{\eta}(t) = \mathcal{L}(t)\,\eta(t)$ with $\eta(0) = \sigma(0) = \rho_0$

$$\Rightarrow \quad \eta(t) = \int_\Omega \sigma(t)\,\mathrm{d}\mathbb{Q} = \mathbb{E}_\mathbb{Q}\Big[\sigma(t)\Big]$$

Instruments, a posteriori states & output processes

For all *t* define the instrument on $\mathcal{E}_t = \sigma\Big(B_j(s)\Big|\, j = 1,2;\ 0 \le s \le t\Big)$

$$\mathcal{I}_t(E)[\rho_0] := \int_E \sigma(t)\,\mathrm{d}\mathbb{Q} = \int_E \rho(t)\,\mathrm{d}\mathbb{P}^t_{\rho_0} \qquad \forall E \in \mathcal{E}_t$$

Markov process satisfying

non linear SME

$\Rightarrow \quad \sigma(t)$ non normalized *a posteriori* state

$$\rho(t) := \frac{\sigma(t)}{\mathrm{Tr}\,\sigma(t)}$$ normalized *a posteriori* state

$$\mathbb{P}^t_{\rho_0}(E) := \mathrm{Tr}\Big[\mathcal{I}_t(E)[\rho_0]\Big] = \int_E \mathrm{Tr}\Big[\sigma(t)\Big]\,\mathrm{d}\mathbb{Q} \qquad \forall E \in \mathcal{E}_t$$

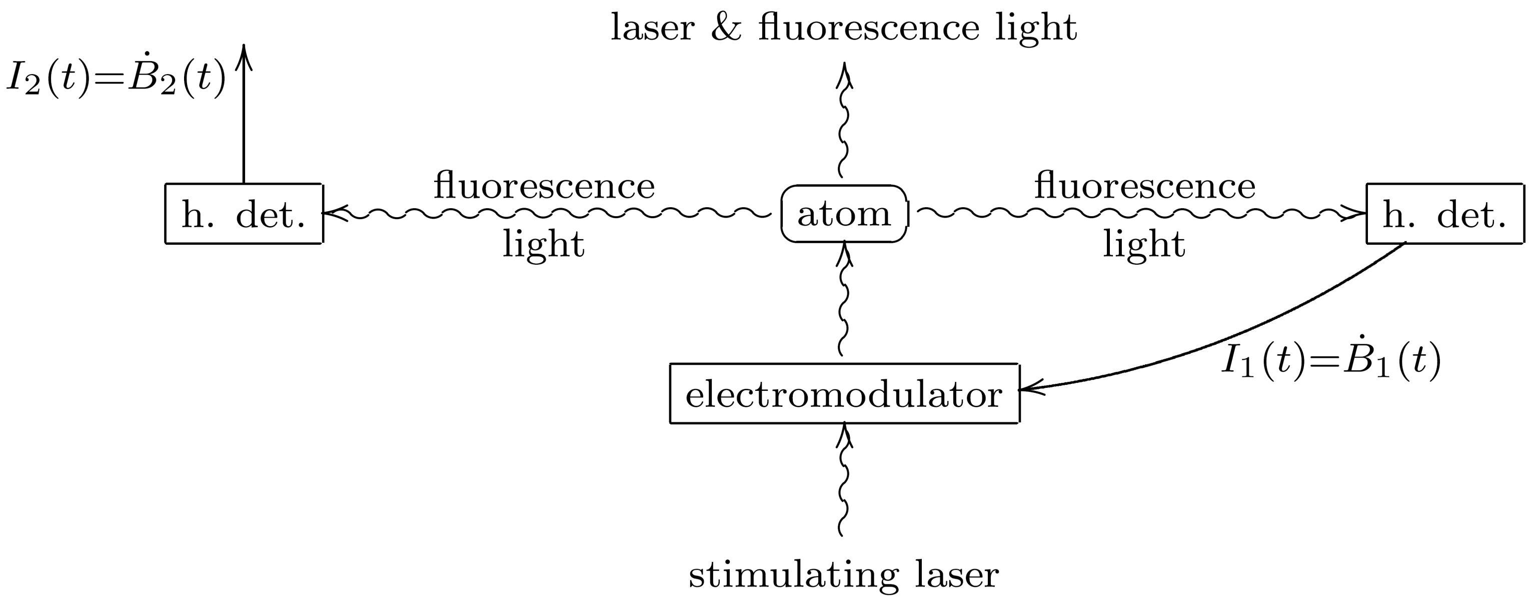
distribution of the outputs

White noise

$$\dot{B}_j(t) = \mathrm{Tr}\Big[\Big(L_j(t) + L_j(t)^*\Big)\rho(t)\Big] + \dot{W}_j(t)$$

a priori state

Imperfect laser & feedback



$$f(t) = \frac{\Omega}{2} \exp\left\{ -\mathrm{i}\Big(\vartheta + \omega t + k_0 B_0(t) + k_1 B_1(t) \Big) \right\}$$

non Markovian feedback

independent Brownian motion modelling non perfectly monochromatic laser

Imperfect laser & feedback

$$d\sigma(t) = \mathcal{L}(t)[\sigma(t)]\,dt + \sum_{i=1}^{2}\Big(L_i(t)\,\sigma(t) + \sigma(t)\,L_i(t)^*\Big)\,dB_i(t)$$

adapted stochastic processes

$\exists!$ strong solution $\sigma(t)$ for every initial condition $\sigma(0)$

$$\mathcal{I}_t(E)[\rho_0] := \int_E \sigma(t)\,d\mathbb{Q} \qquad \forall E \in \mathcal{E}_t$$

$$\mathbb{P}^t_{\rho_0}(E) := \operatorname{Tr}\Big[\mathcal{I}_t(E)[\rho_0]\Big] \qquad \forall E \in \mathcal{E}_t$$

$\sigma(t)$ and $\rho(t)$ **non** Markov processes

$$\hat{\rho}(t) = \mathbb{E}_{\mathbb{P}^t_{\rho_0}}\Big[\rho(t)\Big|\mathcal{E}_t\Big] = \frac{\mathbb{E}_{\mathbb{Q}}\Big[\sigma(t)\Big|\mathcal{E}_t\Big]}{\operatorname{Tr}\Big\{\mathbb{E}_{\mathbb{Q}}\Big[\sigma(t)\Big|\mathcal{E}_t\Big]\Big\}}$$

a posteriori state

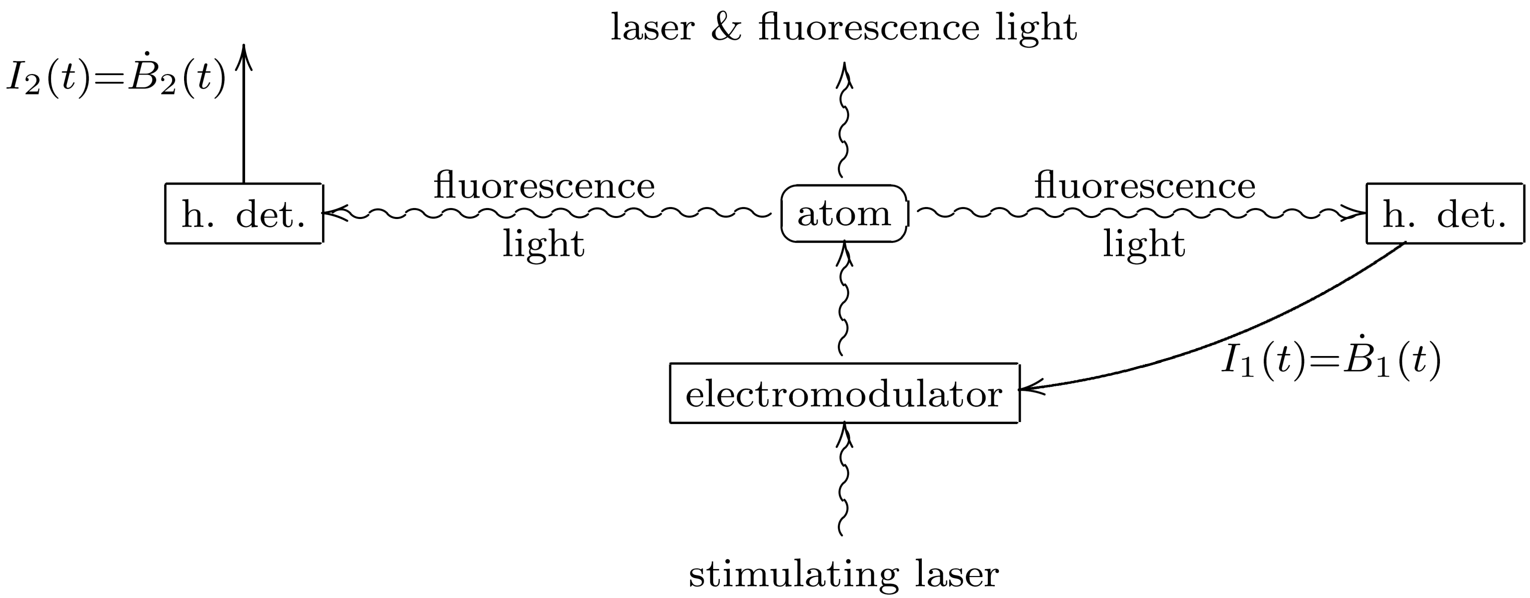
$$\eta(t) = \int_\Omega \sigma(t)\,d\mathbb{Q} = \int_\Omega \rho(t)\,d\mathbb{P}^t_{\rho_0} = \mathcal{I}_t(\Omega)[\rho_0] = \int_\Omega \hat{\rho}(t)\,d\mathbb{P}^t_{\rho_0}$$

a priori state

$\hat{\rho}(t)$ and $\eta(t)$ non Markov processes

do not satisfy closed differential equations

Homodyne spectrum (on the left)



$$S_2^{\mathrm{inel}}(\mu) = 1 + \begin{pmatrix}\cos\vartheta_2 & \sin\vartheta_2 & 0\end{pmatrix} \frac{2\,|\alpha_2|^2}{A^2+\mu^2} \left(A \begin{pmatrix} \cos\vartheta_2(1+d_3) \\ \sin\vartheta_2(1+d_3) \\ -v \end{pmatrix} + v\vec{u} \right)$$

Homodyne spectrum (on the left)

$$S_2^{\mathrm{inel}}(\mu) = 1 + \begin{pmatrix}\cos\vartheta_2 & \sin\vartheta_2 & 0\end{pmatrix} \frac{2\left|\alpha_2\right|^2}{A^2+\mu^2} \left(A \begin{pmatrix} \cos\vartheta_2(1+d_3) \\ \sin\vartheta_2(1+d_3) \\ -v \end{pmatrix} + v\vec{u} \right)$$

$$A = \begin{pmatrix} \frac{\Gamma}{2} & \Delta\omega & -k_1\left|\alpha_1\right|\sin\vartheta_1 \\ -\Delta\omega & \frac{\Gamma}{2} & \Omega + k_1\left|\alpha_1\right|\cos\vartheta_1 \\ 0 & -\Omega & (2\bar{n}+1)\,\gamma \end{pmatrix} \qquad \vec{u} = \begin{pmatrix} -k_1\left|\alpha_1\right|\sin\vartheta_1 \\ k_1\left|\alpha_1\right|\cos\vartheta_1 \\ \gamma \end{pmatrix}$$

$$\Delta\omega = \omega_0 - \omega \qquad \vec{d} = -A^{-1}\vec{u} \qquad \Gamma = (2\bar{n}+1)\,\gamma + {k_0}^2 + {k_1}^2 \qquad v = \cos\vartheta_2\, d_1 + \sin\vartheta_2\, d_2$$

Experimental constraints: $\bar{n},\ \gamma,\ \alpha_1,\ \alpha_2,\ k_0$

Control parameters: $\Omega,\ \Delta\omega,\ \theta_1,\ \theta_2,\ k_1$

$$\gamma = 1,\ k_0 = \bar{n} = 0,\ \left|\alpha_1\right|^2 = \left|\alpha_2\right|^2 = 0.45$$

Control of the squeezing of the fluorescence light

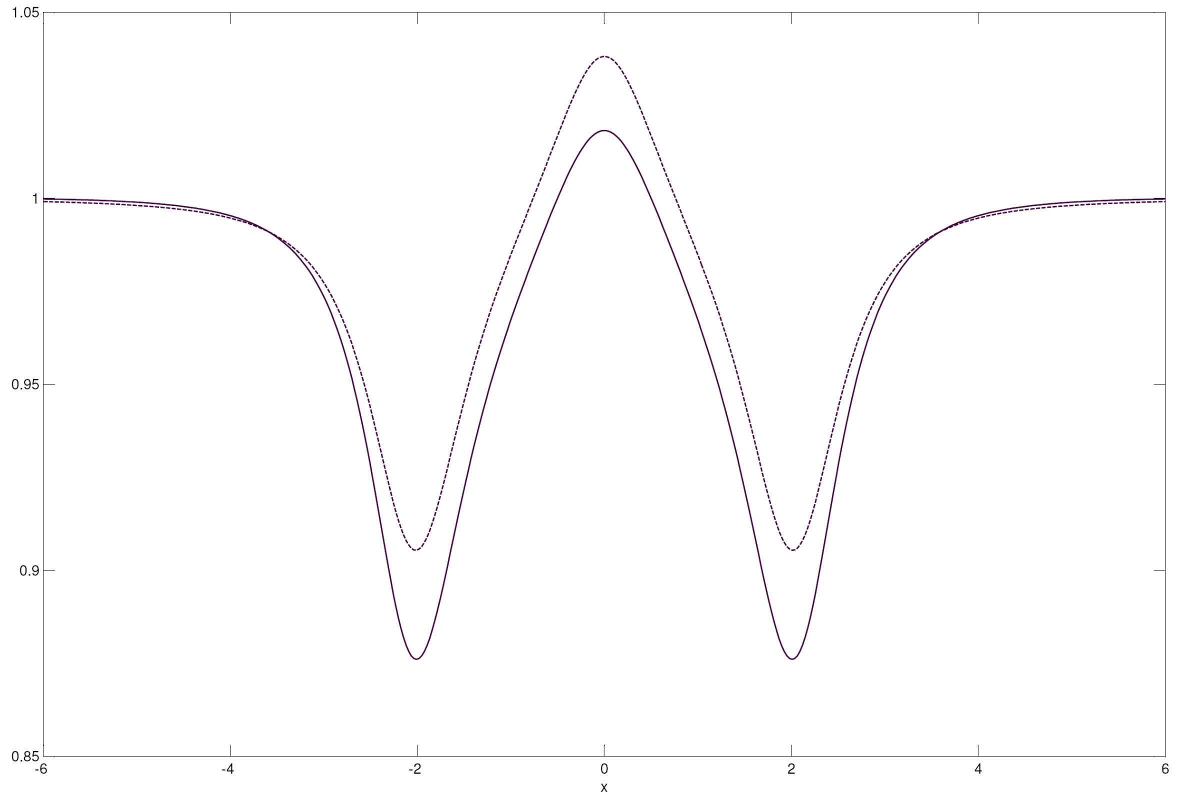


Figure 1: Squeezing control. $S_2^{\text{inel}}(\mu)$ with and without feedback for $\gamma = 1$, $k_0 = \bar{n} = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$ and: (solid line) $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\vartheta_2 = -0.1540$, $\Delta\omega = 1.3833$, $\Omega = 1.6150$; (dotted line) $k_1 = 0$, $\vartheta_2 = -0.1784$, $\Delta\omega = 1.4937$, $\Omega = 1.4360$.

Effect of the Heisenberg uncertainty on the homodyne spectrum

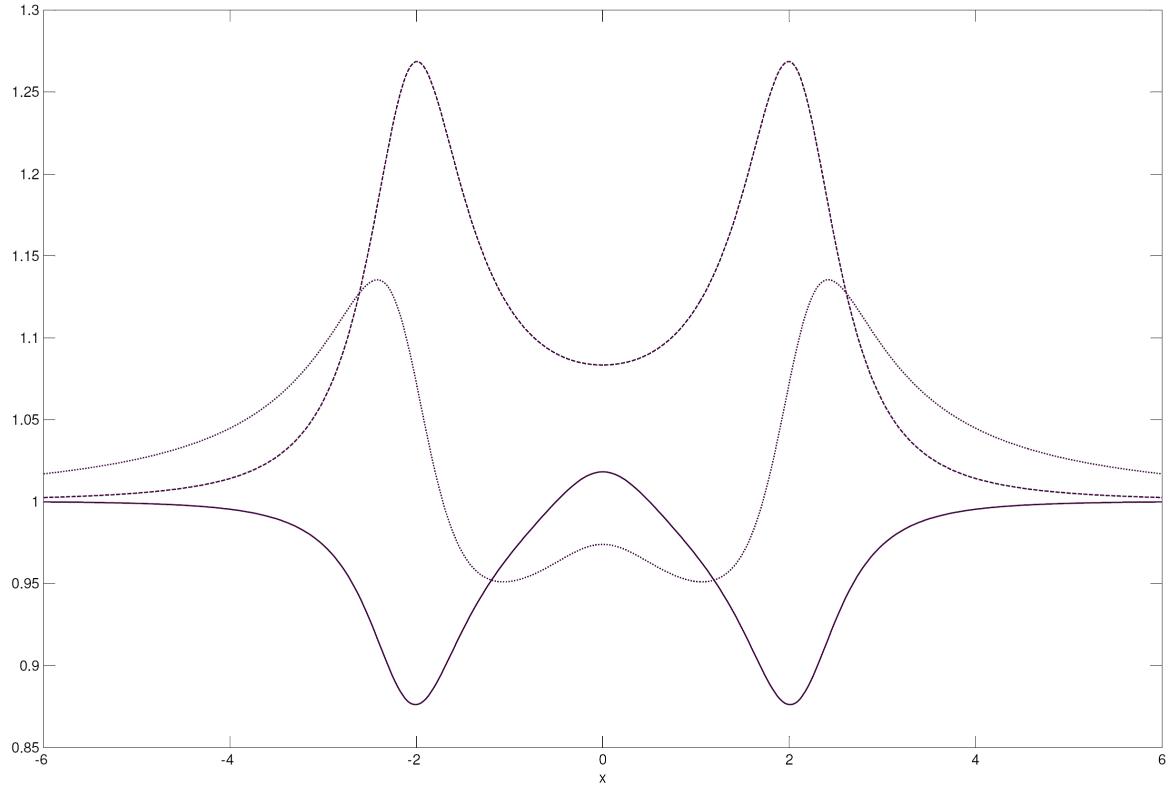


Figure 2: Effect of Heisenberg uncertainty on $S_2^{\rm inel}(\mu)$. The parameters are $\gamma = 1$, $k_0 = \bar{n} = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\Delta\omega = 1.3833$, $\Omega = 1.6150$ and: (solid line) $\vartheta_2 = -0.1540$; (dotted line) $\vartheta_2 = \frac{\pi}{4} - 0.1540$; (dashed line) $\vartheta_2 = \frac{\pi}{2} - 0.1540$.

Conclusions

- Linear SSE and/or SME with stochastic coefficients allow to model continuous measurements also in the case of non Markovian noise and measurement based feedback, in agreement with axiomatic structure of QM

  (a) distribution of the outputs

  (b) non Markovian *a posteriori* evolution

  (c) non Markovian *a priori* evolution

- Introduce and compute the spectrum of the homodyne detection, in agreement with the probability theory of stochastic processes

- Study the effects on the homodyne spectrum of the feedback and of the other control parameters, in presence of non perfect efficiency, detuning, thermal effects, non perfectly monochromatic laser.

References

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